THE DEGREE OF PROPER HYPERSUBSTITUTIONS

K. Denecke, J. Koppitz and R. Srithus

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Abstract. Let V be a variety of type τ. A hypersubstitution which preserves all identities of V is called a V-proper hypersubstitution. The set $P(V)$ of all V-proper hypersubstitutions forms a monoid, which is a submonoid of the monoid of all hypersubstitutions of type τ. The hypersubstitutions in $P(V)$ can be partitioned according to an equivalence relation $\sim_V$ first introduced by P łonka. The authors introduce the name "degree of proper hypersubstitutions with respect to V" for the number $d_p(V)$ of distinct equivalence classes under this relation, and study the properties of this parameter.

1 Introduction Let $(f_i)_{i\in I}$ be an indexed set of operation symbols of type τ and assume that f_i is n_i-ary and $n_i \geq 1$ for all $i \in I$. We denote by $W_\tau(X)$ the set of all terms built up from variables from a set X and operation symbols from $\{f_i \mid i \in I\}$. *Hypersubstitutions* are mappings from $\{f_i \mid i \in I\}$ into $W_\tau(X)$ preserving the arities. Any hypersubstitution σ induces a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$. The mapping $\hat{\sigma}$ is defined in the following inductive way :

(i) $\hat{\sigma}[x] := x$ for every variable $x \in X$.

(ii) $\hat{\sigma}[f_i(t_1, \cdots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_{n_i}])$ for any operation symbol f_i and terms $t_1, \cdots, t_{n_i}$.

Using this extension one defines a multiplication $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of functions. Using the identity hypersubstitution defined by $\sigma_{id}(f_i) = f_i(x_1, \cdots, x_{n_i})$ for all $i \in I$, one obtains the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$ of all hypersubstitutions of type τ. An identity $s \approx t$ in the variety V of all algebras $Alg(\tau)$ of type τ is a *hyperidentity* of V if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in V for any $\sigma \in Hyp(\tau)$. A variety V is called *solid* if each of its identities is a hyperidentity.

We will use the following notation: Let IdV be set of all identities satisfied in the variety V : $\mathcal{A} \models s \approx t$ means that the algebra $\mathcal{A}$ of type τ satisfies $s \approx t$ as identity. Let $vb(s)$ be the set of variables occurring in s. Then $s \approx t$ is *regular* if $vb(s) = vb(t)$. Let $leftmost(s)$ and $rightmost(s)$ be the first and the last variables, respectively occurring in s. Then $s \approx t$ is called *outermost* if $leftmost(s) = leftmost(t)$ and $rightmost(s) = rightmost(t)$.

Let $P(V)$ be the set of all V-proper hypersubstitutions ([13]), i.e. the set of all hypersubstitutions of $Hyp(\tau)$ which preserve all identities in the variety V, so $P(V) := \{\sigma \mid \forall s \approx t \in IdV(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV)\}$. It is easy to see that $P(V)$ forms a submonoid of $Hyp(\tau)$. The variety V is *solid* iff $P(V) = Hyp(\tau)$. For a hypersubstitution $\sigma \in Hyp(\tau)$ and an algebra $\mathcal{A} = (A; ({f_i}^{A})_{i\in I})$ of type τ we define the derived algebra $\sigma(\mathcal{A}) = (A; (\sigma(f_i)^{\mathcal{A}})_{i\in I})$, meaning that, for the fundamental operations ${f_i}^{\sigma(\mathcal{A})}$ of the derived algebra we have ${f_i}^{\sigma(\mathcal{A})} = \sigma(f_i)^{\mathcal{A}}$ for all $i \in I$.

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We also need the following " conjugate property", which holds for all algebras $\mathcal{A}$ and all hypersubstitutions σ.

$$\mathcal{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t] \Leftrightarrow \sigma(\mathcal{A}) \models s \approx t.$$

In [13] the following binary relation on sets of hypersubstitutions was introduced.

Definition 1.1 Let V be a variety of algebras of type τ. Then

$$\sigma_1 \sim_V \sigma_2 :\Longleftrightarrow \forall i \in I(\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV).$$

Clearly, $\sim_V$ is an equivalence relation on the set $Hyp(\tau)$, but in general it is not a congruence relation. By induction on the complexity of term definition one shows that $\sigma_1 \sim_V \sigma_2$ implies that there follows $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in IdV$ for any term $t \in W_\tau(X)$ ([4]). Therefore, from $\sigma_1 \sim_V \sigma_2$ we obtain

$$(\sigma_1 \circ_h \sigma)(f_i) = \hat{\sigma}_1[\sigma(f_i)] \approx \hat{\sigma}_2[\sigma(f_i)] = (\sigma_2 \circ_h \sigma)(f_i) \in IdV$$

for all $i \in I$, and thus $\sigma_1 \circ_h \sigma \sim_V \sigma_2 \circ_h \sigma$. This shows that $\sim_V$ is a right congruence. If V is a solid variety, then $\sim_V$ is both a left and a right congruence and therefore it is a congruence. The following proposition is also well-known.

Proposition 1.2 ([13]) *Let V be a variety of type τ. Let $s \approx t \in IdV$, $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ and $\sigma_1 \sim_V \sigma_2$. Then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.*

This result means that $P(V)$ is a union of equivalence classes with respect to $\sim_V$. The elements of $P_0(V) := [\sigma_{id}]_{\sim_V}$ are called *inner hypersubstitutions.* The inner hypersubstitutions form a submonoid of $P(V)$. In this paper, we are interested in the cardinality of the quotient set $P(V)/{\sim_V}$ where $\sim_V$ is the restriction of the relation $\sim_V$ defined on $Hyp(\tau)$ to the subset $P(V)$.

Definition 1.3 The cardinal number $d_p(V) := |P(V)/{\sim_V}\ |$ is called the *degree* of proper hypersubstitutions with respect to the variety V.

One extreme case is $d_p(V) = 1$. In this case $P(V) = P_0(V) = [\sigma_{id}]_{\sim_V}$. In [6] varieties having this property are called *unsolid.* This suggests a classification of varieties using the degree of proper hypersubstitutions.

2 Derived Algebras

The class $Alg(\tau)$ of all algebras of type τ together with the homomorphisms between them forms a category. If we map each algebra $\mathcal{A} \in Alg(\tau)$ to the derived algebra $\sigma(\mathcal{A})$ and each homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$ to itself, then we get an endofunctor F_σ of the category $Alg(\tau)$.

Proposition 2.1 *For every hypersubstitution σ of type τ we get a functor $F_\sigma : Alg(\tau) \longrightarrow Alg(\tau)$.*

Proof. We prove first that $F_\sigma(h) : \sigma(\mathcal{A}) \longrightarrow \sigma(\mathcal{B})$ is a homomorphism. For the homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$, for every n-ary term t of type τ and for the induced term operations $t^{\mathcal{A}}$ and $t^{\mathcal{B}}$ we have $h(t^{\mathcal{A}}(a_1, \cdots, a_n)) = t^{\mathcal{B}}(h(a_1), \cdots, h(a_n))$ for every n-tuple $(a_1, \cdots, a_n)$ of elements from A. Then for the term $\sigma(f_i)$ we have:

$$\begin{aligned} &h({f_i}^{\sigma(\mathcal{A})}(a_1, \cdots, a_{n_i})) \\ &\quad = \quad h(\sigma(f_i)^{\mathcal{A}}(a_1, \cdots, a_{n_i})) \\ &\quad = \quad \sigma(f_i)^{\mathcal{B}}(h(a_1), \cdots, h(a_{n_i})) \\ &\quad = \quad f_i^{\sigma(\mathcal{B})}(h(a_1), \cdots, h(a_{n_i})) \end{aligned}$$

and this shows that $F_\sigma(h) = h : \sigma(\mathcal{A}) \longrightarrow \sigma(\mathcal{B})$ is a homomorphism for every homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$. Now we check the conditions which a functor has to satisfy. Let $h_1 : \mathcal{A} \longrightarrow \mathcal{B}$ and $h_2 : \mathcal{B} \longrightarrow \mathcal{C}$ be homomorphisms. Then we have $F_\sigma(h_2 \circ h_1) = h_2 \circ h_1 = F_\sigma(h_2) \circ F_\sigma(h_1)$ and for the identity homomorphism $id_A : \mathcal{A} \longrightarrow \mathcal{A}$ we have $F_\sigma(id_A) = id_A = id_{\sigma(\mathcal{A})}$ since the algebras $\mathcal{A}$ and $\sigma(\mathcal{A})$ have the same universes. ∎

Since any functor preserves isomorphisms, we have

Lemma 2.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of type τ and let σ be a hypersubstitution. If $\mathcal{A} \cong \mathcal{B}$, then $\sigma(\mathcal{A}) \cong \sigma(\mathcal{B})$.*

Using the isomorphism of derived algebras we define a binary relation $\sim_{iso}$ on the set $Hyp(\tau)$.

Definition 2.3 $\sigma_1 \sim_{iso} \sigma_2 :\Longleftrightarrow \forall \mathcal{A} \in Alg(\tau)(\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A}))$.

Clearly $\sim_{iso}$ is an equivalence relation on $Hyp(\tau)$. There arises the question of whether $\sim_{iso}$ is a congruence relation on the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$. To prove this we need some preparation.

Lemma 2.4 *Let $\sigma_1, \sigma_2 \in Hyp(\tau)$, let $\mathcal{A}$ be an algebra of type τ and let t be an n-ary term of type τ. Then from $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$ there follows*

$$h(\hat{\sigma}_1[t]^{\mathcal{A}}(a_1, \cdots, a_n)) = \hat{\sigma}_2[t]^{\mathcal{A}}(h(a_1), \cdots, h(a_n)) \qquad (*)$$

for an isomorphism $h : \sigma_1(\mathcal{A}) \longrightarrow \sigma_2(\mathcal{A})$.

Proof. We will give a proof by induction on the complexity of the term t. Let $t = x_i \in X_n$ be a variable. Then $h(\hat{\sigma}_1[x_i]^{\mathcal{A}}(a_1, \cdots, a_n)) = h(e_i^{n,\mathcal{A}}(a_1, \cdots, a_n)) = h(a_i) = \hat{\sigma}_2[x_i]^{\mathcal{A}}(h(a_1), \cdots, h(a_n))$. Let $t = f_i(t_1, \cdots, t_{n_i})$. Assume now inductively that $(*)$ is satisfied for $t_j, j = 1, \cdots, n_i$. Then

$$\begin{aligned}
&h(\hat{\sigma}_1[f_i(t_1, \cdots, t_{n_i})]^{\mathcal{A}}(a_1, \cdots, a_n)) \\
&= h(\sigma_1(f_i)^{\mathcal{A}}[\hat{\sigma}_1[t_1]^{\mathcal{A}}, \cdots, \hat{\sigma}_1[t_{n_i}]^{\mathcal{A}}](a_1, \cdots, a_n)) \\
&= \sigma_2(f_i)^{\mathcal{A}}(h(\hat{\sigma}_1[t_1]^{\mathcal{A}}(a_1, \cdots, a_n)), \cdots, h(\hat{\sigma}_1[t_{n_i}]^{\mathcal{A}}(a_1, \cdots, a_n))) \\
&= \sigma_2(f_i)^{\mathcal{A}}(\hat{\sigma}_2[t_1]^{\mathcal{A}}(h(a_1), \cdots, h(a_n)), \cdots, \\
&\quad \hat{\sigma}_2[t_{n_i}]^{\mathcal{A}}(h(a_1), \cdots, h(a_n))) \\
&= \sigma_2(f_i)^{\mathcal{A}}(\hat{\sigma}_2[t_1]^{\mathcal{A}}, \cdots, \hat{\sigma}_2[t_{n_i}]^{\mathcal{A}})(h(a_1), \cdots, h(a_n)) \\
&= (\hat{\sigma}_2(f_i(t_1, \cdots, t_{n_i})))^{\mathcal{A}}(h(a_1), \cdots, h(a_n)) \\
&= \hat{\sigma}_2[t]^{\mathcal{A}}(h(a_1), \cdots, h(a_n)).
\end{aligned}$$

■

Theorem 2.5 *The relation $\sim_{iso}$ is a congruence relation on the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$.*

Proof. We prove that $\sim_{iso}$ is a left and a right congruence on $(Hyp(\tau); \circ_h, \sigma_{id})$. Assume that $\sigma_1 \sim_{iso} \sigma_2$ and $\sigma \in Hyp(\tau)$. Then for every algebra $\mathcal{A}$ we have $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$, so by Lemma 2.2 it follows that $(\sigma \circ_h \sigma_2)(\mathcal{A}) = \sigma(\sigma_1(\mathcal{A})) = \sigma(\sigma_2(\mathcal{A})) = (\sigma \circ_h \sigma_2)(\mathcal{A})$, and thus $\sigma \circ_h \sigma_1 \sim_{iso} \sigma \circ_h \sigma_2$. If we substitute the term $\sigma(f_i)$ for t in $(*)$ from Lemma 2.4 , then we get from $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$ that $h((\hat{\sigma}_1[\sigma(f_i)])^{\mathcal{A}}(a_1, \cdots, a_{n_i})) = (\hat{\sigma}_2[\sigma(f_i)])^{\mathcal{A}}(h(a_1), \cdots, h(a_{n_i}))$ for an isomorphism $h : \sigma_1(\mathcal{A}) \longrightarrow \sigma_2(\mathcal{A})$. From this equation we obtain $h(((\sigma_1 \circ_h \sigma)(f_i))^{\mathcal{A}}(a_1, \cdots, a_{n_i})) = ((\sigma_2 \circ_h \sigma)(f_i))^{\mathcal{A}}(h(a_1), \cdots, h(a_{n_i}))$. Since $((\sigma_j \circ_h \sigma)(f_i))^{\mathcal{A}}, j = 1, 2$, are the fundamental operations of the algebras $(\sigma_j \circ_h \sigma)(\mathcal{A})$ and since $h : A \longrightarrow A$ is bijective, we have $(\sigma_1 \circ_h \sigma)(\mathcal{A}) \cong (\sigma_2 \circ_h \sigma)(\mathcal{A})$ and then $\sigma_1 \circ_h \sigma \sim_{iso} \sigma_2 \circ_h \sigma$. ■

Let V be a variety of type τ. Then for hypersubstitutions $\sigma_1, \sigma_2 \in Hyp(\tau)$ we define :

Definition 2.6 $\sigma_1 \sim_{V-iso} \sigma_2 :\Longleftrightarrow \forall \mathcal{A} \in V(\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A}))$.

Clearly, $\sim_{iso}=\sim_{Alg(\tau)-iso}$. If V_1, V_2 are varieties of type τ and $V_1 \subseteq V_2$, then $\sim_{iso}\subseteq\sim_{V_2-iso}\subseteq\sim_{V_1-iso}$. Since from $\sigma_1 \sim_V \sigma_2$ there follows $\sigma_1(\mathcal{A}) = \sigma_2(\mathcal{A})$ for all $\mathcal{A} \in V$ and since from $\sigma_1(\mathcal{A}) = \sigma_2(\mathcal{A})$ we get $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$, we have also $\sim_V\subseteq\sim_{V-iso}$ for every variety V of algebras of type τ. Moreover, we have $P_0(V) \subseteq [\sigma_{id}]_{V-iso} \subseteq P(V)$.

Proposition 2.7 *Let V be a variety of type τ and let $\sigma_1, \sigma_2 \in Hyp(\tau)$. Then the following holds : If $s \approx t \in IdV$, if $\sigma_1 \sim_{V-iso} \sigma_2$ and if $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.*

Proof. From $\sigma_1 \sim_{V-iso} \sigma_2$ there follows $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$ for every $\mathcal{A}$ in V. By the conjugate property from $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{A}$ we get $s \approx t \in Id\sigma_1(\mathcal{A})$ and by the isomorphism $\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})$ we have also $s \approx t \in Id\sigma_2(\mathcal{A})$ or $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{A}$. Therefore $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$. ∎

As a consequence we have

Corollary 2.8 *The monoid $P(V)$ of all V-proper hypersubstitutions of type τ is a union of equivalence classes with respect to $\sim_{V-iso}$.*

The relation $\sim_{V-iso}$ is an equivalence relation and a left-congruence, but in general not a congruence on $(Hyp(\tau); \circ_h, \sigma_{id})$.

Definition 2.9 The cardinality $isd_p(V) := |P(V)/\!\sim_{V-iso} |$ is called the *isomorphism degree* of proper hypersubstitutions with respect to the variety V.

3 Some General Results We mentioned already that for a solid variety V the relation $\sim_V$ is a congruence on $(Hyp(\tau); \circ_h, \sigma_{id})$ and therefore, the restriction of $\sim_V$ is a congruence on $(P(V); \circ_h, \sigma_{id})$. If V is solid, then by Lemma 2.2 the relation $\sim_{V-iso}$ is also a congruence. Now we want to characterize solid varieties with $d_p(V) = 1$ and with $d_p(V) = 2$, respectively. For $d_p(V) = 1$ in [6] was proved:

Proposition 3.1 *A non-trivial variety V is solid and has $d_p(V) = 1$ iff $\tau = (1, \cdots, 1, \cdots)$ and $V = Mod\{f_i(x) \approx x \mid i \in I\}$. (This means that every operation symbol is unary and satisfies the same identity $f_i(x) \approx x$.)*

Now for $d_p(V) = 2$ we have :

Proposition 3.2 *A non-trivial variety V of type τ is solid and has $d_p(V) = 2$ iff $\tau = (1)$ and $V = Mod\{f(x) \approx f(f(x))\}$ or $V = Mod\{x \approx f(f(x))\}$.*

Proof. If $\tau = (1)$ and $V = Mod\{x \approx f(f(x))\}$, then $Hyp(\tau) = [\sigma_x]_{\sim_V} \cup [\sigma_{f(x)}]_{\sim_V}$. It is easy to check that $\sigma_x, \sigma_{f(x)} \in P(V)$ and $x \approx f(x) \notin IdV$. Thus $|P(V)/\!\sim_V| = |\{[\sigma_x]_{\sim_V}, [\sigma_{f(x)}]_{\sim_V}\}| = 2$ and $P(V) = Hyp(\tau)$, i.e. V is solid and $d_p(V) = 2$. By similar arguments we get that $V = Mod\{f(x) \approx f(f(x))\}$ is solid and $d_p(V) = 2$. Suppose now that V is solid and $d_p(V) = 2$. Let $\tau = (n_i)_{i\in I}$ with $n_i \geq 2$ for some $i \in I$. Then $[\sigma_{x_1}]_{\sim_V} = [\sigma_{x_2}]_{\sim_V}$ or $[\sigma_{x_1}]_{\sim_V} = [\sigma_{f_i(x_1,x_2,\cdots,x_2)}]_{\sim_V}$ or $[\sigma_{x_2}]_{\sim_V} = [\sigma_{f_i(x_1,x_2,\cdots,x_2)}]_{\sim_V}$. Then $x_1 \approx x_2 \in IdV$ or $x_1 \approx f_i(x_1, x_2, \cdots, x_2) \in IdV$ or $x_2 \approx f_i(x_1, x_2, \cdots, x_2) \in IdV$. Since V is non-trivial $x_1 \approx x_2 \in IdV$ is not possible. Since V is solid, it contains the variety RA_τ of all rectangular algebras (see e.g. [4]), i.e. the variety RA_τ which is generated by all projection algebras of type τ. But this is a contradiction since neither $x_1 \approx f_i(x_1, x_2, \cdots, x_2) \in IdRA_\tau$ nor $x_2 \approx f_i(x_1, x_2, \cdots, x_2) \in IdRA_\tau$. This shows that $\tau = (1, \cdots, 1)$. Since $d_p(V) \neq 1$ by Proposition 3.1 there is an $i \in I$ such that $x \approx f_i(x) \notin IdV$. This shows that $Hyp(\tau) = [\sigma_x]_{\sim_V} \cup [\sigma_{f_i(x)}]_{\sim_V}$ since $d_p(V) = 2$ and $P(V) = Hyp(\tau)$. Assume that $\mid I \mid> 1$. Then there is a $j \in I \setminus \{i\}$ and $\sigma_{f_j(x)} \in [\sigma_x]_{\sim_V}$ or $\sigma_{f_j(x)} \in [\sigma_{f_i(x)}]_{\sim_V}$, i.e. $x \approx f_j(x) \in IdV$

or $f_i(x) \approx f_j(x) \in IdV$. Applying the hypersubstitutions σ_1, σ_2 which map f_j to $f_i(x)$ and f_k to $f_k(x)$ for all $k \neq j$ and f_j to x and f_k to $f_k(x)$, respectively for all $k \neq j$ to $x \approx f_j(x)$ or to $f_i(x) \approx f_j(x)$ we get $x \approx f_i(x) \in IdV$, a contradiction. Thus $\tau = (1)$. Further we have $\sigma_{f_i(f_i(x))} \in [\sigma_x]_{\sim_V}$ or $a_{f_i(f_i(x))} \in [\sigma_{f_i(x)}]_{\sim_V}$, i.e. $x \approx f_i(f_i(x)) \in IdV$ or $f_i(x) \approx f_i(f_i(x)) \in IdV$. In the first case we obtain ${f_i}^k(x) \approx {f_i}^{k+2l}(x) \in IdV$ (where ${f_i}^k(x) := f_i(\cdots f_i(x) \cdots))$ for $k, l \in \mathbb{N}$ and in the second case we have ${f_i}^k(x) \approx f^l(x) \in IdV$ for $k, l \geq 1, k, l \in \mathbb{N}$. Assume that $x \approx f_i(f_i(x)) \in IdV$ and ${f_i}^k(x) \approx {f_i}^{k+2l+1}(x) \in IdV$ for some $k, l \in \mathbb{N}$. Then both identities give $x \approx f_i(x) \in IdV$, a contradiction. Assume that $f_i(x) \approx {f_i}^2(x) \in IdV$ and $x \approx {f_i}^k(x) \in IdV$ for some $1 \leq k \in \mathbb{N}$. Then both identities give $x \approx f_i(x) \in IdV$, a contradiction. Altogether, this shows that $V = Mod\{f(x) \approx f(f(x))\}$ or $V = Mod\{x \approx f(f(x))\}$. ∎

Clearly, if V, V' are solid varieties of the same type and $V' \subseteq V$, then $d_p(V') \leq d_p(V)$. The following lemma provides a lower bound of $d_p(V)$ for a solid variety V. Let H_n be the set of all functions defined on $\{1, \cdots, n\}$ and let Ims be the image of $s \in H_n$.

Lemma 3.3 *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i \in I}$ such that $n := max\{n_i \mid i \in I\}$ exists. Then $d_p(V) \geq (\Pi_{i \in I} n_i) + n^n - n$.*

Proof. case 1: $n = max\{n_i \mid i \in I\} = 1$.
Clearly, the variety $V' = Mod\{f_i(x) \approx x \mid i \in I\}$ is contained in any non-trivial solid variety of type $\tau = (1, \cdots, 1)$ and has $d_p(V') = 1$. Therefore $d_p(V) \geq 1$.
case 2: $n := max\{n_i \mid i \in I\} > 1$.
Then there is a $j \in I$ such that $n_j = n$. A hypersubstitution which maps each operation symbol to a variable is called a projection hypersubstitution. There are exactly $\Pi_{i \in I} n_i$ different projection hypersubstitutions of type $\tau = (n_i)_{i \in I}$. Since V is non-trivial for any pair σ, σ' of distinct projection hypersubstitutions we have $\sigma \not\sim_V \sigma'$. Since V is solid, every projection hypersubstitution is V-proper and therefore $P(V)/{\sim_V}$ contains the $\Pi_{i \in I} n_i$ pairwise different blocks generated by the projection hypersubstitutions.
Now we consider the hypersubstitutions σ_s^j which map the n-ary operation symbol f_j to $f_j(x_{s(1)}, \cdots, x_{s(n)})$ for any $s \in H_n$ and $f_i, i \neq j, i \in I$, to the (fundamental) term $f_i(x_1, \cdots, x_{n_i})$. We show that these hypersubstitutions generate pairwise different blocks with respect to $\sim_V$. We will verify that
Claim: $f_j(x_{s(1)}, \cdots, x_{s(n)}) \approx f_j(x_{s'(1)}, \cdots, x_{s'(n)}) \notin IdV$ for all $s, s' \in H_n, s \neq s'$.
Suppose that there are distinct mappings $s, s' \in H_n$ such that $f_j(x_{s(1)}, \cdots, x_{s(n)}) \approx f_j(x_{s'(1)}, \cdots, x_{s'(n)}) \in IdV$. From $s \neq s'$ there follows that there is a $k \in \{1, \cdots, n\}$ such that $s(k) \neq s'(k)$. Let σ be a projection hypersubstitution with $\sigma(f_j) = x_k$. Then $\hat{\sigma}[f_j(x_{s(1)}, \cdots, x_{s(n)})] = x_{s(k)} \approx x_{s'(k)} = \hat{\sigma}[f_j(x_{s'(1)}, \cdots, x_{s'(n)})] \in IdV$. But this means that V is trivial, a contradiction. The claim shows that $\sigma_s^j \not\sim_V \sigma_{s'}^j$ for $s \neq s'$. Therefore $P(V)/{\sim_V}$ contains the n^n pairwise different blocks generated by these hypersubstitutions. Now we want to verify that no projection hypersubstitution can collapse (with respect to $\sim_V$) with a hypersubstitution of the form σ_s^j as above if the mapping s is non-constant. Suppose that there are a projection hypersubstitution σ and a non-constant mapping $s \in H_n$ with $\sigma \sim_V \sigma_s^j$. Then we have $\sigma(f_j) = x_{j_l} \approx f_j(x_{s(1)}, \cdots, x_{s(n)}) = \sigma_s^j(f_j) \in IdV$. Since $|Ims| > 1$, there is a $k \in \{1, \cdots, n\}$ such that $s(k) \neq j_l$ and thus $x_{j_l} \neq x_{s(k)}$. Let σ' be a projection hypersubstitution with $\sigma'(f_i) = x_{s(k)}$. Then $\hat{\sigma}'[x_{j_l}] = x_{j_l} \approx x_{s(k)} = \hat{\sigma}'[f_j(x_{s(1)}, \cdots, x_{s(n)})] \in IdV$, a contradiction since V is non-trivial. Since there are exactly n hypersubstitutions mapping f_j to a term of the form $f_j(x_c, \cdots, x_c)$ for some $c \in \{1, \cdots, n\}$ we get $d_p(V) \geq (\Pi_{i \in I} n_i) + n^n - n$. ∎

The following generalization of Propositions 3.1 and 3.2 shows how $d_p(V)$ for a solid variety V can influence the type of V as well as the identities valid in V. We consider the case that $d_p(V)$ is minimal.

Proposition 3.4 *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i\in I}, n_i \geq 1$ for all $i \in I$ such that $n = max\{n_i \mid i \in I\}$ exists. If $d_p(V) = (\Pi_{i\in I} n_i) + n^n - n < \aleph_0$, then there is a $j \in I$ such that $n_j = n$, while for all other $i \neq j$ we have $n_i = 1$ and the identity $f_i(x) \approx x$ hold in V. Moreover, for all terms $t \in W_\tau(X_n)$ one of the following conditions is satisfied*

(i) $\exists\, l \in \{1, \cdots, n\}(t \approx x_l \in IdV)$,

(ii) $\exists\, s \in H_n, s$ *is non-constant and* $t \approx f_j(x_{s(1)}, \cdots, x_{s(n)}) \in IdV$.

Proof. Since $max\{n_i \mid i \in I\}$ exists there is a $j \in I$ such that $n_j = n$. We show that $n_i = 1$ for all $i \in I$ with $i \neq j$. We may assume first that $n > 1$. Suppose that there is a $k \in I$ with $k \neq j$ and $n_k > 1$. Let $id_n \in H_n$ be the identity mapping and let σ'' be the hypersubstitution defined by $\sigma''(f_j) = f_j(x_1, \cdots, x_n)$ and $\sigma''(f_i) = x_{n_i}$ for all $i \in I\setminus\{j\}$. By the claim in the proof of Lemma 3.3 we have $\sigma'' \not\sim_V \sigma^j_s$ for all $s \in H_n \setminus \{id_n\}$ where σ^j_s is the hypersubstitution mapping f_j to $f_j(x_{s(1)}, \cdots, x_{s(n)}), s \in H_n$ and f_i to $f_i(x_1, \cdots, x_{n_i})$ for any $i \neq j, i \in I$, which was used in the proof of Lemma 3.3. Since V is solid and $n, n_k > 1$ we have $x_m \approx f_j(x_1, \cdots, x_n) \notin IdV$ for all $m \in \{1, \cdots, n\}$ and $x_l \approx f_k(x_1, \cdots, x_{n_k}) \notin IdV$ for all $l \in \{1, \cdots, n_k\}$. Then there follows $\sigma''(f_k) = x_{n_k} \approx f_k(x_1, \cdots, x_{n_k}) = \sigma^j_{id_n}(f_k) \notin IdV$ for every $k \neq j$ and $\sigma''(f_j) = f_j(x_1, \cdots, x_n) \approx x_{j_l} = \sigma(f_j) \notin IdV$ for $1 \leq j_l \leq n_j \leq n$ and any n_j-ary projection hypersubstitution σ. We obtain $\sigma'' \not\sim_V \sigma^j_{id_n}$ and $\sigma'' \not\sim_V \sigma$. This means, $[\sigma'']_{\sim_V} \notin \{[\sigma]_{\sim_V} \mid \sigma$ is a projection hypersubstitution $\} \cup \{[\sigma^j_s]_{\sim_V} \mid s \in H_n, s \neq id_n\}$ and $d_p(V) > (\Pi_{i\in I} n_i) + n^n - n$, a contradiction. Therefore $n_k = 1$ for all $k \in I \setminus \{j\}$, i.e. $\tau = (1, \cdots, n, 1, \cdots)$. We want to show that V satisfies the identities $f_i(x) \approx x$ for every $i \neq j, i \in I$. Let σ''' be the hypersubstitution defined by $\sigma'''(f_j) = f_j(x_1, \cdots, x_n)$ and $\sigma'''(f_i) = x_1$ for all $i \in I \setminus \{j\}$. Clearly, $\sigma''' \not\sim_V \sigma$ for any projection hypersubstitution σ since $x_m \approx f_j(x_1, \cdots, x_n) \notin IdV$ for all $1 \leq m \leq n$. Further, $\sigma''' \not\sim_V \sigma^j_s$ for all $s \in H_n\setminus\{id_n\}$ by the claim in the proof of Lemma 3.3. Since $Hyp(\tau)/\sim_V = \{[\sigma]_{\sim_V} \mid \sigma$ is a projection hypersubstitution $\} \cup \{[\sigma^j_s]_{\sim_V} \mid s \in H_n$ and $|Ims| > 1\}$ thus (by the proof of Lemma 3.3) we must have $\sigma''' \sim_V \sigma_{id}$. Now, for $t \in W_\tau(X_n)$ we have to verify (i) or (ii). We define the hypersubstitution σ_t by $\sigma_t(f_j) := t$ and $\sigma_t(f_i) := f_i(x_1)$ for all $i \in I \setminus \{j\}$. From $Hyp(\tau)/\sim_V = \{[\sigma]_{\sim_V} \mid \sigma$ is a projection hypersubstitution $\} \cup \{[\sigma^j_s]_{\sim_V} \mid s \in H_n$ and $|Ims| > 1\}$ follows that there is a projection hypersubstitution σ with $\sigma_t \sim_V \sigma$ or that there is a non-constant mapping $s \in H_n$ with $\sigma_t \sim_V \sigma^j_s$. In the first case we have $\sigma_t(f_j) = t \approx x_{j_l} = \sigma(f_j) \in IdV$ for some $1 \leq j_l \leq n$ and in the second case we have $\sigma_t(f_j) = t \approx f_j(x_{s(1)}, \cdots, x_{s(n)}) = \sigma^j_s(f_j) \in IdV$. If $n = 1$, then by Proposition 3.1 we get $V = Mod\{f_i(x) \approx x \mid i \in I\}$ and from these equations for any $t \in W_\tau(X_1)$ we can derive the identities $t \approx x_1$. ∎

We next show that the converse is also satisfied. Altogether we have

Theorem 3.5 *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i\in I}, n_i \geq 1$ for all $i \in I$ such that $n = max\{n_i \mid i \in I\}$ exists. Then $d_p(V) = (\Pi_{i\in I} n_i) + n^n - n < \aleph_0$ if and only if the following conditions are satisfied :*

(i) *There is a $j \in I$ such that $n_j = n$ and $n_i = 1$ for all $i \in I, i \neq j$.*

(ii) $f_i(x) \approx x \in IdV$ *for any unary operation symbol* $f_i, i \neq j$.

(iii) *For any term* $t \in W_\tau(X_n)$ *one of the following conditions holds:*

(a) $\exists\ l \in \{1, \cdots, n\}(t \approx x_l \in IdV)$,

(b) $\exists\ s \in H_n$ *with* $\mid Ims \mid > 1(t \approx f_j(x_{s(1)}, \cdots, x_{s(n)}) \in IdV)$
(where f_j *is the* n*-ary operation symbol).*

Proof. For the type $\tau = (1, \cdots, 1, n, 1, \cdots, 1, \cdots)$ we have $\Pi_{i\in I} n_i + n^n - n = n^n$. Therefore we have to show that $d_p(V) = n^n$ if V satisfies all conditions given in the theorem. Let f_j be the n-ary operation symbol and let $x_l \in \{x_1, \cdots, x_n\}$. For the n-ary term $t = f_j(x_l, \cdots, x_l)$ by (iii)

we have the following possibilities: for each $l \in \{1, \cdots, n\}$, we have $f_j(x_l, \cdots, x_l) \approx x_p \in IdV$ for some $1 \leq p \leq n$ or $f_j(x_l, \cdots, x_l) \approx f_j(x_{s(1)}, \cdots, x_{s(n)}) \in IdV$ for some $s \in H_n$ with $|Ims| > 1$. The latter case is impossible by the solidity of V, and in the former case we can rule out $p \neq l$ again by the solidity of V. Therefore $f_j(x_l, \cdots, x_l) \approx x_l \in IdV$ for $l \in \{1, \cdots, n\}$. The identities $f_i(x) \approx x \in IdV$ for all $i \in I \setminus \{j\}$ and the identity $f_j(x, \cdots, x) \approx x \in IdV$ show that $W_\tau(X_1)/IdV = \{[x]_{IdV}\}$. This means that $Hyp(\tau)/{\sim_V}$ contains precisely all classes of hypersubstitutions mapping f_i to x_1 for every $i \in I, i \neq j$ and f_j to one of the terms $f_j(x_{s(1)}, \cdots, x_{s(n)})$ for $s \in H_n$. Since all these hypersubstitutions are proper we have $d_p(V) = n^n$. Together with Proposition 3.4 we have a proof of Theorem 3.5. ∎

The variety RA_τ generated by the set of all projection algebras of type τ is called the variety of *rectangular algebras*. RA_τ is for any type τ the least non-trivial solid variety of type τ. For type $\tau = (1, \cdots, n, \cdots, 1)$ we obtain $d_p(RA_\tau) = n^n$.
One could also consider the cardinality of $P(V)$ instead of $|P(V)/{\sim_V}|$.

Definition 3.6 $sd_p(V) := |P(V)|$ is called the *strict degree* of proper hypersubstitutions with respect to the variety V.

Example 3.7 Let $cv_\alpha(t)$ be the total number of occurrences of the variable $\alpha \in X$ in the term t. If $\alpha = x_i$ we will write for short $cv_i(t)$ instead of $cv_{x_i}(t)$. Then we will show that for the type (2) variety $V = Mod\{(xy)(zw) \approx x(y(zw))\}$, we have $sd_p(V) = 3$. Let $\sigma_t \in P(V)$ be the hypersubstitution which maps the binary operation symbol f to the binary term t. Then $\hat{\sigma}_t[(xy)(zw)] \approx \hat{\sigma}_t[x(y(zw))] \in IdV$ where $cv_x(\hat{\sigma}_t[(xy)(zw)]) = (cv_1(t))^2$ and $cv_x(\hat{\sigma}_t[x(y(zw)]) = cv_1(t)$. Since for every identity $u \approx v$ in V the numbers $cv_\alpha(u)$ and $cv_\alpha(v)$ for all $\alpha \in X$ are equal, we have $cv_1(t) \leq 1$. Moreover, we have $cv_w(\hat{\sigma}_t[(xy)(zw)]) = (cv_2(t))^2$ and $cv_w(\hat{\sigma}_t[x(y(zw)]) = (cv_2(t))^3$ by the previous argument we get $cv_2(t) \leq 1$. Therefore, $P(V) \subseteq \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}\}$. It is easy to see that only $\sigma_{x_1}, \sigma_{x_2}$ and $\sigma_{x_1x_2}$ are V-proper; $P(V) = \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}\}$ and $sd_p(V) = 3$. Since in V there are only regular identities and since V is non-trivial, none of the equations $x_1 \approx x_2, x_1 \approx x_1x_2, x_2 \approx x_1x_2$ is an identity in V and thus we have also $d_p(V) = 3$.

Now we look for the largest number $d_p(V)$. The case that the index set I is finite is especially important. For this case we want to prove that $d_p(V) \leq \aleph_0$. For the proof we need some preparation. First of all we want to assign to each n-ary term t a natural number which we call the *label* of t.

Definition 3.8 Let $\tau = (n_i)_{i\in I}$ be a type where I is at most countably infinite. Let $s : I \longrightarrow \mathbb{N}$ be an injection. Then we define the sequence lab_s as follows:

(i) $lab_s(x_k) = k$ for each variable $x_k \in X_n$.

(ii) If $x_{k_1}, \cdots, x_{k_{n_i}} \in X_n$ and if f_i is an n_i-ary operation symbol,
then $lab_s(f_i(x_{k_1}, \cdots, x_{k_{n_i}})) = s(i)k_1 \cdots k_{n_i}$.

(iii) If $t_1, \cdots, t_{n_j}$ are n-ary terms for some natural number $n \geq 1$ and if f_j is an n_j-ary operation symbol, then $lab_s(f_j(t_1, \cdots, t_{n_j})) = s(j)lab_s(t_1) \cdots lab_s(t_{n_j})$.

In this way, to each term t we associate a natural number. It is easy to see that $lab_s(t) \neq lab_s(t')$ whenever $t \neq t'$. This means that the mapping which maps each term to its label is one-to-one. Then we have

Proposition 3.9 *Let $\tau = (n_i)_{i \in I}$ be a type which is at most countably infinite. Then $|W_\tau(X_n)| = \aleph_0$ for each $n \geq 1$. Moreover, $|W_\tau(X)| = \aleph_0$.*

Proof. The labelling of every n-ary term defines a one-to-one mapping $lab_s : W_\tau(X_n) \longrightarrow \mathbb{N}$. Therefore, $|W_\tau(X_n)| \leq | \mathbb{N} | = \aleph_0$. But even if the type contains only one operation symbol, for each natural number n there is a term $t \in W_\tau(X_n)$ such that in t the operation symbol occurs n times. Thus, $|W_\tau(X_n)| = \aleph_0$. Since $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is a countable union of the countable sets $W_\tau(X_n)$, it is countable. ∎

It is well-known that for a countable set A and a natural number l the set

$$[A]^l := \{S \subseteq A \mid |S| \leq l\}$$

is also countable. We use this fact to prove the following theorem.

Theorem 3.10 *Let $\tau = (n_i)_{i \in I}$ be a finite type, i.e. I is a finite set. Then $|Hyp(\tau)| \leq \aleph_0$.*

Proof. Since I is finite, there is a natural number l such that $|\{f_i \mid i \in I\}| = l$. Any hypersubstitution of type τ maps each n_i-ary operation symbol to an n_i-ary term. Let $n := max\{n_i \mid i \in I\}$. Then $\sigma(\{f_i \mid i \in I\}) \subseteq W_\tau(X_n)$ and $|\sigma(\{f_i \mid i \in I\})| \leq l$. Since by Proposition 3.9 the set $W_\tau(X_n)$ is countable, we get $|Hyp(\tau)| \leq \aleph_0$. ∎

Since $P(V) \subseteq Hyp(\tau), |P(V)/{\sim_V}| \leq |P(V)|$ and $|P(V)/{\sim_{V-iso}}| \leq |P(V)|$ we have :

Corollary 3.11 *Let $\tau = (n_i)_{i \in I}$ be a finite type. Then for any variety V of type τ, $d_p(V) \leq \aleph_0, sd_p(V) \leq \aleph_0$ and $isd_p(V) \leq \aleph_0$.*

The variety $Alg(\tau)$ of all algebras of type τ is solid, thus $Hyp(\tau) = P(Alg(\tau))$. Assume that $\tau = (n)$. Then $|Hyp(\tau)| = |W_\tau(X_n)| = \aleph_0$ and $|P(Alg(\tau))| = \aleph_0$. Since $s \approx t \in IdAlg(\tau)$ iff $s = t$ the relation $\sim_{Alg(\tau)}$ is the diagonal and thus $|P(Alg(\tau))| = |P(Alg(\tau))/{\sim_{Alg(\tau)}}| = \aleph_0$. This shows $d_p(V) = sd_p(V) = \aleph_0$ and $isd_p(V) \leq \aleph_0$.
Now we want to give an example of a variety V of type $\tau = (2)$ such that $isd_p(V) = |P(Alg(V))/{\sim_{Alg(V)}}| = \aleph_0$.

Proposition 3.12 *For the variety $V = Mod\{f(x,x) \approx x\}$ of type $\tau = (2)$ there holds $isd_p(V) = \aleph_0$.*

Proof. We inductively define a sequence $(\sigma_k)_{k \geq 1, k \in \mathbb{N}}$ of hypersubstitutions as follows

(i) $\sigma_1(f) := f(x_1, x_2)$

(ii) $\sigma_{k+1}(f) := f(\sigma_k(f), x_2)$ for $k \in \mathbb{N}, k \geq 1$.

It is easy to see that V is solid and therefore $P(V) = Hyp(\tau)$ and thus $\sigma_k \in P(V)$ for all $k \geq 1, k \in \mathbb{N}$. Now we prove that for $k \neq l$ we have $\sigma_k \not\sim_{V-iso} \sigma_l$. For each natural number $n \geq 1$ we define the following algebra $\mathcal{S}_n = (S_n; f_n)$. Let $S_n := \{a_1, \cdots, a_{n+2}\}$ be an $(n+2)$-element set and let the binary operation $f_n : S_n \times S_n \longrightarrow S_n$, for each pair $a_i, a_j, i, j \in \{1, \cdots, n+2\}$ be defined as follows

$$f_n(a_i, a_j) = \begin{cases} a_i & \text{if} \quad i = j \\ a_{i+1} & \text{for} \quad 1 \leq j < n \text{ and } j+1 \leq i \leq n, \\ a_{j+1} & \text{for} \quad 1 \leq i < n \text{ and } i+1 \leq j \leq n, \\ a_{n+2} & \text{otherwise.} \end{cases}$$

Since f_n is idempotent $\mathcal{S}_n$ belongs to the variety V. For $1 \leq k < l \in \mathbb{N}$ we want to show that $\sigma_k(\mathcal{S}_l) \not\cong \sigma_l(\mathcal{S}_l)$. Let $1 \leq i < l$ and $i+1 \leq j$, so that we are considering only the last two cases in the general definition for f_l. Then we prove by induction on r that

$$\sigma_r(f)^{S_l}(a_i, a_j) = \begin{cases} a_{l+2} & \text{if} \quad r+j > l+1 \\ a_{r+j} & \text{if} \quad r+j \leq l+1. \end{cases} \qquad (*)$$

For $r = 1$ we have

$$\sigma_1(f)^{S_l}(a_i, a_j) = f_l(a_i, a_j) = \begin{cases} a_{l+2} & \text{if} \quad j > l \text{ , i.e. } r+j > l+1 \\ a_{j+1} & \text{if} \quad j \leq l \text{ , i.e. } r+j \leq l+1. \end{cases}$$

Inductively, we assume now that

$$\sigma_p(f)^{S_l}(a_i, a_j) = \begin{cases} a_{l+2} & \text{if} \quad p+j > l+1 \\ a_{p+j} & \text{if} \quad p+j \leq l+1. \end{cases}$$

Then we have
$\sigma_{p+1}(f)^{S_l}(a_i, a_j) = f_l(\sigma_p(f)^{S_l}(a_i, a_j), a_j)$
$= \begin{cases} f_l(a_{l+2}, a_j) & \text{if} \quad p+j > l+1 \\ f_l(a_{p+j}, a_j) & \text{if} \quad p+j \leq l+1. \end{cases}$

Since $p+j > l+1$ implies $p+1+j > l+1$, we get $f_l(a_{l+2}, a_j) = a_{l+2}$. Moreover, we have

$$f_l(a_{p+j}, a_j) = \begin{cases} a_{l+2} & \text{if} \quad p+j > l \text{ , i.e. } p+1+j > l+1 \\ a_{p+1+j} & \text{if} \quad p+j \leq l \text{ and } p+1+j \leq l+1. \end{cases}$$

This proves $(*)$ and for $i, j \in \{1, \cdots, l+2\}$ there holds

$$\sigma_l(f)^{S_l}(a_i, a_j) = \begin{cases} a_{l+2} & \text{if} \quad i \neq j \\ a_i & \text{if} \quad i = j. \end{cases}$$

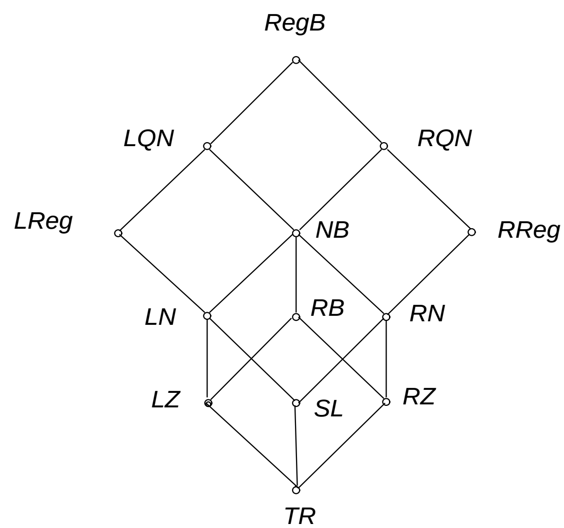
But for $k < l$ we have $a_{l+2} \neq a_{k+2} = \sigma_k(f)^{S_l}(a_1, a_2)$. This prove $\sigma_k(\mathcal{S}_l) \not\cong \sigma_l(\mathcal{S}_l)$ and thus $\sigma_k \not\sim_{V-iso} \sigma_l$. Because of Corollary 3.11 we have $isd_p(V) = \aleph_0$. ■

Since $\sim_V$ is a subrelation of $\sim_{V-iso}$ we have

Corollary 3.13 *For $V = Mod\{f(x,x) = x\}$ we have $d_p(V) = \aleph_0$.*

In the next section we consider varieties of bands.

4 The Degree of Proper Hypersubstitutions for Varieties of Bands and Medial Semigroups Bands are idempotent semigroups. Let B be the variety of all bands. There are a countably infinite number of subvarieties of B, and the structure of the lattice they form has been completely determined ([2], [8], [9]). The diagram of the lattice of all varieties of bands shown below is due to Gerhard and Petrich ([10]). Each of these varieties is defined by the associative and idempotent laws, plus one additional identity. We will be particularly interested in several of these varieties listed here and shown by the diagram



$TR = Mod\{x_1 \approx x_2\}$,
$LZ = Mod\{x_1x_2 \approx x_1\}$,
$RZ = Mod\{x_1x_2 \approx x_2\}$,
$SL = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2 \approx x_2x_1\}$,
$RB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx x_1x_3, {x_1}^2 \approx x_1\}$,
$NB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3x_4 \approx x_1x_3x_2x_4\}$,
$RegB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_1x_3x_1$
$\approx x_1x_2x_3x_1\}$,
$LN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3 \approx x_1x_3x_2\}$,
$RN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3 \approx x_2x_1x_3\}$,
$LReg = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2 \approx x_1x_2x_1\}$,
$RReg = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2 \approx x_2x_1x_2\}$,
$LQN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3 \approx x_1x_2x_1x_3\}$,
$RQN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3 \approx x_1x_3x_2x_3\}$.

First of all we show that every variety V of bands satisfies $\sim_V = \sim_{V-iso}$.

Proposition 4.1 *For each variety of bands we have* $\sim_V = \sim_{V-iso}$.

Proof. We have to show that $\sim_{V-iso} \subseteq \sim_V$. We denote by $\mathcal{S}_1(\mathcal{S}_2)$ a two-element left-zero (right-zero) semigroup and by $\mathcal{S}_3$ a two-element semilattice. We denote the hypersubstitution by σ_t which maps our binary operation symbol to the term t. If B is the variety of all bands, then $Hyp(2)/{\sim_B} = \{[\sigma_{x_1}]_{\sim_B}, [\sigma_{x_2}]_{\sim_B}, [\sigma_{x_1x_2}]_{\sim_B}, [\sigma_{x_2x_1}]_{\sim_B}, [\sigma_{x_1x_2x_1}]_{\sim_B}, [\sigma_{x_2x_1x_2}]_{\sim_B}\}$. The variety NB is the least variety of bands which contains $\mathcal{S}_1$ and $\mathcal{S}_2$ and $\mathcal{S}_3$. Therefore, for each variety V of bands we have either $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \in V$ (if $NB \subseteq V$) or $V \in \{TR, LZ, RZ, SL, RB, LN, RN, LReg, RReg\}$ (if $NB \nsubseteq V$). The following isomorphisms are easy to check:

$$\begin{array}{lll}
\sigma_{x_1}[\mathcal{S}_1] \cong \mathcal{S}_1 & \sigma_{x_1}[\mathcal{S}_2] \cong \mathcal{S}_1 & \sigma_{x_1}[\mathcal{S}_3] \cong \mathcal{S}_1 \\
\sigma_{x_2}[\mathcal{S}_1] \cong \mathcal{S}_2 & \sigma_{x_2}[\mathcal{S}_2] \cong \mathcal{S}_2 & \sigma_{x_2}[\mathcal{S}_3] \cong \mathcal{S}_2 \\
\sigma_{x_1x_2}[\mathcal{S}_1] \cong \mathcal{S}_1 & \sigma_{x_1x_2}[\mathcal{S}_2] \cong \mathcal{S}_2 & \sigma_{x_1x_2}[\mathcal{S}_3] \cong \mathcal{S}_3 \\
\sigma_{x_2x_1}[\mathcal{S}_1] \cong \mathcal{S}_2 & \sigma_{x_2x_1}[\mathcal{S}_2] \cong \mathcal{S}_1 & \sigma_{x_2x_1}[\mathcal{S}_3] \cong \mathcal{S}_3 \\
\sigma_{x_1x_2x_1}[\mathcal{S}_1] \cong \mathcal{S}_1 & \sigma_{x_1x_2x_1}[\mathcal{S}_2] \cong \mathcal{S}_1 & \sigma_{x_1x_2x_1}[\mathcal{S}_3] \cong \mathcal{S}_3 \\
\sigma_{x_2x_1x_2}[\mathcal{S}_1] \cong \mathcal{S}_2 & \sigma_{x_2x_1x_2}[\mathcal{S}_2] \cong \mathcal{S}_2 & \sigma_{x_2x_1x_2}[\mathcal{S}_3] \cong \mathcal{S}_3.
\end{array}$$

Using this list of isomorphisms it is easy to check that for any two hypersubstitutions $\sigma_1, \sigma_2 \in \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}, \sigma_{x_1x_2x_1}, \sigma_{x_2x_1x_2}\}$ with $\sigma_1 \not\sim_V \sigma_2$ there is an $i \in \{1,2,3\}$ such that $\sigma_1[\mathcal{S}_i] \ncong \sigma_2[\mathcal{S}_i]$, i.e. with $\sigma_1 \not\sim_{V-iso} \sigma_2$. Let $\sigma_1, \sigma_2 \in Hyp(2)$ with $\sigma_1 \sim_{V-iso} \sigma_2$. Then σ_1, σ_2 belong to one of the six classes with respect to $\sim_V$, i.e. there are $\sigma_1', \sigma_2' \in \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}, \sigma_{x_1x_2x_1}, \sigma_{x_2x_1x_2}\}$ with $\sigma_1 \sim_V \sigma_1'$ and $\sigma_2 \sim_V \sigma_2'$. From $\sim_V \subseteq \sim_{V-iso}$ we obtain $\sigma_1 \sim_{V-iso} \sigma_1'$ and $\sigma_2 \sim_{V-iso} \sigma_2'$ and thus $\sigma_1' \sim_{V-iso} \sigma_2'$. By the previous considerations we have $\sigma_1' \sim_V \sigma_2'$ and then also $\sigma_1 \sim_V \sigma_2$. This shows $\sim_{V-iso} \subseteq \sim_V$. ■

Now we determine $d_p(V)$ (and thus $isd_p(V)$) for each variety V of bands. Since the 2-generated free algebra over the variety of bands contains 6 elements, for each variety V of bands we have $d_p(V) \leq 6$. In the following theorem we need the concept of a dualsolid variety of type $\tau = (2)$. The variety V of type $\tau = (2)$ is dualsolid if $\hat{\sigma}_{x_2x_1}[s] \approx \hat{\sigma}_{x_2x_1}[t] \in IdV$ for every identity $s \approx t$ satisfied in V.

Theorem 4.2 *Let V be a variety of bands. Then*
$d_p(V) = 1$ *iff* $V \in \{TR, LZ, RZ, SL\}$,
$d_p(V) = 2$ *iff* $V \in \{LN, RN, LReg, RReg\}$,
$d_p(V) = 3$ *iff V is not dual solid and* $V \notin \{LZ, RZ,$
$LN, RN, LReg, RReg, LQN, RQN\}$,
$d_p(V) = 4$ *iff V is dual solid and either* $V \notin \{TR, SL, NB, RegB\}$ *or* $V \in \{LQN, RQN\}$,
$d_p(V) = 6$ *iff* $V \in \{NB, RegB\}$.

Proof. It is easy to see that $d_p(TR) = d_p(LZ) = d_p(RZ) = d_p(SL) = 1$. Further, $Hyp(2) = [\sigma_{x_1}]_{\sim_{LN}} \cup [\sigma_{x_2}]_{\sim_{LN}} \cup [\sigma_{x_1x_2}]_{\sim_{LN}} \cup [\sigma_{x_2x_1}]_{\sim_{LN}}$ since $x_1x_2x_1 \approx x_1x_2, x_2x_1x_2 \approx x_2x_1 \in IdLN$ and $x_1 \approx x_2, x_1 \approx x_1x_2, x_1 \approx x_2x_1, x_1x_2 \approx x_2x_1 \notin IdLN$. The hypersubstitutions σ_{x_1} and $\sigma_{x_1x_2}$ are LN-proper. If we apply $\hat{\sigma}_{x_2}$ to $x_1x_2x_3 \approx x_1x_3x_2$ we obtain $x_3 \approx x_2$ which is not satisfied in LN and applying $\hat{\sigma}_{x_2x_1}$ to $x_1x_2x_3 \approx x_1x_3x_2$ give $x_3x_2x_1 \approx x_2x_3x_1$ which is also not satisfied. Therefore $P(LN)/\sim_{LN|_{P(LN)}} = \{[\sigma_{x_1}]_{\sim_{LN}}, [\sigma_{x_1x_2}]_{\sim_{LN}}\}$ and $d_p(LN) = 2$. Similarly we show that $d_p(RN) = 2$. We show in a similar way that $d_p(LReg) = d_p(RReg) = 2$. From the properties of the basis of the identities in NB there follows that in NB there are only outermost and regular identities. It is easy to check that $Hyp(2) = [\sigma_{x_1}]_{\sim_{NB}} \cup [\sigma_{x_2}]_{\sim_{NB}} \cup [\sigma_{x_1x_2}]_{\sim_{NB}} \cup [\sigma_{x_2x_1}]_{\sim_{NB}} \cup [\sigma_{x_1x_2x_1}]_{\sim_{NB}} \cup [\sigma_{x_2x_1x_2}]_{\sim_{NB}}$ and that this is a partition of $Hyp(2)$. Since NB is solid, we get $d_p(NB) = 6$. In a similar way we prove that $d_p(RegB) = 6$. It is clear that $Hyp(2) = [\sigma_{x_1}]_{\sim_{RB}} \cup [\sigma_{x_2}]_{\sim_{RB}} \cup [\sigma_{x_1x_2}]_{\sim_{RB}} \cup [\sigma_{x_2x_1}]_{\sim_{RB}}$ and that there is no collapsing of these classes since all identities in RB are outermost. Since RB is solid, we get $P(RB) = Hyp(2)$ and this gives $d_p(RB) = 4$. Now we consider the varieties LQN and RQN. It is easy to check that $Hyp(2) = [\sigma_{x_1}]_{\sim_{LQN}} \cup [\sigma_{x_2}]_{\sim_{LQN}} \cup [\sigma_{x_1x_2}]_{\sim_{LQN}} \cup [\sigma_{x_2x_1}]_{\sim_{LQN}} \cup [\sigma_{x_1x_2x_1}]_{\sim_{LQN}} \cup [\sigma_{x_2x_1x_2}]_{\sim_{LQN}}$.

Since from the identity basis we can only derive regular and outermost identities $\sigma_{x_1}, \sigma_{x_2}$ are LQN-proper. The identity hypersubstitution is also LQN-proper. We show that $\sigma_{x_1x_2x_1}$ is LQN-proper. If we apply $\hat{\sigma}_{x_1x_2x_1}$ to the associative law or to $x_1x_2x_3 \approx x_1x_2x_1x_3$ we obtain $x_1x_2x_3x_2x_1 \approx x_1x_2x_1x_3x_1x_2x_1$ which is a consequence of $x_1x_2x_3 \approx x_1x_2x_1x_3 \in IdLQN$. If we apply $\hat{\sigma}_{x_1x_2x_1}$ to the idempotent law we get an identity which is satisfied in LQN. The hypersubstitution $\sigma_{x_2x_1x_2}$ is not LQN-proper since the application to $x_1x_2x_3 \approx x_1x_2x_1x_3$ gives $x_3x_2x_1x_2x_3 \approx x_3x_1x_2x_1x_2x_1x_3$. ¿From this identity we can derive the medial law which is not satisfied in LQN. Altogether we have $|P(LQN)/{\sim_{LQN|P(LQN)}}| = |\{[\sigma_{x_1}]_{\sim_{LQN}}, [\sigma_{x_2}]_{\sim_{LQN}}, [\sigma_{x_1x_2}]_{\sim_{LQN}}, [\sigma_{x_1x_2x_1}]_{\sim_{LQN}}\}| = 4 = d_p(LQN)$. In a similar way we show that $d_p(LQN) = 4$. Let now V be a dual solid variety different from TR, SL, RB, NB and $RegB$. Then we have $Hyp(2) = [\sigma_{x_1}]_{\sim_V} \cup [\sigma_{x_2}]_{\sim_V} \cup [\sigma_{x_1x_2}]_{\sim_V} \cup [\sigma_{x_2x_1}]_{\sim_V} \cup [\sigma_{x_1x_2x_1}]_{\sim_V} \cup [\sigma_{x_2x_1x_2}]_{\sim_V}$. Since V is dual solid, the hypersubstitutions $\sigma_{x_1x_2}$ and $\sigma_{x_2x_1}$ are V-proper. As a consequence of $V \neq TR, SL$ and since V is dual solid we have $\sigma_{x_1}, \sigma_{x_2} \in P(V)$. The application of $\sigma_{x_1x_2x_1}$ to the associative law provides $x_1x_2x_3x_2x_1 \approx x_1x_2x_1x_3x_1x_2x_1$. ¿From this equation we derive $x_1x_2x_3x_1 \approx x_1x_2x_1x_3x_1$ in the following way

$$\begin{aligned}
x_1x_2x_3x_1 &\approx x_1x_2x_3x_3x_2x_3x_1 \\
&\approx x_1x_2x_3x_1x_3x_1x_2x_3x_1 \\
&\approx x_1x_2x_3x_1x_3x_1x_2x_1x_3x_1 \\
&\approx x_1x_2x_1x_3x_1x_3x_1x_2x_1x_3x_1 \\
&\approx x_1x_2x_1x_3x_1x_2x_1x_3x_1 \\
&\approx x_1x_2x_1x_3x_1x_1x_2x_1x_3x_1 \\
&\approx x_1x_2x_1x_3x_1.
\end{aligned}$$

This shows $V \subseteq RegB$. But TR, SL, RB, NB and $RegB$ are the only dual solid subvarieties of $RegB$. Since V is different from these varieties we have $\sigma_{x_1x_2x_1} \notin P(V)$. The same argument shows $\sigma_{x_2x_1x_2} \notin P(V)$. Since $RB \subseteq V$ the set IdV of all identities satisfied in V consists only of outermost identities and this shows $|P(V)/{\sim_{V|P(V)}}| = |\{[\sigma_{x_1}]_{\sim_V}, [\sigma_{x_2}]_{\sim_V}, [\sigma_{x_1x_2}]_{\sim_V}, [\sigma_{x_2x_1}]_{\sim_V}\}| = 4$, *i.e.* $d_p(V) = 4$. Finally, if V is a not dual solid variety different from $LZ, RZ, LN, RN, LReg, RReg, LQN, RQN$, then $Hyp(2) = [\sigma_{x_1}]_{\sim_V} \cup [\sigma_{x_2}]_{\sim_V} \cup [\sigma_{x_1x_2}]_{\sim_V} \cup [\sigma_{x_2x_1}]_{\sim_V} \cup [\sigma_{x_1x_2x_1}]_{\sim_V} \cup [\sigma_{x_2x_1x_2}]_{\sim_V}$. We can prove that $\sigma_{x_2x_1}, \sigma_{x_1x_2x_1}, \sigma_{x_2x_1x_2} \notin P(V)$. Then $|P(V)/{\sim_{V|P(V)}}|$
$= |\{[\sigma_{x_1}]_{\sim_V}, [\sigma_{x_2}]_{\sim_V}, [\sigma_{x_1x_2}]_{\sim_V}\}| = 3$, i.e. $d_p(V) = 3$. Since there are no more varieties of bands, in each case we have also the opposite direction. ∎

A semigroup is called *medial* if the medial law $x_1x_2x_3x_4 \approx x_1x_3x_2x_4$ is satisfied as an identity. We consider varieties of medial semigroups satisfying the identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2$, i.e. all subvarieties of the variety $V_{big} := Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1x_2x_3x_4 \approx x_1x_3x_2x_4, x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2\}$. The two-generated free algebra over this variety consists exactly of the classes $[x_1]_{\sim_{V_{big}}}, [x_2]_{\sim_{V_{big}}}$,
$[x_1x_2]_{\sim_{V_{big}}}, [x_2x_1]_{\sim_{V_{big}}}, [x_1^2]_{\sim_{V_{big}}}, [x_2^2]_{\sim_{V_{big}}}, [x_1x_2x_1]_{\sim_{V_{big}}}, [x_2x_1x_2]_{\sim_{V_{big}}}$.
Therefore, for any $V \subseteq V_{big}$ we have $|Hyp(2)/{\sim_V}| \leq 8$ and then also $d_p(V) \leq 8$. If V is also a variety of bands, then $V \subseteq NB$ since NB is the greatest medial and idempotent variety of semigroups. In this case the degree of proper hypersubstitution is given by Theorem 4.2. Therefore we may assume that $V \not\subseteq NB$. Therefore $[\sigma_{x_1}]_{\sim_V} \neq [\sigma_{x_1x_2}]_{\sim_V}, [\sigma_{x_1}]_{\sim_V} \neq [\sigma_{x_2x_1}]_{\sim_V}, [\sigma_{x_1}]_{\sim_V} \neq [\sigma_{x_1^2}]_{\sim_V}, [\sigma_{x_1}]_{\sim_V} \neq [\sigma_{x_2}]_{\sim_V}$. The identities of V_{big} have a particular form. An equation $s \approx t$ of terms of an arbitrary type τ is called *normal* if either both terms s and t are equal to the same variable or none of them is a variable. A variety in

which all identities are normal is called a *normal variety*. The concept of normalization was first studied by Mel'nik([12]) and Płonka ([13]) and later by Graczyńska ([11]), Chajda ([3] and Denecke/Wismath ([7]).

Let $N(\tau)$ be the set of all normal equations of type τ. If V is a variety of type τ, we may consider all normal identities valid in V, i.e. $IdV \cap N(\tau)$ and the variety $N(V) := Mod\{IdV \cap N(\tau)\}$. The variety $N(V)$ is called the *normalization* of V. By definition of $N(V)$ we have $N(V) = V \vee ModN(\tau)$. Here $ModN(\tau)$ is the least normal variety of type τ. It is easy to see that for type $\tau = (2)$ the variety $N(\tau)$ agrees with the variety Z of all zero-semigroups. Therefore the normalization of any variety V of semigroups is given by $N(V) = V \vee Z$. Let $\mathcal{L}(V)$ be the lattice of all subvarieties of the non-normal variety V. E. Graczyńska proved in [11] that the lattice $\mathcal{L}(N(V))$ is isomorphic to the direct product of the lattice $\mathcal{L}(V)$ and a two-element chain. Our first observation is:

Proposition 4.3 *V_{big} is the normalization of the variety NB, i.e. $V_{big} = NB \vee Z$.*

Proof. Since any identity in V_{big} is normal, outermost and regular we have $NB \vee Z \subseteq V_{big}$. It is easy to see that every normal, outermost and regular equation is an identity in V_{big}. Therefore, $Id(NB \vee Z) \subseteq IdV_{big}$, i.e $NB \vee Z \supseteq V_{big}$. ∎

By Graczyńska's result ([11]) the subvariety lattice $\mathcal{L}(V_{big})$ is given by the direct product of $\mathcal{L}(NB)$ and $\{TR, Z\}$. Since the subvariety lattice of NB is completely known, $\mathcal{L}(V_{big})$ consists of the following varieties. It is easy to calculate the degree of proper hypersubstitutions for every subvariety of V_{big}.

variety	defining system of identities	degree of proper hyper-substitution
$Z \vee TR$	$= Z = Mod\{xy \approx zt\}$	1
$Z \vee LZ$	$= Mod\{ass., xy \approx xz\}$	2
$Z \vee RZ$	$= Mod\{ass., xy \approx zy\}$	2
$Z \vee SL$	$= Mod\{ass., xy \approx yx, xy \approx x^2y \approx xy^2\}$	1
$Z \vee RB$	$= Mod\{ass., xy \approx xzy\}$	6
$Z \vee LN$	$= Mod\{ass., zxy \approx zyx, xy \approx x^2y \approx xy^2\}$	3
$Z \vee RN$	$= Mod\{ass., xyz \approx yxz, xy \approx x^2y \approx xy^2\}$	3
$Z \vee NB$	$= V_{big}$	8.

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Klaus Denecke, Jörg Koppitz, Rattana Srithus,
Universität Potsdam,
Institute of Mathematics,
Am Neuen Palais,
14415 Potsdam, Germany.
e-mail: kdenecke@rz.uni-potsdam.de