COMMUTATION OF INTERNAL COLIMIT AND FINITE LIMITS

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Abstract. It is well known that if $\mathcal{E}$ is a topos, $Colim : \mathcal{E}^C \longrightarrow \mathcal{E}$ preserves finite limits if and only if C is a filtered internal category.

The main result of this paper is that for $\mathcal{E}$ belonging to a wider class of categories than topoi, $Colim : \mathcal{E}^C \longrightarrow \mathcal{E}$ preserves finite limits if C is smooth, proper, and a universal extremal filtered internal category.

This result is simplified for topological categories $\mathcal{E}$ over Set that satisfy certain conditions. It is shown that for these categories $Colim : \mathcal{E}^C \longrightarrow \mathcal{E}$ preserves finite limits if C is filtered and d_{0C} is initial.

1. Preliminaries

Let $\mathcal{E}$ be a category with finite limits and coequalizers of reflexive pairs. Throughout the paper we refer to a pullback diagram:

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "p.b."] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

Diagram I

as the pullback $f\pi_1 = g\pi_2 : P \longrightarrow Z$. Given a morphism $f : X \longrightarrow Y$ in $\mathcal{E}$, form the kernel pair, $K_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$, of f. Let $q_f : X \longrightarrow I$ be the coequalizer of the reflexive pair $K_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$. Since f coequalizes π_1 and π_2, it factors through q_f by a unique morphism i_f. Hence to each morphism f in $\mathcal{E}$ there corresponds unique morphisms q_f and i_f such that $f = i_f q_f$, where q_f is the coequalizer of the kernel pair of f. We refer to q_f, respectively i_f, as the *q-map*, respectively *i-map* of f.

Let $Cat(\mathcal{E})$ denote the category of the internal categories in $\mathcal{E}$, see [2] p 47. For C in $Cat(\mathcal{E})$, define the maps d_C, t_C, and s_C as follows, see [2], p66.

Form the pullback

$$d_1\pi_1 = d_1\pi_2 : K_C \longrightarrow C_0$$

Let $\pi = \langle \pi_1, \pi_2 \rangle : K_C \longrightarrow C_1^2$ and $d_C = d_0^2\pi : K_C \longrightarrow C_0^2$.

Form the pullbacks

$$\langle d_0, d_1 \rangle \pi_1 = \langle d_0, d_1 \rangle \pi_2 : R \longrightarrow C_0^2$$

and

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$$\langle \pi_2 m, m\rangle \pi_1 = \langle \pi_2 m, m\rangle \pi_2 : T \longrightarrow C_1^2$$

Let $t_C = (\pi_1\pi_1, \pi_1\pi_2) : T \longrightarrow R$.

Finally form the pullbacks

$$d_0\pi_1 = d_0\pi_2 : Q \longrightarrow C_0 \quad \text{and} \quad m\pi_1 = m\pi_2 : S \longrightarrow C_1$$

Let $s_C = (\pi_1\pi_1, \pi_1\pi_2) : S \longrightarrow Q$.

We write *e.e.*, for an *extremal epimorphism*, i.e., a morphism that whenever it factors through a monomorphism with the same codomain, the monomorphism is an isomorphism and we write *u.e.e.*, for a *universal extremal epimorphism*, i.e. an e.e. whose pullback along every morphism is an e.e.

Consider the following (internal) conditions on an internal category C in $\mathcal{E}$.

1.1. **Conditions:**

(a) The unique morphism $C_0 \xrightarrow{!} 1$ is an e.e.

(b) The morphism $d_C : K_C \longrightarrow C_0^2$ is a u.e.e.

(c) The morphism $t_C : T_C \longrightarrow R_C$ is a u.e.e.

(d) The morphism $s_C : S_C \longrightarrow Q_C$ is a u.e.e.

Also consider the following (external) conditions on C.

1.2. **Conditions:**

(a)$'$ The unique morphism $C_0 \longrightarrow 1$ is an e.e.

(b)$'$ For all pair of morphisms $U \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} C_0$ in $\mathcal{E}$, there are morphisms $V \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} C_1$ and $\epsilon : V \longrightarrow U$ such that ϵ is a u.e.e., $d_1\lambda_1 = d_1\lambda_2$ and $d_0\lambda_j = \gamma_j\epsilon$, for $j = 1, 2$.

(c)$'$ For all pair of morphisms $U \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} C_1$ coequalized by d_0 and d_1, there are morphisms $\mu : V \longrightarrow C_1$ and $\epsilon : V \longrightarrow U$ such that ϵ is a u.e.e., $d_0\mu = d_1\gamma_j\epsilon$, for $j = 1, 2$, and $m(\gamma_1\epsilon, \mu) = m(\gamma_2\epsilon, \mu)$.

(d)$'$ For all pair of morphisms $U \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} C_1$ coequalized by d_0, there are morphisms $V \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} C_1$, and $\epsilon : V \longrightarrow U$ such that ϵ is a u.e.e., $d_0\delta_j = d_1\gamma_j\epsilon$, for $j = 1, 2$, and $m(\gamma_1\epsilon, \delta_1) = m(\gamma_2\epsilon, \delta_2)$.

1.3. **Theorem:** Conditions (a), (b), (c), and (d) of 1.1 are equivalent to conditions (a)$'$, (b)$'$, (c)$'$ and (d)$'$ of 1.2, respectively.

Proof: (a) and (a)$'$ are the same.

(b) $\Rightarrow$ (b)$'$: Given $U \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} C_0$, pullback d_C along $\langle \gamma_1, \gamma_2\rangle : U \longrightarrow C_0^2$ to get $\lambda : V \longrightarrow K_C$ and $\epsilon : V \longrightarrow U$. (b) implies that d_C, and hence ϵ, is a u.e.e. Let $\lambda_j = \pi_j\lambda, j = 1, 2$, where $K_C \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} C_1$ is the kernel pair of $d_1 : C_1 \longrightarrow C_0$. $d_1\pi_1 = d_1\pi_2$ implies $d_1\lambda_1 = d_1\pi_1\lambda = d_1\pi_2\lambda = d_1\lambda_2$. Also the above pullback diagram implies that $d_C\lambda = \langle \gamma_1, \gamma_2\rangle\epsilon$, where $d_C = d_0^2\pi = d_0^2\langle \pi_1, \pi_2\rangle = \langle d_0\pi_1, d_0\pi_2\rangle$, thus $\langle d_0\pi_1, d_0\pi_2\rangle\lambda = \langle \gamma_1, \gamma_2\rangle\epsilon$, which implies that $d_0\pi_j\lambda = \gamma_j\epsilon, j = 1, 2$. This in turn implies that $d_0\lambda_j = \gamma_j\epsilon, j = 1, 2$ as desired.

(b)$'$ $\Rightarrow$ (b): Let $C_0^2 \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} C_0$ be the projections. (b)$'$ implies the existence of morphisms $V \overset{\lambda_1}{\underset{\lambda_2}{\rightrightarrows}} C_1$, and $\epsilon : V \longrightarrow C_0^2$ such that ϵ is u.e.e., $d_1\lambda_1 = d_1\lambda_2$, and $d_0\lambda_j = pr_j\epsilon, j = 1, 2$. Since $d_1\lambda_1 = d_1\lambda_2$, the pullback $d_1\pi_1 = d_1\pi_2 : K_C \longrightarrow C_0$ implies there is a unique $\lambda = (\lambda_1, \lambda_2) : V \longrightarrow K_C$ such that $\pi_j\lambda = \lambda_j, j = 1, 2$. We have $\pi\lambda = \langle\pi_1, \pi_2\rangle\lambda = \langle\pi_1\lambda, \pi_2\lambda\rangle = \langle\lambda_1, \lambda_2\rangle$, and therefore $d_C\lambda = d_0^2\pi\lambda = d_0^2\langle\lambda_1, \lambda_2\rangle = \langle d_0\lambda_1, d_0\lambda_2\rangle = \langle pr_1\epsilon, pr_2\epsilon\rangle = \langle pr_1, pr_2\rangle\epsilon = \epsilon$. Now since ϵ is a u.e.e., so is d_C.

The proofs that (c) $\Leftrightarrow$ (c)$'$ and (d) $\Leftrightarrow$ (d)$'$ are similar.

□

1.4. **Definition:** C in $Cat(E)$ is said to be a *universal extremal weakly filtered* (*u.e.w.f.*) category if conditions (c) and (d) of 1.1 are met, and it is said to be a *universal extremal filtered* (*u.e.f.*) category if conditions (a), (b), and (c) of 1.1 are satisfied.

1.5. **Lemma:** Let C be in $Cat(E)$ and $\gamma : F \longrightarrow C$ be a discrete opfibration.

(a) If C is a u.e.f. category, then it is a u.e.w.f. category.

(b) If C is a u.e.w.f. category, then so is F.

(c) If C is a u.e.f. category, then C satisfies the following condition:

(d)$''$ For all pairs of morphisms $U \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} C_1$ and $U \overset{\gamma'_1}{\underset{\gamma'_2}{\rightrightarrows}} C_1$ such that $d_0\gamma_1 = d_0\gamma_2, d_0\gamma'_1 = d_0\gamma'_2$ and $d_1\gamma_j = d_1\gamma'_j, j = 1, 2$, there are morphisms $V \overset{\zeta_1}{\underset{\zeta_2}{\rightrightarrows}} C_1$, and $\epsilon : V \longrightarrow U$ such that ϵ is a u.e.e., $d_0\zeta_j = d_1\gamma_j\epsilon = d_1\gamma'_j\epsilon, m(\gamma_1\epsilon, \zeta_1) = m(\gamma_2\epsilon, \zeta_2)$, and $m(\gamma'_1\epsilon, \zeta_1) = m(\gamma'_2\epsilon, \zeta_2)$.

Proof: (a) See [2], p 68, 2.53 Lemma.

(b) See [2], p 69, 2.56 Lemma.

(c) Straightforward.

□

2. Proper Internal Category

A category $\mathcal{E}$ with finite limits and coequalizers of reflexive pairs is said to be admissible if for all $f \in \mathcal{E}$, the i-map of f is mono.

In this and the next section we let $\mathcal{E}$ be an admissible category.

2.1. **Lemma:** In $\mathcal{E}$:

(a) a morphism is an e.e. iff it is a coequalizer of its kernel pair.

(b) the composition of two e.e.'s as an e.e..

Proof: Straightforward.

□

2.2. **Definition:** Let C be in $Cat(\mathcal{E})$.

(a) C is said to be *weakly proper* if the q-map of d_C is a u.e.e., and the i-map of d_C is effective, (see [2], p 16) if it is an equivalence relation (see [2], p 16).

(b) C is said to be *proper* if for all discrete opfibrations $\gamma : F \longrightarrow C$, F is weakly proper.

2.3. **Lemma:** Let $\gamma : F \longrightarrow C$ be a discrete opfibration in $Cat(\mathcal{E})$.

(a) If C is proper, then it is weakly proper.

(b) If C is proper, then so is F.

Proof: (a) Consider the discrete opfibration $1 : C \longrightarrow C$ to conclude C is weakly proper.

(b) Let $\alpha : E \longrightarrow F$ be a discrete opfibration. The composite $\gamma\alpha : E \longrightarrow C$ is a discrete opfibration. So E is weakly proper as desired.

$\square$

2.4. **Lemma:** Let C in $Cat(\mathcal{E})$ be weakly proper. If C satisfies Condition 1.1(d), then the i-map of d_C is effective.

Proof: By Definition 2.2(a), it is sufficient to show the i-map of d_C is an equivalence relation. A straightforward computation shows that the relation is symmetric and reflexive. To show transitivity form the pullbacks $d_1\pi_1 = d_1\pi_2 : K_C \longrightarrow C_0$ and

$d_C a = d_C b : R \longrightarrow C_0^2$. Let $q : K_C \longrightarrow I$ be the coequalizer of $R \underset{b}{\overset{a}{\rightrightarrows}} K_C$. We have $iq = d_C = d_0^2\pi$. Form the pullback $i_2 r = i_1 s : T \longrightarrow C_0$, where i_1 and i_2 are the components of i. We need to show $\langle i_1 r, i_2 s\rangle : T \longrightarrow C_0^2$ factors through i. Form the following pullbacks to get the maps α_1, α_2, and β.

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left=50, "\alpha_2"] \arrow[d] \arrow[dr, "\beta"] \arrow[dd, bend right=50, "\alpha_1"'] & T_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & K \arrow[d, "q"] \arrow[dd, bend left=50, "d_0\pi_1"] \\
T_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & T \arrow[r, "s"] \arrow[d, "r"] \arrow[dr, phantom, "p.b."] & I \arrow[d, "i_1"] \\
K \arrow[r, "q"'] \arrow[rr, bend right=50, "d_0\pi_2"'] & I \arrow[r, "i_2"'] & C_0
\end{tikzcd}$$

Diagram II

The fact that C is weakly proper implies q is a u.e.e., and therefore so is β.

By Diagram II we have, $d_0(\pi_2\alpha_1) = d_0(\pi_1\alpha_2)$, so we get the unique morphism $(\pi_2\alpha_1, \pi_1\alpha_2) : A \longrightarrow Q$, such that $\pi_1(\pi_2\alpha_1, \pi_1\alpha_2) = \pi_2\alpha_1$, and $\pi_2(\pi_2\alpha_1, \pi_1\alpha_2) = \pi_1\alpha_2$. pullback s_C along $(\pi_2\alpha_1, \pi_1\alpha_2)$ to get:

(1) $$(\pi_2\alpha_1, \pi_1\alpha_2)\epsilon = s_C\delta : V \longrightarrow Q$$

C satisfies Condition (d), so by Theorem 1.3 it satisfies Condition (d)$'$, hence s_C, and therefore ϵ is a u.e.e.. (1), and the definition of K_C as the pullback of d_1 along d_1 imply that $d_1\pi_1\alpha_1\epsilon = d_1\pi_2\alpha_1\epsilon = d_1\pi_1\pi_1\delta = d_0\pi_2\pi_1\delta$ and $d_1\pi_2\alpha_2\epsilon = d_1\pi_1\alpha_2\epsilon = d_1\pi_1\pi_2\delta = d_0\pi_2\pi_2\delta$. Now the definition of C_2 as the pullback of d_0 along d_1 yields the existence of the maps:

$$(\pi_1\alpha_1\epsilon, \pi_2\pi_1\delta), (\pi_2\alpha_2\epsilon, \pi_2\pi_2\delta) : V \rightrightarrows C_2$$

Define $\eta_j = m(\pi_j\alpha_j\epsilon, \pi_2\pi_j\delta) : V \longrightarrow C_1$, where m is the multiplication. Since $m\pi_1 = m\pi_2$ and $d_1 m = d_1\pi_2$, we have $d_1\pi_2\pi_1 = d_1 m\pi_1 = d_1 m\pi_2 = d_1\pi_2\pi_2$, and so $d_1\eta_1 = d_1 m(\pi_1\alpha_1\epsilon, \pi_2\pi_1\delta) = d_1\pi_2(\pi_1\alpha_1\epsilon, \pi_2\pi_1\delta) = d_1\pi_2\pi_1\delta = d_1\pi_2\pi_2\delta = d_1\pi_2(\pi_2\alpha_2\epsilon, \pi_2\pi_2\delta) = d_1 m(\pi_2\alpha_2\epsilon, \pi_2\pi_2\delta) = d_1\eta_2$. Hence η_1 and η_2 are coequalized by d_1, which yields the existence of the unique map $(\eta_1, \eta_2) : V \longrightarrow K$. Now we have $iq(\eta_1, \eta_2) = d(\eta_1, \eta_2) = d_0^2\pi(\eta_1, \eta_2) = \langle d_0\pi_1, d_0\pi_2\rangle(\eta_1, \eta_2) = \langle d_0\pi_1(\eta_1, \eta_2), d_0\pi_2(\eta_1, \eta_2)\rangle = \langle d_0\eta_1, d_0\eta_2\rangle$, and by Diagram II, $\langle i_1 r, i_2 s\rangle\beta\epsilon = \langle i_1 r\beta\epsilon, i_2 s\beta\epsilon\rangle = \langle i_1 q\alpha_1\epsilon, i_2 q\alpha_2\epsilon\rangle = \langle d_0\pi_1\alpha_1\epsilon, d_0\pi_2\alpha_2\epsilon\rangle = \langle d_0\eta_1, d_0\eta_2\rangle$. Therefore we have

(2) $$iq(\eta_1,\eta_2) = \langle i_1 r, i_2 s\rangle \beta\epsilon$$

Since β and ϵ are e.e.'s, so is $\beta\epsilon$, and so by Lemma 2.1, $\beta\epsilon$ is the coequalizer of its kernel pair. Let $W \underset{g}{\overset{f}{\rightrightarrows}} V$ be the kernel pair of $\beta\epsilon$. It follows that $\beta\epsilon f = \beta\epsilon g$, and so $\langle i_1 r, i_2 s\rangle\beta\epsilon f = \langle i_1 r, i_2 s\rangle\beta\epsilon g$. Equation (2) implies that $iq(\eta_1,\eta_2)f = iq(\eta_1,\eta_2)g$. Since i is a mono $q(\eta_1,\eta_2)f = q(\eta_1,\eta_2)g$. Thus $q(\eta_1,\eta_2)$ coequalizes f and g, and hence there is a unique map h such that $q(\eta_1,\eta_2) = h\beta\epsilon$. Therefore $ih\beta\epsilon = iq(\eta_1,\eta_2)$, and so, by (2), $ih\beta\epsilon = \langle i_1 r, i_2 s\rangle\beta\epsilon$. $\beta\epsilon$ is an e.e., and thus an epi. Hence $ih = \langle i_1 r, i_2 s\rangle$, that is, $\langle i_1 r, i_2 s\rangle$ factors through i as desired.

□

For an object X of $\mathcal{E}$, let $\mathcal{E}/X$ denote the comma category, see [3] p 47. Let C be in $Cat(\mathcal{E})$, and $\sigma : C_0 \longrightarrow L$ the coequalizer of $C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0$. C can be regarded as an object of $Cat(\mathcal{E}/L)$ by means of the morphism $\sigma : C_0 \longrightarrow L$, in which case we denote it by C/L.

2.5. **Proposition:** Let C in $Cat(\mathcal{E})$ be weakly proper, and $\sigma : C_0 \longrightarrow L$ be the coequalizer of $C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0$. If C is a u.e.w.f. category in $Cat(\mathcal{E})$, then C/L is a u.e.f. category in $Cat(\mathcal{E}/L)$.

Proof: We need to show Conditions 1.1(a)-(c) hold. Since a morphism of $\mathcal{E}/L$ is a mono, respectively an e.e., in $\mathcal{E}/L$ if and only if it is so when regarded as a morphism of $\mathcal{E}$, Conditions (a) and (c) follow trivially.

To show (b) holds in $\mathcal{E}/L$, we need to show $d_{C/L}$ is a u.e.e.. But $d_{C/L} = \partial$ is obtained as follows:

$$\begin{tikzcd}
K_C \arrow[dr, "\partial"] \arrow[drr, bend left, "d_0\pi_2"] \arrow[ddr, bend right, "d_0\pi_1"'] & & \\
& C_0 \times_L C_0 \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "p.b."] & C_0 \arrow[d, "\sigma"] \\
& C_0 \arrow[r, "\sigma"'] & L
\end{tikzcd}$$

Diagram III

The commutativity of the outer square in Diagram III follows from the fact that $\sigma d_0 = \sigma d_1$. We have:

(1) $$d_C = \langle a, b\rangle\partial : K_C \longrightarrow C_0^2 \text{ and } iq = d_C : K_C \longrightarrow C_0^2$$

Since q is the q-map of d_C, it is a u.e.e. On the other hand since C is weakly proper, C is a u.e.w.f. category in $Cat(\mathcal{E})$, so it satisfies Condition 1.1(d). Lemma 2.4 now implies that, the i-map of d_C is effective. Therefore there is a morphism $f : C_0 \longrightarrow Y$ for some object Y of $\mathcal{E}$, such that:

(2) $$fi_1 = fi_2 : I \longrightarrow Y \text{ is a pullback}$$

Let $i : C_0 \longrightarrow C_1$ be the inclusion of identities. Since $d_1 = d_1(id_1)$, the pullback $d_1\pi_1 = d_1\pi_2 : K_C \longrightarrow C_0$ yields a unique morphism $(1, id_1) : C_1 \longrightarrow K_C$ such that:

(3) $$\pi_1(1, id_1) = 1 : C_1 \longrightarrow C_1 \text{ and } \pi_2(1, id_1) = id_1 : C_1 \longrightarrow C_1$$

(1), (2), (3), and the equation $d_0 i = 1_{C_0}$ imply that $fd_0 = fd_0\pi_1(1, id_1) = fi_1 q(1, id_1) = fi_2 q(1, id_1) = fd_0\pi_2(1, id_1) = fd_0 id_1 = fd_1$. So f coequalizes d_0 and d_1. Hence there is a unique morphism $t : L \longrightarrow Y$ such that $t\sigma = f$. This and Diagram III yield $fa = t\sigma a = t\sigma b = fb$. Now the pullback $fi_1 = fi_2 : I \longrightarrow Y$ yields a unique $m : C_0 \underset{L}{\times} C_0 \longrightarrow I$ such that $i_1 m = a$, and $i_2 m = b$.

On the other hand, (1), Diagram III, and the definition of σ yield $\sigma i_1 q = \sigma d_0 \pi_1 = \sigma d_1 \pi_1 = \sigma d_1 \pi_2 = \sigma d_0 \pi_2 = \sigma i_2 q$ i.e., $\sigma i_1 q = \sigma i_2 q$. But q is epi, hence $\sigma i_1 = \sigma i_2$, and since $\sigma a = \sigma b : C_0 \underset{L}{\times} C_0 \longrightarrow L$ is a pullback, it follows that there is a unique n such that $an = i_1$, and $bn = i_2$. Pullbacks $fi_1 = fi_2 : I \longrightarrow Y$ and $\sigma a = \sigma b : C_0 \underset{L}{\times} C_0 \longrightarrow L$, imply $mn = 1$ and $nm = 1$. This and (1) imply that $d_C = \langle a, b\rangle\partial$, $im = \langle a, b\rangle$, $\langle a, b\rangle n = i$ $nq = \partial$, and $q = m\partial$. Since n is an isomorphism, and q a u.e.e., ∂ is a u.e.e..

$\square$

3. Internal Colimit

For any category $\mathcal{E}$, with finite limits and coequalizers of reflexive pairs, and any C in $Cat(\mathcal{E})$, $\mathcal{E}^C$ denotes the full subcategory of the comma category $Cat(\mathcal{E})/C$ whose objects are discrete opfibrations. See [2] p 49 and 50.

3.1. **Definition:** Define:

(a) $Colim : Cat(\mathcal{E}) \longrightarrow \mathcal{E}$ by $Colim(C) = Coeq(C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0)$.

(b) For C in $Cat(\mathcal{E})$, $Colim_C : \mathcal{E}^C \longrightarrow \mathcal{E}$ by $Colim_C(F \xrightarrow{C} \gamma) = Colim(F)$. We refer to this functor as the *internal colimit.*

3.2. **Definition:** An object X of $\mathcal{E}$ is said to be *separated* if in $\mathcal{E}/X$, the product of e.e.'s is an e.e..

3.3. **Remark:** Since products in $\mathcal{E}/X$ are pullbacks in $\mathcal{E}$, it follows that X in $\mathcal{E}$ is separated if and only if for any two e.e.'s $\sigma_1 : X_1 \longrightarrow Y_1$, and $\sigma_2 : X_2 \longrightarrow Y_2$; and any two morphisms $f_1 : Y_1 \longrightarrow X$, and $f_2 : Y_2 \longrightarrow X$, the morphism $\sigma_1 \underset{X}{\times} \sigma_2$ is an e.e., where $\sigma_1 \underset{X}{\times} \sigma_2$ is obtained by the following pullbacks :

$$
\begin{tikzcd}
X_1 \underset{X}{\times} X_2 \arrow[r] \arrow[d] \arrow[dr, "\sigma_1 \underset{X}{\times} \sigma_2"] & E_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & X_2 \arrow[d, "\sigma^2"] \\
E_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & Y_1 \underset{L}{\times} Y_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "p.b."] & Y_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "\sigma^1"'] & Y_1 \arrow[r, "f_1"'] & X
\end{tikzcd}
$$

Diagram IV

3.4. **Definition:** C in $Cat(\mathcal{E})$ is said to be:
(a) *weakly smooth* if $Colim(C)$ is a separated object of $\mathcal{E}$.
(b) *smooth* if for all discrete opfibrations $F \longrightarrow C$, F is weakly smooth.

3.5. **Lemma:** (a) If C in $Cat(\mathcal{E})$ is smooth, then it is weakly smooth.
(b) If $\gamma : F \longrightarrow C$ is a discrete opfibration, and C is smooth, then so is F.

Proof: The proof is similar to the proof of Lemma 2.3.

□

3.6. **Theorem:** Let C in $Cat(\mathcal{E})$ be weakly smooth and proper. If C is u.e.w.f., then $Colim : (\mathcal{E}/L)^{C/L} \longrightarrow \mathcal{E}/L$ preserves finite products, where $L = Colim(C)$.

Proof: For $\alpha^i : F^i/L \longrightarrow C/L$ in $(\mathcal{E}/L)^{C/L}$, the product of α^1 by α^2 in $(\mathcal{E}/L)^{C/L}$ is $\alpha^1\pi^1 : G/L \longrightarrow C/L$, where G is obtained by the pullback:

(1) $$\alpha^1\pi^1 = \alpha^2\pi^2 : G \longrightarrow C$$

and G/L is G regarded as an object over L by passing to C, and regarding C as an object over L.

The internal colimit of$G/L \longrightarrow C/L$ in $\mathcal{E}/L$ is, by definition, the coequalizer of $(G/L)_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} (G/L)_0$ in $\mathcal{E}/L$, which is simply the coequalizer $G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0 \xrightarrow{\tau} M$ in $\mathcal{E}$, regarded as an object over L. So M/L is the internal colimit of the product of α^1 and α^2.

On the other hand, the internal colimit of $\alpha^i : F^i/L \longrightarrow C/L$ in $\mathcal{E}/L$ is the coequalizer of $(F^i/L)_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} (F^i/L)_0$ in $\mathcal{E}/L$, which is again simply the coequalizer $F^i_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} F^i_0 \xrightarrow{\sigma^i} L^i$ regarded as an object over L by l^i. The product of L^1/L by L^2/L in $\mathcal{E}/L$ is $(L^1 \underset{L}{\times} L^2)$/L, where $L^1 \underset{L}{\times} L^2$ is the pullback of $l^2 : L^2 \longrightarrow L$ along $l^1 : L^1 \longrightarrow L$. So the product of the internal colimit of α^1 and the internal colimit of α^2 is $(L^1 \underset{L}{\times} L^2)$/L. We need to show M/L and $(L^1 \underset{L}{\times} L^2)$/L are canonically isomorphic.

To this end we first note that the morphism $\langle \sigma^1\pi^1_0, \sigma^2\pi^2_0 \rangle : G_0 \longrightarrow L^1 \times L^2$ is coequalized by d_0 and d_1 of G, where d_0 and d_1 are obtained by the pullback (1). Therefore there is a unique morphism Φ such that:

(2) $$\Phi\tau = \langle \sigma^1\pi^1_0, \sigma^2\pi^2_0 \rangle$$

It is straightforward to verify that $l^1\sigma^1\pi^1_0 = l^2\sigma^2\pi^2_0$. Hence $\langle \sigma^1\pi^1_0, \sigma^2\pi^2_0 \rangle$ factors through $L^1 \underset{L}{\times} L^2$. The factoring map is denoted by $(\sigma^1\pi^1_0, \sigma^2\pi^2_0)$. Note that

(3) $$\pi(\sigma^1\pi^1_0, \sigma^2\pi^2_0) = \langle \sigma^1\pi^1_0, \sigma^2\pi^2_0 \rangle$$

where $\pi = \langle \pi_1, \pi_2 \rangle$. Since π is a mono, and $\langle \sigma^1\pi^1_0, \sigma^2\pi^2_0 \rangle$ is coequalized by d_0 and d_1, it follows that $(\sigma^1\pi^1_0, \sigma^2\pi^2_0)$ is coequalized by d_0 and d_1. Hence there is a unique morphism ϕ (canonical morphism) such that

(4) $$\phi\tau = (\sigma^1\pi^1_0, \sigma^2\pi^2_0)$$

(2), (3), and (4) imply that $\Phi\tau = \langle \sigma^1\pi^1_0, \sigma^2\pi^2_0 \rangle = \pi(\sigma^1\pi^1_0, \sigma^2\pi^2_0) = \pi\phi\tau$. τ, being a coequalizer, is an epi, so $\Phi = \pi\phi$. It follows that ϕ is mono if and only if Φ is mono. We will show that ϕ is an isomorphism by showing it is mono and e.e..

To show ϕ is mono, we show that Φ is mono by showing for arbitrary morphisms $U \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} G_0$, if $\Phi\tau\gamma_1 = \Phi\tau\gamma_2$ then $\tau\gamma_1 = \tau\gamma_2$. But first let us show this is sufficient. Form the pullback $(\Phi\tau)\pi_1 = (\Phi\tau)\pi_2 : R \longrightarrow L^1 \times L^2$ and the coequalizer $R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} G_0 \xrightarrow{Q} q$ and let $i : Q \longrightarrow L^1 \times L^2$ be the i-map of $\Phi\tau$, so that $iq = \Phi\tau$. The morphism i is mono since it is the i-map of $\Phi\tau$. $\Phi\tau\pi_1 = \Phi\tau\pi_2$ implies by our hypothesis that $\tau\pi_1 = \tau\pi_2$. So there is a unique morphism $\alpha : Q \longrightarrow M$ such that $\alpha q = \tau$. Therefore $iq = \Phi\tau$ and $\alpha q = \tau$, and consequently $\Phi\alpha q = iq$. Since q is an epi, $\Phi\alpha = i$. Since i is mono, so is α. Since the e.e. τ factors through α, α is an isomorphism. Thus Φ is a mono, since $\Phi\alpha = i$, α is an isomorphism, and i is a mono. So suppose we are given $U \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} G_0$ such that

$$\Phi\tau\gamma_1 = \Phi\tau\gamma_2 \tag{5}$$

Define morphisms $\gamma^i_j : U \longrightarrow F^i_0$, $i, j = 1, 2$, by $\gamma^i_j = \pi^i_0\gamma_j$. (2), and (5) imply $\sigma^i\gamma^i_1 = \sigma^i\gamma^i_2$, from which it follows that U can be regarded as an object of $\mathcal{E}/L^i$, for each i, and that γ^i_1 and γ^i_2 are morphisms in $\mathcal{E}/L^i$. By Lemma 1.5, F^i is u.e.w.f., and by Lemma 2.3, F^i is weakly proper. Hence by Proposition 2.5, F^i/L^i is u.e.f. in $Cat(\mathcal{E}/L^i)$, and therefore satisfies Condition 1.2(b)$'$ in $\mathcal{E}/L^i$. Applying (b)$'$ to γ^i_1 and γ^i_2 we get morphisms $V^i \overset{\lambda^i_1}{\underset{\lambda^i_2}{\rightrightarrows}} F^i_1$ and $\epsilon^i : V^i \longrightarrow U$ such that:

$$\epsilon^i \text{ is a u.e.e., } d_1\lambda^i_1 = d_1\lambda^i_2, \text{ and } d_0\lambda^i_j = \gamma^i_j\epsilon^i \tag{6}$$

Pullback ϵ^2 along ϵ^1 to get $t^i : V \longrightarrow V^i$ for $i = 1, 2$. Define $\epsilon : V \longrightarrow U$, and $\Lambda^i_j : V \longrightarrow F^i_1$ by $\epsilon = \epsilon^1 t^1$, and $\Lambda^i_j = \lambda^i_j t^i$. (6) implies that:

$$\epsilon \text{ is a u.e.e., } d_1\Lambda^i_1 = d_1\Lambda^i_2, \text{ and } d_0\Lambda^i_j = \gamma^i_j\epsilon \tag{7}$$

Define $\mu^i_j : V \longrightarrow C_1$ by $\mu^i_j = \alpha^i_1\Lambda^i_j$. The definition of γ^1_j, γ^2_j, and (1) imply that $\alpha^1_0\gamma^1_j = \alpha^2_0\gamma^2_j$. This together with the definition of μ^i_j, the fact that $d_0\alpha^1_1 = \alpha^1_0 d_0$, and (7) imply that:

$$d_0\mu^1_j = d_0\mu^2_j, \text{ and } d_1\mu^i_1 = d_1\mu^i_2 \tag{8}$$

The fact that $\sigma d_0 = \sigma d_1$, and (8) imply that $\sigma d_0\mu^i_j$ does not depend on i or j. This shows that V can be regarded as an object over L, and μ^i_j as a morphism in $\mathcal{E}/L$.

By Lemma 2.3, and Proposition 2.5, C is a u.e.f. in $Cat(\mathcal{E}/L)$, therefore by Lemma 1.5, it satisfies Condition (d)$''$ in $\mathcal{E}/L$. Applying (d)$''$ to μ^i_j, we get morphisms $W \overset{\rho^1}{\underset{\rho^2}{\rightrightarrows}} C_1$ and $\xi : W \longrightarrow V$ such that the following hold simultaneously for both $j = 1$ and 2.

$$\xi \text{ is a u.e.e., } d_0\rho^i = d_i\mu^i_j\xi, \text{ and } m(\mu^1_j\xi, \rho^1) = m(\mu^2_j\xi, \rho^2) \tag{9}$$

The definition of μ^i_j, and (9) imply that $d_0\rho^i = d_1\mu^i_j\xi = d_1\alpha^i_1\Lambda^i_j\xi = \alpha^i_0 d_1\Lambda^i_j\xi$. Since $d_0\alpha^i_1 = \alpha^i_0 d_0 : F^i_1 \longrightarrow C_0$ is a pullback there is a unique map $\theta^i : W \longrightarrow F^i_1$, for each i, such that:

$$\alpha^i_1\theta^i = \rho^i, \text{ and } d_0\theta^i = d_1\Lambda^i_j\xi \tag{10}$$

Note that by (7), $d_1\Lambda^i_1 = d_1\Lambda^i_2$, so that θ^i does not depend on j.

By (7), and (10) we have $d_0\theta^i = d_1\Lambda^i_1\xi = d_1\Lambda^i_2\xi$. Since $d_1\pi_1 = d_0\pi_2 : F^i_2 \longrightarrow F^i_0$ is a pullback there exists a unique map $(\Lambda^i_j\xi, \theta^i)$ such that:

$$\pi_1(\Lambda^i_j\xi, \theta^i) = \Lambda^i_j\xi, \text{ and } \pi_2(\Lambda^i_j\xi, \theta^i) = \theta^i \tag{11}$$

Define morphisms $\Psi^i_j : W \longrightarrow F^i_1$ by $\Psi^i_j = m(\Lambda^i_j\xi, \theta^i)$. (11), and the definition of μ^i_j imply that $\pi_1\alpha^i_2(\Lambda^i_j\xi, \theta^i) = \alpha^i_1\pi^i_1(\Lambda^i_j\xi, \theta^i) = \alpha^i_1\Lambda^i_j\xi = \mu^i_j\xi = \pi_1(\mu^i_j\xi, \rho^i)$. Similarly we can show $\pi_2\alpha^i_2(\Lambda^i_j\xi, \theta^i) = \pi_2(\mu^i_j\xi, \rho^i)$. The pullback $d_1\pi_1 = d_0\pi_2 : C_2 \longrightarrow C_0$ then shows that $\alpha^i_2(\Lambda^i_j\xi, \theta^i) = (\mu^i_j\xi, \rho^i)$. This and the definition of Ψ^i_j imply that $\alpha^i_1\Psi^i_j = \alpha^i_1 m(\Lambda^i_j\xi, \theta^i) = m\alpha^i_2(\Lambda^i_j\xi, \theta^i) = m(\mu^i_j\xi, \rho^i)$.

On the other hand by (9) $m(\mu^1_j\xi, \rho^1) = m(\mu^2_j\xi, \rho^2)$. Therefore $\alpha^1_1\Psi^1_j = \alpha^2_1\Psi^2_j$, and since $\alpha^1_1\pi^1_1 = \alpha^2_1\pi^2_1 : G_1 \longrightarrow C_1$ is a pullback there is a unique morphism (Ψ^1_j, Ψ^2_j) such that $\pi^1_1(\Psi^1_j, \Psi^2_j) = \Psi^1_j$ and $\pi^2_1(\Psi^1_j, \Psi^2_j) = \Psi^2_j$. By the definition of Ψ^i_j, and (11), we get $d_1\Psi^i_j = d_1 m(\Lambda^i_j\xi, \theta^i) = d_1\pi_2(\Lambda^i_j\xi, \theta^i) = d_1\theta^i$. It follows that:

(12) $$d_1(\Psi_1^1, \Psi_1^2) = (d_1\theta^1, d_1\theta^2) = d_1(\Psi_2^1, \Psi_2^2)$$

On the other hand $d_0\Psi_j^i = d_0 m(\Lambda_j^i\xi, \theta^i) = d_0\pi_1(\Lambda_j^i\xi, \theta^i) = d_0\Lambda_j^i\xi$. This and equation (7) imply $d_0(\Psi_j^1, \Psi_j^2) = (d_0\Lambda_j^1\xi, d_0\Lambda_j^2\xi) = (d_0\Lambda_j^1, d_0\Lambda_j^2)\xi = (\gamma_j^1\epsilon, \gamma_j^2\epsilon)\xi = (\gamma_j^1, \gamma_j^2)\epsilon\xi = \gamma_j\epsilon\xi$. This and equation (12) imply that $\tau\gamma_1\epsilon\xi = \tau d_0(\Psi_1^1, \Psi_1^2) = \tau d_1(\Psi_1^1, \Psi_1^2) = \tau d_1(\Psi_2^1, \Psi_2^2) = \tau d_0(\Psi_2^1, \Psi_2^2) = \tau\gamma_2\epsilon\xi$. ϵ and ξ are e.e.'s and therefore epi's, hence $\tau\gamma_1 = \tau\gamma_2$ as desired. This proves Φ, and therefore ϕ, is mono.

To show ϕ is e.e., form the pullback

(13) $$(\sigma\alpha_0^1)\pi_0^1 = (\sigma\alpha_0^2)\pi_0^2 : F_0^1 \underset{L}{\times} F_0^2 \longrightarrow L$$

and note that the morphisms $F_0^1 \underset{L}{\times} F_0^2 \underset{\alpha_0^2\pi_0^2}{\overset{\alpha_0^1\pi_0^1}{\rightrightarrows}} C_0$ are coequalized by σ, and therefore are morphisms in $\mathcal{E}/L$. C/L is u.e.f. in $Cat(\mathcal{E}/L)$, and so satisfies Condition (b)$'$ of 1.2 in $\mathcal{E}/L$. Applying (b)$'$ to $\alpha_0^1\pi_0^1$ and $\alpha_0^2\pi_0^2$, we get morphisms $V \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} C_1$ and $\epsilon : V \longrightarrow F_0^1 \underset{L}{\times} F_0^2$ such that

(14) $$\epsilon \text{ is a u.e.e., } d_1\lambda_1 = d_1\lambda_2, \text{ and } d_0\lambda_i = \alpha_0^i\pi_0^i\epsilon$$

The last equation of (14) and the fact that $d_0\alpha_1^i = \alpha_0^i d_0 : F_1^i \longrightarrow C_0$ is a pullback imply the existence of the unique morphism $\mu_i : V \longrightarrow F_1^i$ such that:

(15) $$\alpha_1^i\mu_i = \lambda_i, \text{ and } d_0\mu_i = \pi_0^i\epsilon$$

By (14) and (15) we have $\alpha_0^1 d_1\mu_1 = d_1\alpha_1^1\mu_1 = d_1\lambda_1 = d_1\lambda_2 = d_1\alpha_1^2\mu_2 = \alpha_0^2 d_1\mu_2$. The pullback $\alpha_0^1\pi_0^1 = \alpha_0^2\pi_0^2 : G_0 \longrightarrow C_0$ yields a unique morphism $\rho : V \longrightarrow G_0$ such that:

(16) $$\pi_0^1\rho = d_1\mu_1, \text{ and } \pi_0^2\rho = d_1\mu_2$$

(4), (15), and (16) imply that:

(17) $$\begin{aligned} \phi\tau\rho = (\sigma^1\pi_0^1, \sigma^2\pi_0^2)\rho = (\sigma^1\pi_0^1\rho, \sigma^2\pi_0^2\rho) = (\sigma^1 d_1\mu_1, \sigma^2 d_1\mu_2) = \\ (\sigma^1 d_0\mu_1, \sigma^2 d_0\mu_2) = (\sigma^1\pi_0^1\epsilon, \sigma^2\pi_0^2\epsilon) = (\sigma^1\pi_0^1, \sigma^2\pi_0^2)\epsilon \end{aligned}$$

Using (13), and the fact that $l^i\sigma^i = \sigma\alpha_0^i$, which holds by the definition of l^i, form the following pullback to get the morphism $\sigma^1 \underset{L}{\times} \sigma^2$.

$$\begin{tikzcd}
F_0^1 \underset{L}{\times} F_0^2 \arrow[r] \arrow[d] \arrow[drr, phantom, "p.b." near start] \arrow[dr, "\sigma_1 \underset{L}{\times} \sigma_2"] \arrow[rr, bend left, "\pi_0^2"] \arrow[dd, bend right, "\pi_0^1"'] & E^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & F_0^2 \arrow[d, "\sigma^2"] \arrow[dd, bend left, "\sigma\alpha_0^2"] \\
E^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & L^1 \underset{L}{\times} L^2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "p.b."] & L^2 \arrow[d, "l^2"] \\
F_0^1 \arrow[r, "\sigma^1"'] \arrow[rr, bend right, "\sigma\alpha_0^1"'] & L^1 \arrow[r, "l^1"'] & L
\end{tikzcd}$$

Diagram V

By Diagram V, it is obvious that $\sigma^1 \underset{L}{\times} \sigma^2 = (\sigma^1\pi_0^1, \sigma^2\pi_0^2)$, and so by (17) we have $\phi\tau\rho = (\sigma^1 \underset{L}{\times} \sigma^2)\epsilon$. C is weakly smooth, so by Remark 3.3, $\sigma^1 \underset{L}{\times} \sigma^2$ is an e.e.. Also by (15) ϵ is an e.e., therefore $\phi\tau\rho$ is an e.e., and hence so is ϕ. This concludes the proof.

$\square$

3.7. **Corollary:** Let C in $Cat(\mathcal{E})$ be smooth and proper. If C is u.e.w.f., then $Colim_C : \mathcal{E}^C \longrightarrow \mathcal{E}$ preserves pullbacks.

Proof: Let $\gamma : F \longrightarrow C$ be an arbitrary discrete opfibration. We need to show that $Colim_C : \mathcal{E}^C \longrightarrow \mathcal{E}$ takes pullbacks over γ in $\mathcal{E}^C$ to pullbacks over $Colim_C(\gamma)$ in $\mathcal{E}$.

Form the coequalizer $F_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} F_0 \xrightarrow{\sigma} L$, so that $L = Colim_C(\gamma)$. A straightforward computation shows that $U \circ Colim_{F/L} = Colim_F \circ \theta : (\mathcal{E}/L)^{F/L} \longrightarrow \mathcal{E}$, where the functor $\theta : (\mathcal{E}/L)^{F/L} \longrightarrow \mathcal{E}^F$ is the obvious equivalence of categories and $U : \mathcal{E}/L \longrightarrow \mathcal{E}$ is the forgetful functor. Since C is smooth, proper, and u.e.w.f., Lemmas 1.5, 2.3, and 3.5 imply that F is weakly smooth, proper, and u.e.w.f.. Therefore by Theorem 3.6, $Colim_{F/L} : (\mathcal{E}/L)^{F/L} \longrightarrow \mathcal{E}/L$ preserves finite products. A pullback over γ in $\mathcal{E}^C$ is a product in $\mathcal{E}^F$, which is taken to a product in $(\mathcal{E}/L)^{F/L}$ by θ, which is then taken to a product in $\mathcal{E}/L$ by $Colim_{F/L}$, which is finally taken to a pullback over L in $\mathcal{E}$ by U. We conclude the proof by noting that $Colim_C$ of a pullback diagram over γ in $\mathcal{E}^C$, and $Colim_F$ of the same diagram regarded as a product in $\mathcal{E}^F$ are the same.

□

3.8. **Lemma:** If C in $Cat(\mathcal{E})$ is u.e.f., then $Colim(C) \simeq 1$, and the functor $Colim_C : \mathcal{E}^C \longrightarrow \mathcal{E}$ preserves the terminal object.

Proof: Form the coequalizer $C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0 \xrightarrow{\tau} L$, so that $L = Colim(C)$. Given morphisms $U \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} C_0$, since C is u.e.f., and so satisfies Condition 1.2(b)$'$, we get morphisms $V \underset{\lambda_2}{\overset{\lambda_1}{\rightrightarrows}} C_1$ and $\epsilon : V \longrightarrow U$ such that ϵ is a u.e.e., $d_1\lambda_1 = d_1\lambda_2$, and $d_0\lambda_i = \gamma_i\epsilon$. Therefore $\tau\gamma_1\epsilon = \tau d_0\lambda_1 = \tau d_1\lambda_1 = \tau d_1\lambda_2 = \tau d_0\lambda_2 = \tau\gamma_2\epsilon$. ϵ being epi implies $\tau\gamma_1 = \tau\gamma_2$. Hence τ coequalizes any pair of maps to C_0, in particular the kernel pair $R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} C_0$, of the unique morphism $! : C_0 \longrightarrow 1$.

By Condition 1.1(a), $! : C_0 \longrightarrow 1$ is an e.e., and so by Lemma 2.1, it is the coequalizer of its kernel pair. But τ coequalizes π_1 and π_2, therefore there is a unique map $\sigma : 1 \longrightarrow L$ such that $\sigma! = \tau$. The map τ, being a coequalizer is by Lemma 2.1 an e.e.. $\sigma! = \tau$ implies that σ is an e.e.. On the other hand σ is a mono, since its domain is the terminal object. Therefore σ is an isomorphism, and $L \simeq 1$. The other assertion of the lemma is obvious.

□

3.9. **Theorem:** Let C in $Cat(\mathcal{E})$ be smooth and proper. If C is u.e.f., then $Colim_C : \mathcal{E}^C \longrightarrow \mathcal{E}$ preserves finite limits.

Proof: By Corollary 3.7, and Lemma 3.8, pullbacks and the terminal object are preserved, which implies the preservation of finite limits (see [3], Chapter 5).

□

4. Examples

It can be easily verified that any topos, $\mathcal{E}$, is an admissible category, and any internal category C in $\mathcal{E}$ is smooth and proper, see [2], and [6]. Also C is u.e.f. iff it is filtered, see [2], p66. So theorem 4.9 contains the known result in a topos as a special case, see [2], p70.

Now let $\mathcal{E}$ denote any category that is topological over the category Set of sets, see [1], in which product of extremal epis is an extremal epi (e.g. if $\mathcal{E}$ is cartesian closed then this condition is satisfied, see [5]) and the pullback of an extremal epi along an initial mono is an extremal epi. It follows that $\mathcal{E}$ is an admissible category. In what follows we denote by $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ the underlying map of the morphism $f : X \longrightarrow Y$ of $\mathcal{E}$.

4.1. **Theorem:** In $\mathcal{E}$ initial maps are universal.

Proof: Pullback the initial map $f : X \longrightarrow Y$ along $g : Z \longrightarrow Y$ to get:

(1) $$g\pi_1 = f\pi_2 : T \longrightarrow Y$$

Given $\alpha : R \longrightarrow Z$ and $h : \underline{R} \longrightarrow \underline{T}$ such that $\underline{\pi_1}h = \underline{\alpha}$, we have $\underline{f}(\underline{\pi_2}h) = \underline{g\alpha}$. Initiality of f implies there is a unique morphism $k : R \longrightarrow X$ such that:

(2) $$\underline{k} = \underline{\pi_2}h \text{ and } fk = g\alpha$$

Pullback (1) yields a morphism $h^* : R \longrightarrow T$ such that $\pi_1 h^* = \alpha$ and $\pi_2 h^* = k$. Since both $\underline{h^*}$ and h followed by $\underline{\pi_1}$ and $\underline{\pi_2}$ are equal to $\underline{\alpha}$ and $\underline{k}$ respectively and since $\underline{g}\underline{\pi_1} = \underline{f}\underline{\pi_2} : \underline{T} \longrightarrow \underline{Y}$ is a pullback in $Sets$ we get $\underline{h^*} = h$.

Now suppose there is h^{**} such that $\pi_1 h^{**} = \alpha$ and $\underline{h^{**}} = h$. This implies $f(\pi_2 h^{**}) = g\alpha$ and $\underline{\pi_2}\underline{h^{**}} = \underline{\pi_2}h$. By (2) $\pi_2 h^{**} = k$ and by pullback (1) $h^{**} = h^*$.

□

4.2. **Corollary:** An initial epi in $\mathcal{E}$ is a split epi and therefore a u.e.e.

Proof: straightforward.

□

4.3. **Theorem:** Let the morphism $f : X \longrightarrow Y$, the epimorphism $q : X \longrightarrow Z$, and the monomorphism $i : Z \longrightarrow Y$ be in $\mathcal{E}$ such that $f = iq$. f is initial iff both q and i are.

Proof: $\Rightarrow$: To show q is initial, suppose $\alpha : R \longrightarrow Z$ and $h : \underline{R} \longrightarrow \underline{X}$ are given such that $\underline{q}h = \underline{\alpha}$. We have $\underline{f}h = \underline{i\alpha}$. Initiality of f implies there is a unique $h^* : R \longrightarrow X$ such that:

(1) $$fh^* = i\alpha \text{ and } \underline{h^*} = h$$

This implies $iqh^* = i\alpha$. i is a mono, therefore $qh^* = \alpha$.

Now suppose there is h^{**} such that $qh^{**} = \alpha$ and $\underline{h^{**}} = h$. This implies $iqh^{**} = i\alpha$ and $\underline{h^{**}} = h$. Thus $fh^{**} = i\alpha$ and $\underline{h^{**}} = h$. Therefore by (1) $h^{**} = h^*$.

To show i is initial suppose $\alpha : R \longrightarrow Y$ and $h : \underline{R} \longrightarrow \underline{Z}$ are given such that $\underline{i}h = \underline{\alpha}$. Let $q' : \underline{Z} \longrightarrow \underline{X}$ split $\underline{q} : \underline{X} \longrightarrow \underline{Z}$, i.e. $\underline{q}q' = 1$ (every epi in $Sets$ is split epi), so that $\underline{f}q'h = \underline{\alpha}$. Initiality of f implies there is a unique $k^* : R \longrightarrow X$ such that $fk^* = \alpha$ and $\underline{k^*} = q'h$. Let $h^* = qk^*$. Since $fk^* = \alpha$ we have $iqk^* = \alpha$, which implies $ih^* = \alpha$ and $\underline{h^*} = \underline{q}\underline{k^*} = \underline{q}q'h = h$.

Now suppose there is h^{**} such that $ih^{**} = \alpha$. This implies $ih^{**} = ih^*$, and since i is a mono we have $h^{**} = h^*$.

$\Leftarrow$: Easy.

□

4.4. **Theorem:** Let the morphisms $f : X \longrightarrow Y$, $g : Z \longrightarrow Y$, $\pi_1 : T \longrightarrow Z$, and $\pi_2 : T \longrightarrow X$ be morphisms in $\mathcal{E}$. The diagram $g\pi_1 = f\pi_2 : T \longrightarrow Y$ is a pullback in $\mathcal{E}$ iff the underlying diagram $\underline{g\pi_1} = \underline{f\pi_2} : T \longrightarrow Y$ is a pullback in Set and $\pi : T \longrightarrow Z \times X$ is initial.

Proof: See [1] and [5].

□

4.5. **Theorem:** Let C be an internal category in $\mathcal{E}$. If $d_{0C} : C_1 \longrightarrow C_0$ is initial, then C is proper.

Proof: Let $\gamma : F \longrightarrow C$ be a discrete opfibration. Since $d_{0F} : F_1 \longrightarrow F_0$ is the pullback of $d_{0C} : C_1 \longrightarrow C_0$ along $\gamma_0 : F_0 \longrightarrow C_0$, it is initial by Theorem 4.1. This implies $d_{0F}^2 : F_1^2 \longrightarrow F_0^2$ is initial. Also $\pi_F : K_F \longrightarrow F_1^2$, where K_F is the kernel pair of d_{1F}, is initial by Theorem 4.4. So $d_F = d_{0F}^2 \pi_F$ is initial by Theorem 4.3. On the other hand $d_F = i_F q_F$, so by Theorem 4.3 both q and i are initial. Therefore by Corollary 4.2 q is a u.e.e. and i is effective if it is an equivalence relation, see [2], p16. This shows C is proper.

□

4.6. **Theorem:** Every internal category C in $\mathcal{E}$ is smooth.

Proof: To show C is smooth, we show that every object $X \in \mathcal{E}$ is separated. Given morphisms $\sigma_1 : X_1 \longrightarrow Y_1$ and $\sigma_2 : X_2 \longrightarrow Y_2$ over X, see Remark 3.3, find the product of σ_1 by σ_2 in $\mathcal{E}/X$ to get $\sigma_1 \underset{X}{\times} \sigma_2 : X_1 \underset{X}{\times} X_2 \longrightarrow Y_1 \underset{X}{\times} Y_2$. It can be shown that $\sigma_1 \underset{X}{\times} \sigma_2$ is the pullback of $\sigma_1 \times \sigma_2$ along the inclusion (initial mono) $\pi : Y_1 \underset{X}{\times} Y_2 \longrightarrow Y_1 \times Y_2$. Since product of extremal epis in $\mathcal{E}$ is an extremal epi, $\sigma_1 \times \sigma_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ is an e.e., and since $\sigma_1 \underset{X}{\times} \sigma_2 : X_1 \underset{X}{\times} X_2 \longrightarrow Y_1 \underset{X}{\times} Y_2$ is the pullback of the morphism $\sigma_1 \times \sigma_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ along the initial mono $\pi : Y_1 \underset{X}{\times} Y_2 \longrightarrow Y_1 \times Y_2$, it is an e.e..

□

4.7. **Theorem:** Let $C \in Cat(\mathcal{E})$ be filtered. If $d_{0C} : C_1 \longrightarrow C_0$ is initial, then C is universal extremal filtered.

Proof: To show that C is u.e.f., condition 1.1(a) holds because C is filtered, see [2], p66. To prove condition 1.1(b), the initiality of d_{0C} implies that of $d_C : K_C \longrightarrow C_0^2$. Since d_C is epi, therefore it is a u.e.e.. Finally to prove condition 1.1(c), it can be shown that $t_C : T_C \longrightarrow R_C$ satisfies the equation $\pi t_C = \pi_1^2 \pi : T_C \longrightarrow C_1^2$, where $\pi_1 : C_2 \longrightarrow C_1$ is the pullback of $d_{0C} : C_1 \longrightarrow C_0$ along $d_{1C} : C_1 \longrightarrow C_0$. By Theorem 4.1, initiality of d_{0C} implies that of π_1 and hence that of π_1^2, and since $\pi : T_C \longrightarrow C_2^2$ is initial, it follows that $\pi_1^2 \pi$ is. However $\pi t_C = \pi_1^2 \pi$, so πt_C is initial. Since t_C is epi and π is mono, by Theorem 4.3, t_C is initial. Since t_C is epi, by Corollary 4.2 it is a u.e.e..

□

4.8. **Corollary:** If $C \in Cat(\mathcal{E})$ is filtered and $d_{0C} : C_1 \longrightarrow C_0$ is initial, then $Colim_C : \mathcal{E}^C \longrightarrow \mathcal{E}$ preserves finite limits.

Proof: From Theorems 4.5, 4.6, 4.7, it follows that C is smooth, proper, and u.e.f.. The proof then follows by Theorem 3.9.

$\square$

We conclude the paper by remarking that $\mathcal{E}$ can be taken to be the categories constructed in [4] as topological completions with universal final epi sinks.

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