FINDING COMMON FIXED POINTS OF A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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Abstract. The main purpose of this paper is to study an iteration procedure for finding a common fixed point of a countable family of nonexpansive mappings in Banach spaces. We introduce a Mann type iteration procedure. Then we prove that such a sequence converges weakly to a common fixed point of a countable family of nonexpansive mappings. Moreover, we apply our result to the problem of finding a common fixed point of a pair of nonexpansive mappings and the problem of finding a common solution of the fixed point problem and the variational inequality problem.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E and T a nonexpansive mapping of C into itself, that is, $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. In this paper, we deal with an approximation of fixed points of nonexpansive mappings.

Mann [9] introduced an iteration procedure for approximating fixed points of a mapping T in a Hilbert space as follows: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later, Reich [11] discussed this iteration procedure in a uniformly convex Banach space whose norm is Fréchet differentiable; see also [10]. For two nonexpansive mappings S and T, Takahashi and Tamura [13] considered the following iteration procedure: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S(\beta_n x_n + (1 - \beta_n)Tx_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$; see also [4]. They obtained weak convergence theorems for this procedure in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable.

The main purpose of this paper is to study an iteration procedure for finding a common fixed point of a countable family of nonexpansive mappings in Banach spaces. We introduce the following iteration procedure: Let $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings. Then we prove that this sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}$. Further we apply our result to the problem of finding a common fixed point of a pair of nonexpansive mappings. We deal with the iteration procedure treated in [13] and another type of sequence for a pair of nonexpansive mappings. Finally, we discuss the problem of finding a common solution of the fixed point problem for a nonexpansive

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mapping and the variational inequality problem for an inverse-strongly-monotone mapping. Similarly, we deal with two types of sequences.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and C a subset of E. The dual of E is denoted by E^*, the set of all positive integers by $\mathbb{N}$, and the set of all real numbers by $\mathbb{R}$. Let $\{x_n\}$ be a sequence in E. Strong convergence of $\{x_n\}$ to x is indicated by $x_n \to x$, weak convergence of $\{x_n\}$ to x by $x_n \rightharpoonup x$, and the closure of the convex hull of C by $\overline{\mathrm{co}}\, C$.

Let $U = \{x \in E : \|x\| = 1\}$. The norm $\|\cdot\|$ of E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. In this case a Banach space E is said to be smooth. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is said to satisfy Opial's condition [10] if $x_n \rightharpoonup x$ and $x \neq y$ imply

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| .$$

A Banach space E is said to be strictly convex if $\|x\| = \|y\| = 1$ and $x \neq y$ imply $\|(x+y)/2\| < 1$. If E is strictly convex, then

$$\|x\| = \|\lambda x + (1-\lambda)y\| = \|y\| \text{ and } \lambda \in (0,1) \text{ imply } x = y. \tag{2.2}$$

A Banach space E is said to be uniformly convex if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$ imply $\|(x+y)/2\| \leq 1 - \delta$. It is known that if E is uniformly convex, then E is reflexive and strictly convex. It is also known that if E is uniformly convex, then the function $\|\cdot\|^2$ is uniformly convex [16] on every bounded convex subset B of E, that is, for each $\epsilon > 0$, there is $\delta > 0$ such that

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\delta$$

for all $\lambda \in (0,1)$ and $x, y \in B$ with $\|x - y\| \geq \epsilon$; see, for example, [5, 16]. To prove our results, we need several theorems:

Theorem 2.1 (Browder [2])**.** *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E, T a nonexpansive mapping of C into itself, and $\{x_n\}$ a sequence of C. If $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ as $n \to \infty$, then $x \in F(T)$.*

Reich stated the following; see also [15].

Theorem 2.2 (Reich [11])**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space whose norm is Fréchet differentiable. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_1$ for $n \in \mathbb{N}$. Then the set*

$$\bigcap_{n=1}^\infty \overline{\mathrm{co}}\{S_m x : m \geq n\} \cap \bigcap_{n=1}^\infty F(T_n)$$

consists of at most one point.

Theorem 2.3 (Bruck [3])**.** *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into E and $\{\beta_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^\infty \beta_n = 1$. If $\bigcap_{n=1}^\infty F(S_n)$ is nonempty, then the mapping $T = \sum_{n=1}^\infty \beta_n S_n$ is well defined and $F(T) = \bigcap_{n=1}^\infty F(S_n)$.*

It is easily seen that this theorem is applied to a finite family of nonexpansive mappings, that is, we conclude the following: Let $\{S_1, S_2, \dots, S_n\}$ be a finite family of nonexpansive mappings of C into a strictly convex Banach space E and $\{\beta_1, \beta_2, \dots, \beta_n\}$ a finite family of positive real numbers such that $\sum_{k=1}^{n} \beta_k = 1$. If $\bigcap_{k=1}^{n} F(S_k)$ is nonempty, then $F(T) = \bigcap_{k=1}^{n} F(S_k)$, where $T = \sum_{k=1}^{n} \beta_k S_k$.

3. The main result

In this section, we consider the problem of approximating a common fixed point of a countable family of nonexpansive mappings. To obtain our main result, we need the following:

Lemma 3.1. *Let E be a uniformly convex Banach space. Let $\{\alpha_n\}$ be a sequence of $(0,1)$ such that $0 < a \le \alpha_n \le b < 1$ for some $a, b \in \mathbb{R}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E which satisfy the following:*

1. *$x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n$;*
2. *there exists $u \in E$ such that $\|y_n - u\| \le \|x_n - u\|$ for every $n \in \mathbb{N}$.*

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. By (1) and (2), we have

$$\|x_{n+1} - u\| \le \alpha_n \|x_n - u\| + (1-\alpha_n) \|y_n - u\| \le \|x_n - u\|$$

for some $u \in E$. This implies that $x_n, y_n \in B = \{z \in E : \|z\| \le \|x_1 - u\| + \|u\|\}$ for every $n \in \mathbb{N}$ and $\{\|x_n - u\|\}$ is convergent. Suppose that $\lim_{n\to\infty} \|x_n - y_n\| \ne 0$. Then there exist $\epsilon > 0$ and a subsequence $\{x_{n_i} - y_{n_i}\}$ of $\{x_n - y_n\}$ such that $\|x_{n_i} - y_{n_i}\| \ge \epsilon$ for each $i \in \mathbb{N}$. Since E is uniformly convex, $\|\cdot\|^2$ is uniformly convex on B, so that there exists $\delta > 0$ such that $\|x - y\| \ge \epsilon$ implies

$$\|\lambda x + (1-\lambda) y\|^2 \le \lambda \|x\|^2 + (1-\lambda) \|y\|^2 - \lambda(1-\lambda)\delta$$

whenever x, $y \in B$ and $\lambda \in (0,1)$. Thus we have

$$\begin{aligned}\|x_{n_i+1} - u\|^2 &= \|\alpha_{n_i}(x_{n_i} - u) + (1-\alpha_{n_i})(y_{n_i} - u)\|^2 \\ &\le \alpha_{n_i} \|x_{n_i} - u\|^2 + (1-\alpha_{n_i}) \|y_{n_i} - u\|^2 - \alpha_{n_i}(1-\alpha_{n_i})\delta.\end{aligned}$$

Therefore we obtain

$$0 < a(1-b)\delta \le \alpha_{n_i}(1-\alpha_{n_i})\delta \le \|x_{n_i} - u\|^2 - \|x_{n_i+1} - u\|^2$$

for each $i \in \mathbb{N}$. Since the right side of the inequality above converges to 0 as $i \to \infty$, we have a contradiction. Therefore we conclude that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. □

Further we need the following:

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and $\{\alpha_n\}$ a sequence of $(0,1)$ such that $0 < a \le \alpha_n \le b < 1$ for some $a, b \in \mathbb{R}$. Let $\{x_n\}$ be a sequence of C defined as follows: $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_n x_n$$

for every $n \in \mathbb{N}$. Suppose that for any nonempty bounded closed convex subset B of C and any increasing sequence $\{n_i\}$ of $\mathbb{N}$, there exist a nonexpansive mapping T of C into itself *and a subsequence* $\{T_{n_{i_j}}\}$ *of* $\{T_{n_i}\}$ *such that*

$$\lim_{j\to\infty}\sup_{y\in B}\left\|Ty - T_{n_{i_j}}y\right\| = 0 \text{ and } F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

Then $\{x_n\}$ *converges weakly to some point of* $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Let $u \in \bigcap_{n=1}^{\infty} F(T_n)$. It is clear that

$$\|T_n x_n - u\| = \|T_n x_n - T_n u\| \le \|x_n - u\|$$

for every $n \in \mathbb{N}$. Thus Lemma 3.1 implies that

$$\lim_{n\to\infty}\|T_n x_n - x_n\| = 0. \tag{3.1}$$

Since

$$\begin{aligned}\|x_{n+1} - u\| &\le \alpha_n \|x_n - u\| + (1-\alpha_n)\|T_n x_n - u\| \\ &\le \alpha_n \|x_n - u\| + (1-\alpha_n)\|x_n - u\| \\ &= \|x_n - u\|\end{aligned} \tag{3.2}$$

for every $n \in \mathbb{N}$, we have $\|x_n - u\| \le \|x_1 - u\|$. So $\{x_n\}$ is bounded and, without loss of generality, we may assume that C is bounded. Since E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. For C and a subsequence $\{T_{n_i}\}$ of $\{T_n\}$, there exist a nonexpansive mapping T of C into itself and a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ such that

$$\lim_{j\to\infty}\sup_{y\in C}\left\|Ty - T_{n_{i_j}}y\right\| = 0 \tag{3.3}$$

and

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

Since

$$\begin{aligned}\left\|x_{n_{i_j}} - Tx_{n_{i_j}}\right\| &\le \left\|x_{n_{i_j}} - T_{n_{i_j}}x_{n_{i_j}}\right\| + \left\|T_{n_{i_j}}x_{n_{i_j}} - Tx_{n_{i_j}}\right\| \\ &\le \left\|x_{n_{i_j}} - T_{n_{i_j}}x_{n_{i_j}}\right\| + \sup_{y\in C}\left\|T_{n_{i_j}}y - Ty\right\|,\end{aligned}$$

we have $\lim_{j\to\infty}\left\|x_{n_{i_j}} - Tx_{n_{i_j}}\right\| = 0$ from (3.1) and (3.3). By Theorem 2.1, we obtain $v \in F(T)$.

Suppose that E satisfies Opial's condition and $x_{n_k} \rightharpoonup w$. From (3.2) and $v, w \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, we know that $\lim_{n\to\infty}\|x_n - v\|$ and $\lim_{n\to\infty}\|x_n - w\|$ exist. If $v \ne w$, then we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - v\| &= \liminf_{i\to\infty}\|x_{n_i} - v\| < \liminf_{i\to\infty}\|x_{n_i} - w\| = \lim_{n\to\infty}\|x_n - w\| \\ &= \liminf_{k\to\infty}\|x_{n_k} - w\| < \liminf_{k\to\infty}\|x_{n_k} - v\| = \lim_{n\to\infty}\|x_n - v\|.\end{aligned}$$

This is a contradiction. Therefore we conclude that $v = w$.

Suppose that the norm of E is Fréchet differentiable. For each $n \in \mathbb{N}$, let S_n be a nonexpansive mapping of C into itself defined by $S_n z = \alpha_n z + (1-\alpha_n)T_n z$ for $z \in C$. Then we know that $x_{n+1} = S_n S_{n-1}\cdots S_1 x$, $v \in \bigcap_{n=1}^{\infty}\overline{\text{co}}\{x_m : m \ge n\}$, and $\bigcap_{n=1}^{\infty} F(S_n) = \bigcap_{n=1}^{\infty} F(T_n) = F(T) \ne \emptyset$. By Theorem 2.2, we have

$$\bigcap_{n=1}^{\infty}\overline{\text{co}}\{x_m : m \ge n\} \cap F(T) = \{v\}.$$

Consequently, we deduce that $\{x_n\}$ converges weakly to some point of $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. $\square$

In Lemma 3.2, we assume that for any nonempty bounded closed convex subset B and any increasing sequence $\{n_i\}$, there exist a nonexpansive mapping T of B into itself and a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ such that $\lim_{j\to\infty} \sup_{y\in B} \left\| Ty - T_{n_{i_j}} y \right\| = 0$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. For an arbitrary countable family $\{S_n\}$ of nonexpansive mappings with a common fixed point, we can generate a sequence $\{T_n\}$ which satisfies this assumption.

Let C be a nonempty closed convex subset of a Banach space E. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself with a common fixed point and $\{\beta_n\}$ a sequence of $(0,1)$ with $\sum_{n=1}^{\infty} \beta_n < \infty$. In this case note that $0 < \prod_{n=1}^{\infty}(1-\beta_n) < 1$. We define a sequence $\{T_n\}$ of mappings of C into itself as follows:

$$\begin{aligned} T_1 &= \beta_1 S_1 + (1-\beta_1) I, \\ T_2 &= \beta_2 S_2 + (1-\beta_2) T_1, \\ &\ \vdots \\ T_n &= \beta_n S_n + (1-\beta_n) T_{n-1}, \end{aligned}$$

that is,

$$T_n = \sum_{k=0}^{n} \beta_k \prod_{i=k+1}^{n} (1-\beta_i) S_k$$

for every $n \in \mathbb{N}$, where $\beta_0 = 1$, I is the identity mapping, $S_0 = I$, and $\prod_{i=m}^{l}(1-\beta_i) = 1$ if $m > l$. Put

$$\gamma_n^k = \beta_k \prod_{i=k+1}^{n} (1-\beta_i) \text{ and } p_k = \prod_{i=1}^{k}(1-\beta_i)$$

for $n \in \mathbb{N}$ and $k = 0, 1, \dots, n$. Note that $\gamma_n^k = \beta_k p_n / p_k$. It is easy to verify that

$$\sum_{k=0}^{n} \gamma_n^k = 1$$

for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \gamma_n^k = \frac{\beta \beta_k}{p_k}$$

for every $k \in \mathbb{N}$, where $\beta = \prod_{n=1}^{\infty}(1-\beta_n)$. Put $\gamma^k = \lim_{n\to\infty} \gamma_n^k$ for $k \in \mathbb{N}$. Since

$$\begin{aligned} \sum_{k=0}^{n} \gamma^k = \sum_{k=0}^{n} \frac{\beta\beta_k}{p_k} &= \beta \left(\frac{\beta_0}{1} + \frac{\beta_1}{1-\beta_1} + \frac{\beta_2}{(1-\beta_1)(1-\beta_2)} + \cdots + \frac{\beta_n}{p_n} \right) \\ &= \beta \frac{\sum_{k=0}^{n} \beta_k \prod_{i=k+1}^{n}(1-\beta_i)}{p_n} = \beta \frac{\sum_{k=0}^{n} \gamma_n^k}{p_n} = \frac{\beta}{p_n}, \end{aligned}$$

we have

$$\sum_{k=0}^{\infty} \gamma^k = 1.$$

Further we obtain that

$$\sum_{k=0}^{n} \left| \gamma_n^k - \gamma^k \right| = \sum_{k=0}^{n} (\gamma_n^k - \gamma^k)$$

$$
\begin{aligned}
&= (p_n - \beta) \sum_{k=0}^{n} \frac{\beta_k}{p_k} \\
&= (p_n - \beta) \frac{1}{p_n} \sum_{k=0}^{n} \beta_k \prod_{i=k+1}^{n} (1 - \beta_i) \\
&\leq (p_n - \beta) \frac{1}{p_n} \sum_{k=0}^{n} \beta_k
\end{aligned}
$$

for every $n \in \mathbb{N}$. Thus we have

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left|\gamma_n^k - \gamma^k\right| = 0.$$

By virtue of Theorem 2.3, we define a nonexpansive mapping T of C into itself by

$$T = \sum_{k=0}^{\infty} \gamma^k S_k,$$

where $S_0 = I$. Since S_k is nonexpansive, we know that

$$\|S_k y\| \leq \|S_k y - S_k u\| + \|S_k u\| \leq \|y - u\| + \|u\|$$

for all $y \in C$, $u \in \bigcap_{n=1}^{\infty} F(S_n)$, and $k \in \mathbb{N}$. Let B be a nonempty bounded closed convex subset of C. From all observations above, we obtain

$$
\begin{aligned}
\|Ty - T_n y\| &\leq \sum_{k=0}^{n} \left|\gamma_n^k - \gamma^k\right| \|S_k y\| + \sum_{k=n+1}^{\infty} \gamma^k \|S_k y\| \\
&\leq (\|y - u\| + \|u\|) \left(\sum_{k=0}^{n} \left|\gamma_n^k - \gamma^k\right| + \sum_{k=n+1}^{\infty} \gamma^k \right)
\end{aligned}
$$

for all $y \in B$. Therefore

$$\lim_{n\to\infty} \sup_{y\in B} \|Ty - T_n y\| \leq \sup_{y\in B} (\|y - u\| + \|u\|) \lim_{n\to\infty} \left(\sum_{k=0}^{n} \left|\gamma_n^k - \gamma^k\right| + \sum_{k=n+1}^{\infty} \gamma^k \right) = 0.$$

Theorem 2.3 also implies that $F(T_n) = \bigcap_{k=1}^{n} F(S_k)$ and $F(T) = \bigcap_{n=1}^{\infty} F(S_n)$. From these facts it is verified that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Now we obtain the following result:

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n)$ is nonempty and $\{\alpha_n\}$ a sequence of $(0,1)$ such that $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Let $\{\beta_n\}$ be a sequence of $(0,1)$ with $\sum_{n=1}^{\infty} \beta_n < \infty$. Let $\{x_n\}$ be a sequence of C defined by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \sum_{k=0}^{n} \beta_k \prod_{i=k+1}^{n} (1 - \beta_i) S_k x_n$$

for every $n \in \mathbb{N}$, where $\beta_0 = 1$, I is the identity mapping, $S_0 = I$, and $\prod_{i=m}^{l} (1 - \beta_i) = 1$ if $m > l$. Then $\{x_n\}$ converges weakly to some point of $\bigcap_{n=1}^{\infty} F(S_n)$.

Remark 3.4. For strong convergence to a common fixed point of a countable family of nonexpansive mappings, see [1].

4. Common fixed points of a pair of nonexpansive mappings

In this section, we discuss the problem of finding a common fixed point of a pair of nonexpansive mappings. This problem was considered in [4, 13].

Lemma 4.1. *Let E be a strictly convex Banach space, C a nonempty closed convex subset of E, S and T two nonexpansive mappings of C into itself, and $\lambda \in (0,1)$. Let U be a nonexpansive mapping of C into itself defined by*

$$U = (\lambda I + (1-\lambda)S)T,$$

where I is the identity mapping on E. If $F(S) \cap F(T) \neq \emptyset$, then $F(U) = F(S) \cap F(T)$.

Proof. By the definition of U, it is clear that $F(U) \supset F(S) \cap F(T)$. Therefore we have $F(U) \neq \emptyset$. So we have to show that $F(U) \subset F(S) \cap F(T)$. Let $z \in F(U)$ and $w \in F(S) \cap F(T)$. Then we have

$$\begin{aligned}\|z-w\| &= \|Uz-w\| = \|(\lambda I + (1-\lambda)S)Tz - w\| \\ &= \|\lambda(Tz-w) + (1-\lambda)(STz-w)\| \\ &\le \lambda\|Tz-w\| + (1-\lambda)\|STz-w\| \\ &\le \lambda\|z-w\| + (1-\lambda)\|Tz-w\| \le \|z-w\|.\end{aligned}$$

Taking into account $1-\lambda > 0$, we obtain

$$\|z-w\| = \|Tz-w\| = \|\lambda(Tz-w) + (1-\lambda)(STz-w)\| = \|STz-w\|.$$

Therefore, it follows that $Tz - w = STz - w$ by (2.2) and hence $Tz \in F(S)$. This yields

$$z = Uz = \lambda Tz + (1-\lambda)STz = \lambda Tz + (1-\lambda)Tz = Tz.$$

Thus we have that $z \in F(T)$ and hence $z = Tz \in F(S)$. Then we conclude that $z \in F(S) \cap F(T)$. □

By using Lemma 3.2, we obtain the following:

Theorem 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let S and T be two nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of $(0,1)$ such that $0 < a \le \alpha_n \le b < 1$ and $0 < c \le \beta_n \le d < 1$ for some $a, b, c, d \in \mathbb{R}$. Let $\{x_n\}$ be a sequence of C defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n T x_n + (1-\beta_n)STx_n)$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to some point of $F(S) \cap F(T)$.

Proof. Put $U_n = (\beta_n I + (1-\beta_n)S)T$ for each $n \in \mathbb{N}$, where I is the identity mapping. Then clearly, U_n is nonexpansive of C into itself and it follows from Lemma 4.1 that $F(U_n) = F(S) \cap F(T)$ for every $n \in \mathbb{N}$. Thus we know that $\bigcap_{n=1}^{\infty} F(U_n) = F(S) \cap F(T) \neq \emptyset$. Let $\{n_i\}_{i=1}^{\infty}$ be an increasing sequence of $\mathbb{N}$. Since $\{\beta_{n_i}\}$ is a sequence of $[c,d]$, there exists a subsequence $\{\beta_{n_{i_j}}\}$ of $\{\beta_{n_i}\}$ such that $\lim_{j\to\infty} \beta_{n_{i_j}} = \beta \in [c,d]$. Put $U = (\beta I + (1-\beta)S)T$. Then it also follows from Lemma 4.1 that $F(U) = F(S) \cap F(T)$ and hence $F(U) = \bigcap_{n=1}^{\infty} F(U_n)$. Further we have

$$\begin{aligned}\left\|Uy - U_{n_{i_j}} y\right\| &= \left\|(\beta I + (1-\beta)S)Ty - (\beta_{n_{i_j}} I + (1-\beta_{n_{i_j}})S)Ty\right\| \\ &= \left|\beta - \beta_{n_{i_j}}\right| \|Ty - STy\| \\ &\le \left|\beta - \beta_{n_{i_j}}\right| (\|Ty-u\| + \|u - STy\|)\end{aligned}$$

$$\leq \left|\beta - \beta_{n_{i_j}}\right| 2 \|y - u\|$$

for all $y \in C$ and $i \in \mathbb{N}$, where $u \in F(S) \cap F(T)$. Let B be a nonempty bounded closed convex subset of C. Then we have

$$\lim_{j\to\infty} \sup_{y \in B} \left\|Uy - U_{n_{i_j}} y\right\| = 0.$$

According to Lemma 3.2, we conclude that $\{x_n\}$ converges weakly to some point of $F(U) = F(S) \cap F(T)$. □

The following result was obtained in [13].

Lemma 4.3 (Takahashi-Tamura [13]). *Let E be a strictly convex Banach space, C a nonempty closed convex subset of E, S and T two nonexpansive mappings of C into itself, and $\lambda \in (0,1)$. Let U be a nonexpansive mapping of C into itself defined by*

$$U = S(\lambda I + (1-\lambda)T),$$

where I is the identity mapping on E. If $F(S) \cap F(T) \neq \emptyset$, then $F(U) = F(S) \cap F(T)$.

By using Lemma 3.2 combined with Lemma 4.3, we also obtain the following result.

Theorem 4.4 (Takahashi-Tamura [13]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let S and T be two nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of $(0,1)$ such that $0 < a \leq \alpha_n \leq b < 1$ and $0 < c \leq \beta_n \leq d < 1$ for some $a, b, c, d \in \mathbb{R}$. Let $\{x_n\}$ be a sequence of C defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S(\beta_n x_n + (1-\beta_n) T x_n)$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to some point of $F(S) \cap F(T)$.

5. Common solutions of a fixed point problem and a variational inequality problem

Finally, we apply Lemma 3.2 to the problem of finding a common solution of the fixed point problem for a nonexpansive mapping and the variational inequality problem for an inverse-strongly-monotone mapping. This problem was discussed in [8, 14].

Let C be a nonempty closed convex subset of a real Hilbert space H and A a mapping of C into H. The variational inequality problem is formulated as follows: Find $x \in C$ such that $\langle y - x, Ax\rangle \geq 0$ for all $y \in C$. In this case, such $x \in C$ is a solution of this problem and the solution set is denoted by $\mathrm{VI}(C, A)$, that is, $\mathrm{VI}(C, A) = \{x \in C : \langle y - x, Ax\rangle \geq 0 \text{ for all } y \in C\}$. For every $x \in H$, there exists a unique nearest point in C, denoted by Px, such that $\|x - Px\| \leq \|x - y\|$ for all $y \in C$. The mapping P is called the metric projection of H onto C. We know that the metric projection P is nonexpansive and firmly nonexpansive, that is,

$$\|Px - Py\|^2 \leq \langle x - y, Px - Py\rangle \tag{5.1}$$

holds for all $x, y \in C$; see [6] for more details. We also know that

$$\mathrm{VI}(C, A) = F(P(I - \lambda A)) \tag{5.2}$$

for all $\lambda > 0$; see [14] for more details. Let $\alpha > 0$. A mapping A of C into H is said to be α-inverse-strongly-monotone if $\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. It is known that

$$\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\| \tag{5.3}$$

for all $x, y \in C$ and

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 \leq \|x-y\|^2-\lambda(2\alpha-\lambda)\|Ax-Ay\|^2 \tag{5.4}$$

holds for all $x, y \in C$ and $\lambda > 0$, where A is an α-inverse-strongly-monotone mapping and I is the identity mapping. From this fact, a mapping $I-\lambda A$ is nonexpansive if $0 < \lambda \leq 2\alpha$; see [14] for more details. To apply Lemma 3.2 to our problem, we need the following:

Lemma 5.1. *Let H be a real Hilbert space and C a nonempty closed convex subset of C. Let $\alpha > 0$ and $0 < \lambda < 2\alpha$. Let A be an α-inverse-strongly-monotone mapping of C into H and S a nonexpansive mapping of C into itself. If $F(S) \cap F(P(I-\lambda A))$ is nonempty, then $F(SP(I-\lambda A)) = F(S) \cap F(P(I-\lambda A))$, where P is the metric projection of H onto C.*

Proof. It is easy to show that $F(SP(I-\lambda A)) \supset F(S) \cap F(P(I-\lambda A))$. Thus $F(SP(I-\lambda A)) \neq \emptyset$. Let us prove that $F(SP(I-\lambda A)) \subset F(S) \cap F(P(I-\lambda A))$. Let $z \in F(SP(I-\lambda A))$ and $w \in F(S) \cap F(P(I-\lambda A))$ be given. From the nonexpansiveness of S and P and (5.4), we obtain the following:

$$\begin{aligned}
\|z-w\|^2 &= \|SP(I-\lambda A)z - SP(I-\lambda A)w\|^2 \\
&\leq \|P(I-\lambda A)z - P(I-\lambda A)w\|^2 \\
&\leq \|(I-\lambda A)z-(I-\lambda A)w\|^2 \\
&\leq \|z-w\|^2 - \lambda(2\alpha-\lambda)\|Az-Aw\|^2 \\
&\leq \|z-w\|^2.
\end{aligned}$$

Thus we have $\lambda(2\alpha-\lambda)\|Az-Aw\|^2 = 0$, that is, it follows that $Az = Aw$. From this fact combined with (5.1) and the nonexpansiveness of P and $I-\lambda A$, we also obtain the following:

$$\begin{aligned}
\|z-w\|^2 &= \|SP(I-\lambda A)z - SP(I-\lambda A)w\|^2 \\
&\leq \|P(I-\lambda A)z - P(I-\lambda A)w\|^2 \\
&\leq \langle P(I-\lambda A)z - P(I-\lambda A)w, (I-\lambda A)z-(I-\lambda A)w\rangle \\
&= \frac{1}{2}(\|P(I-\lambda A)z - P(I-\lambda A)w\|^2 + \|(I-\lambda A)z-(I-\lambda A)w\|^2 \\
&\qquad - \|P(I-\lambda A)z - P(I-\lambda A)w - (I-\lambda A)z + (I-\lambda A)w\|^2) \\
&\leq \frac{1}{2}(2\|z-w\|^2 - \|P(I-\lambda A)z - z\|^2).
\end{aligned}$$

Therefore $\|P(I-\lambda A)z-z\|^2 \leq 0$, that is, $z \in F(P(I-\lambda A))$. This implies that $z = SP(I-\lambda A)z = Sz$ and hence $z \in F(S)$. Consequently we conclude that $z \in F(S) \cap F(P(I-\lambda A))$. □

By Lemma 3.2, we obtain the following:

Theorem 5.2 (Takahashi-Toyoda [14])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly-monotone mapping of C into H and S a nonexpansive mapping of C into itself such that $\mathrm{VI}(C, A) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP(x_n - \lambda_n A x_n),$$

for every $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence of $[a, b]$ for some $a, b \in (0, 2\alpha)$, $\{\alpha_n\}$ is a sequence of $[c, d]$ for some $c, d \in (0, 1)$, and P is the metric projection of H onto C. Then $\{x_n\}$ converges weakly to some point of $\mathrm{VI}(C, A) \cap F(S)$.

Proof. Put $U_n = SP(I - \lambda_n A)$ for each $n \in \mathbb{N}$. It follows from Lemma 5.1 and (5.2) that $F(U_n) = \mathrm{VI}(C, A) \cap F(S)$ and hence $\bigcap_{n=1}^{\infty} F(U_n) = \mathrm{VI}(C, A) \cap F(S) \neq \emptyset$. Let $\{n_i\}_{i=1}^{\infty}$ be an increasing sequence of $\mathbb{N}$. Since $\{\lambda_{n_i}\}$ is a sequence of $[a, b]$, there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\lim_{j\to\infty} \lambda_{n_{i_j}} = \lambda \in [a, b]$. Put $U = SP(I - \lambda A)$. Then it also follows from Lemma 5.1 and (5.2) that $F(U) = \mathrm{VI}(C, A) \cap F(S)$ and hence $F(U) = \bigcap_{n=1}^{\infty} F(U_n)$. Since both S and P are nonexpansive, we have

$$\begin{aligned}
\left\| Uy - U_{n_{i_j}} y \right\| &= \left\| SP(I - \lambda A)y - SP(I - \lambda_{n_{i_j}} A)y \right\| \\
&\leq \left\| P(I - \lambda A)y - P(I - \lambda_{n_{i_j}} A)y \right\| \\
&\leq \left\| (I - \lambda A)y - (I - \lambda_{n_{i_j}} A)y \right\| \\
&= \left| \lambda - \lambda_{n_{i_j}} \right| \|Ay\|
\end{aligned} \tag{5.5}$$

for all $y \in C$ and $j \in \mathbb{N}$. Let B be a nonempty bounded closed convex subset of C. From (5.3) and (5.5) it follows that

$$\lim_{j\to\infty} \sup_{y\in B} \left\| Uy - U_{n_{i_j}} y \right\| = 0.$$

Consequently, Lemma 3.2 implies that $\{x_n\}$ converges weakly to some point of $F(U) = \mathrm{VI}(C, A) \cap F(S)$. □

As in the proof of Lemma 5.1, we also obtain the following:

Lemma 5.3. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $\alpha > 0$ and $0 < \lambda < 2\alpha$. Let A be an α-inverse-strongly-monotone mapping and S a nonexpansive mapping of C onto itself. If $F(P(I - \lambda A)) \cap F(S)$ is nonempty, then $F(P(I - \lambda A)S) = F(P(I - \lambda A)) \cap F(S)$, where P is the metric projection of H onto C.*

Similarly, by using Lemma 3.2 combined with Lemma 5.3, we also obtain the following:

Theorem 5.4 (Iiduka-Takahashi [7])**.** *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H with $\alpha > 0$ and S a nonexpansive mapping of C into itself such that $\mathrm{VI}(C, A) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) P(S x_n - \lambda_n A S x_n),$$

for every $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence of $[a, b]$ for some $a, b \in (0, 2\alpha)$, $\{\alpha_n\}$ is a sequence of $[c, d]$ for some $c, d \in (0, 1)$, and P is the metric projection of H onto C. Then $\{x_n\}$ converges weakly to some point of $\mathrm{VI}(C, A) \cap F(S)$.

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