GEOMETRIC OPERATOR MEAN INDUCED FROM THE RICCATI EQUATION

NOBORU NAKAMURA *

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Abstract. We define the geometric mean of positive operators on a Hilbert space as a solution of the Riccati equation, and show some basic properties from the equation.

1. Introduction

Throughout this note, a capital letter means a (bounded linear) operator on a Hilbert space H. An operator A is said to be positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. Order $A \leq B$ is defined when $B - A \geq 0$.

The quadratic equation

$$XA^{-1}X = B \tag{1.1}$$

for positive invertible operators A and B is said the Riccati equation, which has a unique positive solution

$$A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}, \tag{1.2}$$

defining the geometric mean of A and B. The geometric mean, on the other hand, has been already introduced by Pusz and Woronowicz [16] and Ando [2] as the maximum of positive operators X satisfying a 2×2 operator matrix inequality

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0.$$

In this note, giving equivalence between the above two definitions of the geometric mean, we show some main basic properties of the mean, though they have been stated in various places in the literature. What is of interest is the method we employ on usage of the Riccati equation.

2. Geometric mean induced from Riccati equation

Before we begin with the discussion of the geometric mean, we state about the arithmetic and the harmonic means. For positive operators A and B, let

$$A\nabla B = \frac{1}{2}(A+B) \quad \text{and} \quad A!B = 2(A:B)$$

be their arithmetic and harmonic means, respectively. Here $A : B = (A^{-1} + B^{-1})^{-1} = A(A+B)^{-1}B$ is the parallel sum of A and B with the assumption of invertibility of both A and B or at least $A + B$. It is easily seen that the arithmetic-harmonic mean inequality

$$A\nabla B \geq A!B$$

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holds by $A\nabla B - A!B = \frac{1}{2}(A-B)(A+B)^{-1}(A-B)$. The following is the most basic fact for our discussion.

Lemma 2.1. *Every positive operator A has a unique positive square root $A^{1/2}$. Furthermore, if $0 \le A \le B$, then $A^{1/2} \le B^{1/2}$, that is, the square root is operator monotone.*

For completeness, we sketch the proof. Borrowing the arithmetic-harmonic-mean technique presented by J.I.Fujii [5], [6]. First we show existence of the square root. Let $X_1 = A,\ Y_1 = I$,

$$X_{n+1} = X_n \nabla Y_n \quad \text{and} \quad Y_{n+1} = X_n ! Y_n \quad \text{for} \quad n = 1, 2, \dots .$$

Then we see that

$$X_n - X_{n+1} = \frac{1}{2}(X_n - Y_n) \ge 0$$

and

$$X_2 \ge \cdots \ge X_n \ge Y_n \ge \cdots \ge Y_2 \qquad \text{for all } n \ge 2,$$

so that the sequences $\{X_n\}$ and $\{Y_n\}$ are convergent and have a common limit. Further since $X_nY_n = Y_nX_n = A$, putting X the common limit, we obtain $X^2 = A$ or X as a positive square root of A.

Next for uniqueness, let $\mathcal{S}_A = \{Y \ge 0; Y^2 = A\}$. Then from the above argument the set $\mathcal{S}_A$ is not empty, and all $Y \in \mathcal{S}_A$ commute with A, since $YA = Y^3 = AY$. Let C be the operator obtained as above and $Y \in \mathcal{S}_A$ be arbitrary, then we have to show $Y = C$. Since Y commutes with A, we see that Y commutes with X_n and Y_n for all n and so does C. Further since $\mathcal{S}_C$ is not empty, we have a $D \in \mathcal{S}_C$, that is, $D^2 = C$. Using this D, we have $C^2 \le D\|C\|D = \|C\|C$. From this inequality

$$\begin{aligned} 0 \le (Y-C)C^2(Y-C) &\le \|C\|(Y-C)C(Y-C) \\ &\le \|C\|(Y-C)(Y+C)(Y-C) = 0. \end{aligned}$$

Hence $C(Y-C) = 0$, or $CY = C^2 = A (= YC)$. Consequently, $(Y-C)^2 = Y^2 - 2YC + C^2 = 0$, so that $Y = C$.

For monotonicity, assume that $A \ge B \ge 0$. Put $Z_1 = B, W_1 = I,\ Z_{n+1} = Z_n \nabla W_n$ and $W_{n+1} = Z_n ! W_n$ for $n = 1, 2, ...$ in the same manner as before, to obtain $B^{1/2}$. Then $X_n \ge Z_n$ and $Y_n \ge W_n$, so that for their common limit (denoted by Z) we have $X \ge Z$, which is desired.

There are some other methods to show monotonicity of the square root $f_{1/2}(A) = A^{1/2}$. One of a typical method is to make use of fundamental properties of the spectral radius (cf. [14]).

Note that the Riccati equation (1.1) is equivalent to

$$A^{-1/2}XA^{-1/2}A^{-1/2}XA^{-1/2} = (A^{-1/2}XA^{-1/2})^2 = A^{-1/2}BA^{-1/2},$$

so that from the above lemma

$$A^{-1/2}XA^{-1/2} = (A^{-1/2}BA^{-1/2})^{1/2} \text{ or } \ X = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Hence we have the following:

Theorem 2.2. *For positive invertible operators A and B the Riccati equation $XA^{-1}X = B$ has the geometric mean $A\sharp B$ defined by* (1.2) *as a unique positive solution.*

Corollary 2.3 (cf. [2, Corollary I. 2.1]). *Geometric means have the following properties for positive invertible operators A and B.*

(i) $A\sharp B = B\sharp A$.

(ii) $(\alpha A)\sharp(\alpha B) = \alpha(A\sharp B)$ for $\alpha > 0$.

(iii) $C(A\sharp B)C^* = (CAC^*)\sharp(CBC^*)$ *for any invertible operator* C.

(iv) $A\sharp A = A,\ 1\sharp A = A^{1/2}$.

(v) $A^{-1}\sharp B^{-1} = (A\sharp B)^{-1}$.

Proof. For (i), let $X = A\sharp B$, then $XA^{-1}X = B$. Taking the inverses of both sides, we see that $X^{-1}AX^{-1} = B^{-1}$. (Note that X is invertible.) Hence $XB^{-1}X = A$, or $X = B\sharp A$. Similarly we can induce the other facts from the Riccati equation.

Corollary 2.4 (cf. [4, Corollary 2]). *Let A and B be positive invertible operators. Then*

$$(A + B)\sharp(A : B) = A\sharp B.$$

Proof. Note that

$$\begin{aligned}(A\sharp B)(A : B)^{-1}(A\sharp B) &= (A\sharp B)(A^{-1} + B^{-1})(A\sharp B)\\ &= (A\sharp B)A^{-1}(A\sharp B) + (A\sharp B)B^{-1}(A\sharp B) = B + A.\end{aligned}$$

Hence $A\sharp B$ is the solution of the Riccati equation $X(A : B)^{-1}X = A + B$, which implies the desired identity.

The geometric mean $A\sharp B$ for positive operators A and B, *possibly not invertible*, is defined by the limit of $A_\epsilon\sharp B_\epsilon$ as $\epsilon \to +0$, where $C_\epsilon = C + \epsilon I,\ \epsilon > 0$. We can show that all facts in Corollary 2.3 are still valid without invertibility of A and B.

Remark. The Riccati equation itself has been considered in more general setting: For positive operators A and B with B being nonsingular, that is, the kernel of B is $\{0\}$, the quadratic equation

$$XAX = B$$

was discussed in Pederson-Takesaki [15], Nakamoto [12], and it was shown that the equation has a positive solution if and only if $(A^{1/2}BA^{1/2})^{1/2} \le kB$ for some $k > 0$.

The following general result on monotonicity of $f_p(A) = A^p$ is well-known, and is essentially induced from $p = 1/2$ ([7], [9]):

Löwner-Heinz Theorem. *Let A and B be positive operators. Then*

$$A \ge B \quad \textit{implies} \quad A^p \ge B^p \quad \textit{for} \quad 0 \le p \le 1. \tag{2.1}$$

3. The geometric mean defined by Ando

Recall that Ando's definition of the geometric mean of positive operators A and B is

$$A\sharp B = \max\left\{X \ge 0\ ;\begin{bmatrix} A & X\\ X & B\end{bmatrix} \ge 0\right\}. \tag{3.1}$$

It must be emphasized that the definition does not require invertibility of operators.

Now we state a useful fact to combine the Riccati equation with Ando's definition of the geometric mean :

Lemma 3.1. *If A and B are positive and A is invertible, then*

$$A\sharp B = \max\{X \geq 0\ ; XA^{-1}X \leq B\}. \tag{3.2}$$

Proof. Let X satisfy *the Riccati inequality*

$$XA^{-1}X \leq B.$$

Then $(A^{-1/2}XA^{-1/2})^2 \leq A^{-1/2}BA^{-1/2}$, so that $A^{-1/2}XA^{-1/2} \leq (A^{-1/2}BA^{-1/2})^{1/2}$, or

$$X \leq A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{-1/2} = A\sharp B.$$

On the other hand, $X = A\sharp B$ satisfies the inequality $XA^{-1}X = B \leq B$. Hence we have the desired (3.2).

Remark. From the above lemma the Riccati inequality $XA^{-1}X \leq B$ implies $X \leq A\sharp B$. However, the converse is not necessarily true. In fact, take positive invertible operators C and D such that $C \leq D$ with $C^2 \not\leq D^2$. Putting $A = I,\ B = D^2$ and $X = C$, we have $X \leq D = I\sharp D^2 = A\sharp B$, but $XA^{-1}X = C^2 \not\leq D^2 = B$.

Theorem 3.2 (cf. [2, Theorem I. 2]). *For positive operators A and B with A being invertible, the geometric mean induced from the Riccati equation and one defined by Ando are identical.*

Proof. We may only consider the case that A is invertible. (If necessary, consider A_ϵ in place of A and take the limit as $\epsilon \to +0$. In order to see that (3.1) and (3.2) define the same maximum, we employ the following ingenious device due to Professor R. Nakamoto [13] (and also [11]):

$$\begin{bmatrix} I & 0 \\ -XA^{-1} & I \end{bmatrix}\begin{bmatrix} A & X \\ X & B \end{bmatrix}\begin{bmatrix} I & -A^{-1}X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B - XA^{-1}X \end{bmatrix}.$$

From this relation, we see that $\begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0$ holds if and only if

$$B - XA^{-1}X \geq 0$$

Hence from Lemma 3.1 we have equivalence of the two definitions.

From the above theorem, we can show the following results.

Corollary 3.3 (cf. [2, Corollary I. 2.1]). *Let A and B be positive operators. Then* $C(A\sharp B)C^* \leq (CAC^*)\sharp(CBC^*)$ *for any operator C.*

Corollary 3.4 (cf. [2, Corollary I. 2.1]). *Let $A,\ B,\ C,$ and D be positive operators. Then*

(i) $A \geq B,\ \ C \geq D$ *imply* $A\sharp C \geq B\sharp D$.

(ii) $(A + B)\sharp(C + D) \geq (A\sharp C) + (B\sharp D)$.

For positive operators A and B the arithmetic-geometric-harmonicmean inequalities are

$$A\nabla B \geq A\sharp B \geq A!B.$$

Applying (i) in the above corollary, we give a short proof of them: Since $A\nabla B \geq A!B$, we have,

$$A\nabla B = (A\nabla B)\sharp(A\nabla B) \geq (A\nabla B)\sharp(A!B) \geq (A!B)\sharp(A!B) = A!B.$$

Now since $(A\nabla B)\sharp(A!B) = (A+B)\sharp(A:B) = A\sharp B$ by Corollary 2.4, we have the desired inequalities.

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* Fujikoshi-kogyo Senior Highschool, Higashi-ishigane 7-5, Toyama, 930-0964, Japan
E-mail address: nakamura@fujikoshiko.ac.jp