THE BEST CONSTANT OF L^P SOBOLEV INEQUALITY CORRESPONDING TO THE PERIODIC BOUNDARY VALUE PROBLEM FOR $(-1)^M(D/DX)^{2M}$

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ABSTRACT. The best constant of the Sobolev inequality

$$\| u(x) \|_\infty \le C \left\| u^{(M)}(x) \right\|_p \qquad (p \ge 1)$$

is given by L^q-norm $(1/p+1/q=1)$ of the well-known Bernoulli polynomials, which is a Green fucntion of a certain periodic boundary value problem for M-th order linear ordinary differential equation. The special case of $p=2$ is treated completely in [1].

1 Conclusion Throughout in this paper, we assume that $p,\ q \ge 1, \quad 1/p + 1/q = 1$. For $M = 1, 2, 3, \cdots$ we consider a sequence of function spaces

$$H_M = \Big\{ u(x) \,\Big|\, u^{(M)}(x) = (d/dx)^M u(x) \in L^p(0,1),$$
$$u^{(i)}(1) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1), \quad \int_0^1 u(x)\,dx = 0 \Big\} \tag{1.1}$$

and L^p Sobolev functionals

$$S_M(u) = \| u(x) \|_\infty \Big/ \left\| u^{(M)}(x) \right\|_p \qquad (u(x) \in H_M) \tag{1.2}$$

where $\| \cdot \|_p$ and $\| \cdot \|_\infty$ are the usual L^p and L^∞ norms. That is to say

$$\| u(x) \|_p = \left(\int_0^1 |\, u(x) \,|^p \, dx \right)^{1/p}$$

and

$$\| u(x) \|_\infty = \operatorname*{ess.sup}_{0 \le x \le 1} |\, u(x) \,|$$

To state our conclusion, we need Bernoulli polynomials $b_n(x)$ defined by

$$\begin{cases} b_0(x) = 1 \\ b_n'(x) = b_{n-1}(x), \qquad \displaystyle\int_0^1 b_n(x)\,dx = 0 \qquad (n = 1, 2, 3, \cdots) \end{cases}$$

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They are also defined by the following generating function.

$$\frac{e^{xt}}{t^{-1}(e^t-1)} = \sum_{j=0}^{\infty} b_j(x)\,t^j \qquad (|\,t\,|<2\pi) \tag{1.3}$$

We have

$$\begin{aligned}
b_0(x) &= 1, \qquad b_1(x) = x-\frac{1}{2}, \qquad b_2(x) = \frac{1}{2}\,x^2-\frac{1}{2}\,x+\frac{1}{12}\\
b_3(x) &= \frac{1}{6}\,x^3-\frac{1}{4}\,x^2+\frac{1}{12}\,x, \qquad b_4(x) = \frac{1}{24}\,x^4-\frac{1}{12}\,x^3+\frac{1}{24}\,x^2-\frac{1}{720}\\
b_5(x) &= \frac{1}{120}\,x^5-\frac{1}{48}\,x^4+\frac{1}{72}\,x^3-\frac{1}{720}\,x\\
b_6(x) &= \frac{1}{720}\,x^6-\frac{1}{240}\,x^5+\frac{1}{288}\,x^4-\frac{1}{1440}\,x^2+\frac{1}{30240}\\
b_7(x) &= \frac{1}{5040}\,x^7-\frac{1}{1440}\,x^6+\frac{1}{1440}\,x^5-\frac{1}{4320}\,x^3+\frac{1}{30240}\,x\\
b_8(x) &= \frac{1}{40320}\,x^8-\frac{1}{10080}\,x^7+\frac{1}{8640}\,x^6-\frac{1}{17280}\,x^4+\frac{1}{60480}\,x^2-\frac{1}{1209600}\\
&\cdots
\end{aligned}$$

The main theorem obtained in this paper is as follows.

Theorem 1.1 *The best constant of Sobolev inequality or the supremum of Sobolev functional*

$$C(M) = \sup_{\substack{u\in H_M\\ u\neq 0}} S_M(u) \tag{1.4}$$

is given by the following formula.
(1) *If* $M=2m-1\ (m=1,2,3,\cdots)$ *then we have*

$$C(M) = \|\,b_M(x)\,\|_q = \left(\int_0^1 |\,b_M(x)\,|^q dx\right)^{1/q} \tag{1.5}$$

where the supremum is attained if we put

$$u(x) = \int_0^1 (-1)^{m-1}\mathrm{sgn}(x-y)\,b_M(\,|x-y|\,)\,f(y)\,dy \qquad (0\le x\le 1) \tag{1.6}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1 & (x\ge 0)\\ -1 & (x<0)\end{cases} \tag{1.7}$$

$$f(x) = (-1)^m\,\mathrm{sgn}(\,b_M(x)\,)\,|\,b_M(x)\,|^{q-1} \qquad (0\le x\le 1) \tag{1.8}$$

(2) *If* $M=2m\ (m=1,2,3,\cdots)$ *then we have*

$$C(M) = \min_{0\le\alpha\le 1/2} \|\,b_M(\alpha;x)\,\|_q = \|\,b_M(\alpha_0;x)\,\|_q \tag{1.9}$$

where

$$b_M(\alpha;x) = b_M(x) - b_M(\alpha) \tag{1.10}$$

In (1.9), α_0 *is the unique solution to the equation*

$$\int_0^\alpha \Big((-1)^{m-1}\, b_M(\alpha;x) \Big)^{q-1} dx \;-\; \int_\alpha^{1/2} \Big((-1)^{m}\, b_M(\alpha;x) \Big)^{q-1} dx \;=\; 0 \tag{1.11}$$

in the interval $0 < \alpha < 1/2$.

The supremum of $S_M(u)$ *is attained for*

$$u(x) \;=\; \int_0^1 (-1)^{m-1} b_M(\,|x-y|\,)\, f(y)\, dy \qquad (0 \le x \le 1) \tag{1.12}$$

where

$$f(x) \;=\; (-1)^{m-1}\, \mathrm{sgn}(\, b_M(\alpha_0;x)\,)\, |\, b_M(\alpha_0;x)\,|^{q-1} \qquad (0 \le x \le 1) \tag{1.13}$$

2 Boundary value problems

Concerning the solvability, uniqueness and existence of the classical solution to the boundary value problem

BVP

$$\begin{cases} (-1)^{[\,(M+1)/2\,]}\, u^{(M)} \;=\; f(x) & (0 < x < 1) \qquad (2.1) \\ u^{(i)}(1) \;-\; u^{(i)}(0) \;=\; 0 & (0 \le i \le M-1) \qquad (2.2) \\ \displaystyle\int_0^1 u(x)\, dx \;=\; 0 & \qquad (2.3) \end{cases}$$

we have the following conclusion.

Theorem 2.1 *For any bounded continuous function* $f(x)$ *on an interval* $0 < x < 1$ *, satisfying the solvability condition*

$$\int_0^1 f(y)\, dy \;=\; 0 \tag{2.4}$$

BVP *has one and only one classical solution* $u(x)$ *given by*

$$u(x) \;=\; \int_0^1 G(M;x,y)\, f(y)\, dy \qquad (0 \le x \le 1) \tag{2.5}$$

where Green function $G(M;x,y)$ *is given by*

$$G(x,y) \;=\; G(M;x,y) \;=\; (-1)^{[\,(M-1)/2\,]} \Big(\mathrm{sgn}(x-y) \Big)^{M} b_M(\,|x-y|\,) \tag{2.6}$$

that is

$$G(2M;x,y) \;=\; (-1)^{M-1}\, b_{2M}(\,|x-y|\,) \tag{2.7}$$

$$G(2M-1;x,y) \;=\; (-1)^{M-1}\, \mathrm{sgn}(x-y)\, b_{2M-1}(\,|x-y|\,) \qquad (0 < x,y < 1) \tag{2.8}$$

Theorem 2.1 follows at once from the following Lemma.

Lemma 2.1 *Grren function $G(x,y)$ satisfies the following properties.*

(1) $$G(y,x) = (-1)^M G(x,y) \qquad (0<x,y<1) \tag{2.9}$$

(2) $$\partial_x^M G(x,y) = (-1)^{[(M-1)/2]} \qquad (0<x,y<1, \quad x\neq y) \tag{2.10}$$

(3) $$\partial_x^i G(x,y)\Big|_{x=0,1} = (-1)^{[(M-1)/2]+M+i}\, b_{M-i}(y) \qquad (0<y<1, \quad 0\le i\le M-1) \tag{2.11}$$

(4) $$\partial_x^i G(x,y)\Big|_{y=x-0} - \partial_x^i G(x,y)\Big|_{y=x+0} = \begin{cases} 0 & (0\le i\le M-2) \\ (-1)^{[(M+1)/2]} & (i=M-1) \end{cases} \qquad (0<x<1) \tag{2.12}$$

(5) $$\partial_x^i G(x,y)\Big|_{x=y+0} - \partial_x^i G(x,y)\Big|_{x=y-0} = \begin{cases} 0 & (0\le i\le M-2) \\ (-1)^{[(M+1)/2]} & (i=M-1) \end{cases} \qquad (0<y<1) \tag{2.13}$$

(6) $$\int_0^1 G(x,y)\,dx = 0 \tag{2.14}$$

Before proof of Lemma 2.1, we show the following Lemma.

Lemma 2.2

(1) $$b_j(1-x) = (-1)^j b_j(x) \qquad (j=0,1,2,\cdots) \tag{2.15}$$

(2) $$b_{2j+1}(0) = -1/2 \quad (j=0), \qquad 0 \quad (j=1,2,3,\cdots) \tag{2.16}$$

(3) $$\big(1-(-1)^j\big)\, b_j(0) = 0 \ (j=0), \quad -1\ (j=1), \quad 0\ (j=2,3,4,\cdots) \tag{2.17}$$

Proof of Lemma 2.2 From (1.3), we have

$$\sum_{j=0}^{\infty} b_j(1-x)\,t^j = \frac{e^{(1-x)t}}{t^{-1}(e^t-1)} = \frac{e^{x(-t)}}{(-t)^{-1}(e^{-t}-1)} = \sum_{j=0}^{\infty}(-1)^j b_j(x)\,t^j$$

Thus we have (1). Putting $x=0$ in (2.15), we have

$$b_{2j+1}(1) = -\,b_{2j+1}(0) \quad (j=0,1,2,\cdots) \tag{2.18}$$

From (1.3), we have

$$\sum_{j=0}^{\infty}\Big(b_j(1)-b_j(0)\Big)\,t^j = t$$

that is

$$b_j(1) - b_j(0) \;=\; 0 \;(j=0), \quad 1 \;(j=1), \quad 0 \;(j=2,3,4,\cdots) \tag{2.19}$$

Solving (2.18) and (2.19), we have (2). (3) follows from (2). Lemma 2.2 is proved. ■
Proof of Lemma 2.1 (1) is obious. Differentiating (2.6) i times with respect to x, we have

$$\partial_x^i G(x,y) \;=\; (-1)^{[\,(M-1)/2\,]} \Big(\mathrm{sgn}(x-y)\Big)^{M+i} b_{M-i}(\,|x-y|\,)$$
$$(0<x,y<1,\; x\neq y,\; 0\le i\le M) \tag{2.20}$$

(2) follows (2.20). Substituting $x=1,0$ into (2.20), we have

$$\partial_x^i G(x,y)\Big|_{x=1} \;=\; (-1)^{[\,(M-1)/2\,]}\, b_{M-i}(1-y) \;=\; (-1)^{[\,(M-1)/2\,]+M-i}\, b_{M-i}(y)$$
$$\partial_x^i G(x,y)\Big|_{x=0} \;=\; (-1)^{[\,(M-1)/2\,]+M+i}\, b_{M-i}(y) \qquad (0<y<1,\; 0\le i\le M-1)$$

So we have (3). From (2.20), we have

$$\partial_x^i G(x,y)\Big|_{y=x-0} - \partial_x^i G(x,y)\Big|_{y=x+0} \;=\; (-1)^{[\,(M-1)/2\,]}(1 \,-\, (-1)^{M+i})\, b_{M-i}(0) \;=$$
$$(-1)^{[\,(M-1)/2\,]}(1 \,-\, (-1)^{M-i})\, b_{M-i}(0) \qquad (0<x<1, \quad 0\le i\le M-1)$$

By Lemma 2.2 (3), we have (4). (5) is equivalent to (4).

$$\int_0^1 G(x,y)\,dx \;=\; \int_0^1 (-1)^{[\,(M-1)/2\,]} \Big(\mathrm{sgn}(x-y)\Big)^{M} b_M(\,|x-y|\,)\,dx \;=$$
$$(-1)^{[\,(M-1)/2\,]} \left[\int_0^y (-1)^M\, b_M(y-x)\,dx \,+\, \int_y^1 b_M(x-y)\,dx\right] \;=$$
$$(-1)^{[\,(M-1)/2\,]} \Big[-\, (-1)^M\, b_{M+1}(0) \,+\, (-1)^M\, b_{M+1}(y) \,+\, b_{M+1}(1-y) \,-\, b_{M+1}(0)\Big] \;=$$
$$(-1)^{[\,(M+1)/2\,]}\left(1-(-1)^{M+1}\right) b_{M+1}(0) \;=\; 0$$

which proves (6). This completes the proof of Lemma 2.1. ■

3 Reproducing kernel

In this section, we show that the Green function

$$G(x,y) \;=\; G(2M;x,y) \qquad \text{for} \quad M=1,2,3,\cdots \tag{3.1}$$

is a reproducing kernel for H_M and its inner product

$$(u,v)_M \;=\; \int_0^1 u^{(M)}(x)\,\overline{v}^{(M)}(x)\,dx$$

For any two functions $u=u(x)$ and $v=v(x)$, we have the following identity

$$u^{(M)}\,v^{(M)} \;=\; \left(\sum_{j=0}^{M-1} (-1)^j\, u^{(M-1-j)}\, v^{(M+j)}\right)' \,+\, (-1)^M\, u\, v^{(2M)} \tag{3.2}$$

Putting $v(x)=G(x,y)$ with y arbitrarily fixed in $0<y<1$ and integrating both sides of the above equality on intervals $0<x<y$ and $y<x<1$, we have the following conclusion.

Lemma 3.1 (1) *If* $u^{(M)}(x) \in L^p(0,1)$ *then we have*

$$\int_0^1 u^{(M)}(x)\,\partial_x^M G(x,y)\,dx \;=$$
$$\int_0^1 u^{(M)}(x)\,(-1)^{M-1}\Big(\mathrm{sgn}(x-y)\Big)^M b_M(\,|x-y|\,)\,dx \;=$$
$$u(y) \;-\; \int_0^1 u(x)\,dx \;-\; \sum_{j=0}^{M-1}\Big(u^{(M-1-j)}(1) \;-\; u^{(M-1-j)}(0)\Big)\,b_{M-j}(y)$$
$$(0 \le y \le 1) \tag{3.3}$$

(2) *If* $u(x) \in H_M$ *then we have*

$$u(y) \;=\; \int_0^1 u^{(M)}(x)\,\partial_x^M G(x,y)\,dx \;=$$
$$\int_0^1 u^{(M)}(x)\,(-1)^{M-1}\Big(\mathrm{sgn}(x-y)\Big)^M b_M(\,|x-y|\,)\,dx \qquad (0 \le y \le 1) \tag{3.4}$$

4 The case of odd M Proof of Theorem 1.1 (1) In this section, we treat the case $M = 2m-1$ $(m = 1, 2, 3, \cdots)$. Applying Hölder inequality to (3.4) we have

$$|\,u(y)\,| \;\le\; \|\,b_M(\,|x-y|\,)\,\|_q \,\Big\|\,u^{(M)}(x)\,\Big\|_p \qquad (0 \le y \le 1) \tag{4.1}$$

Owing to the property $b_M(1-x) \;=\; -\,b_M(x)$ $(0 < x < 1)$, we have

$$\|\,b_M(\,|x-y|\,)\,\|_q \;=\; \|\,b_M(x)\,\|_q$$

Hence (4.1) is rewritten as follows.

$$\sup_{0 \le y \le 1} |\,u(y)\,| \;\le\; \|\,b_M(x)\,\|_q \,\Big\|\,u^{(M)}(x)\,\Big\|_p \tag{4.2}$$

This shows that

$$C(M) \;=\; \sup_{\substack{u \in H_M \\ u \not\equiv 0}} S_M(u) \;\le\; \|\,b_M(x)\,\|_q$$

Next we construct $u(x)$ which satisfies

$$S_M(u) \;=\; \|\,b_M(x)\,\|_q$$

The function

$$f(x) \;=\; (-1)^m\,\mathrm{sgn}(\,b_M(x)\,)\,|\,b_M(x)\,|^{q-1} \qquad (0 \le x \le 1) \tag{4.3}$$

satisfies $f(1-x) \;=\; -\,f(x)$ $(0 \le x \le 1)$ and

$$\int_0^1 f(y)\,dy \;=\; 0$$

From Theorem 2.1, the solution $u(x)$ to the boundary value problem

$$\begin{cases} (-1)^m u^{(M)} = f(x) & (0 < x < 1) \\ u^{(i)}(1) - u^{(i)}(0) = 0 & (0 \le i \le M-1) \\ \displaystyle\int_0^1 u(x)\,dx = 0 \end{cases}$$

is given by

$$u(x) = \int_0^1 (-1)^{m-1} \operatorname{sgn}(x-y)\, b_M(|x-y|)\, f(y)\, dy \qquad (0 \le x \le 1) \tag{4.4}$$

Note that $u(x) \in H_M$. Interchanging x and y, we have

$$u(y) = \int_0^1 f(x)\,(-1)^m \operatorname{sgn}(x-y)\, b_M(|x-y|)\, dx \qquad (0 \le y \le 1)$$

Putting $y = 0$, we have

$$u(0) = \int_0^1 f(x)\,(-1)^m\, b_M(x)\, dx$$

Substitution of (4.3) into the above equality gives

$$u(0) = \| b_M(x) \|_q^q = \| b_M(x) \|_q \| b_M(x) \|_q^{q/p} = \| b_M(x) \|_q \| f(x) \|_p = \| b_M(x) \|_q \left\| u^{(M)}(x) \right\|_p$$

Combining this with (4.2) we have

$$\| b_M(x) \|_q \left\| u^{(M)}(x) \right\|_p = u(0) \le \sup_{0 \le y \le 1} |u(y)| \le \| b_M(x) \|_q \left\| u^{(M)}(x) \right\|_p \tag{4.5}$$

This shows that $u(x)$ defined by (4.4) satisfies

$$\sup_{0 \le y \le 1} |u(y)| = \| b_M(x) \|_q \left\| u^{(M)}(x) \right\|_p \tag{4.6}$$

which completes the proof of Theorem 1.1 (1). ■

5 The case of even M In this section, we treat the case of $M = 2m$ ($m = 1, 2, 3, \cdots$). We first investigate the function

$$b_M(\alpha; x) = b_M(x) - b_M(\alpha) \qquad (0 \le x \le 1, \quad 0 < \alpha < 1/2) \tag{5.1}$$

which we have introduced in Theorem 1.1.

Lemma 5.1 *The function $b_M(\alpha; x)$ satisfies the following properties.*

(1) $$b_M(\alpha; 1-x) = b_M(\alpha; x) \qquad (0 \le x \le 1) \tag{5.2}$$

(2) $$(-1)^{m-1} b_M(\alpha; x) \begin{cases} > 0 & (0 < x < \alpha) \\ = 0 & (x = \alpha) \\ < 0 & (\alpha < x < 1-\alpha) \\ = 0 & (x = 1-\alpha) \\ > 0 & (1-\alpha < x < 1) \end{cases} \tag{5.3}$$

This Lemma follows immediately from the following Lemma.

Lemma 5.2

(1) $b_M(1-x) = b_M(x) \qquad (0 \le x \le 1)$ (5.4)

(2) $b_M(0) = (-1)^{m-1}\dfrac{B_m}{M!}$ (5.5)

where B_m are Bernoulli numbers. For example

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30},$$
$$B_5 = \frac{5}{66}, \quad B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \quad B_8 = \frac{3617}{510}, \quad \cdots$$

(3) $(-1)^{m-1}b'_M(x) = (-1)^{m-1}b_{M-1}(x) < 0 \qquad (0 < x < 1/2)$ (5.6)

Proof of Lemma 5.2 The proof of (1) and (2) are easy, we omit them. For $f(x) = -b_{M-3}(x)$ the function $u(x) = b_{M-1}(x)$ satisfies

$$\begin{cases} -u'' = f(x) & (0 < x < 1/2) \\ u(0) = u(1/2) = 0 \end{cases}$$

Using positive-valued Green function

$$x \wedge y - 2xy = \min\{x, y\} - 2xy$$

we have the expression

$$(-1)^m\, b_{M-1}(x) = \int_0^{1/2} \Big(x \wedge y - 2xy \Big) (-1)^{m-1}\, b_{M-3}(y)\, dy > 0$$
$$(0 < x < 1/2, \quad m = 2, 3, 4, \cdots) \tag{5.7}$$

Starting from the fact

$$-b_1(x) > 0 \qquad (0 < x < 1/2)$$

we can show the following inequalities recurrently.

$$(-1)^m\, b_{M-1}(x) > 0 \qquad (0 < x < 1/2)$$

This completes the proof of Lemma 5.2. ■

Proof of Theorem 1.1 (2) For $u(x) \in H_M$ we have $\int_0^1 u^{(M)}(x)\, dx = 0$. From (3.4) we have

$$u(y) = -\int_0^1 u^{(M)}(x)\, b_M(\alpha; |x-y|)\, dx \qquad (0 \le y \le 1, \quad 0 < \alpha < 1/2) \tag{5.8}$$

Applying Hölder inequality, we have

$$|\, u(y)\, | \le \|\, b_M(\alpha; |x-y|)\, \|_q \, \Big\| u^{(M)}(x) \Big\|_p \qquad (0 \le y \le 1, \quad 0 < \alpha < 1/2) \tag{5.9}$$

Due to the property $b_M(\alpha;1-x) = b_M(\alpha;x) \quad (0<x<1,\ 0<\alpha<1/2)$, we have

$$\| b_M(\alpha;|x-y|)\|_q = \| b_M(\alpha;x)\|_q$$

and therefore

$$\sup_{0\le y\le 1} |u(y)| \le \| b_M(\alpha;x)\|_q \left\| u^{(M)}(x) \right\|_p \tag{5.10}$$

At first we investigate the behaviour of

$$g(\alpha) = \| b_M(\alpha;x)\|_q^q = \int_0^1 | b_M(\alpha;x)|^q \, dx \tag{5.11}$$

in an interval $0<\alpha<1/2$. It easy to see

$$g(0) = \int_0^1 \Big((-1)^m \, b_M(0;x) \Big)^q dx > 0$$
$$g(1/2) = \int_0^1 \Big((-1)^{m-1} \, b_M(1/2;x) \Big)^q dx > 0$$

In general

$$g(\alpha) = \int_0^\alpha \Big((-1)^{m-1} \, b_M(\alpha;x) \Big)^q dx + \int_\alpha^{1-\alpha} \Big((-1)^m \, b_M(\alpha;x) \Big)^q dx +$$
$$\int_{1-\alpha}^1 \Big((-1)^{m-1} \, b_M(\alpha;x) \Big)^q dx =$$
$$2\left[\int_0^\alpha \Big((-1)^{m-1} \, b_M(\alpha;x) \Big)^q dx + \int_\alpha^{1/2} \Big((-1)^m \, b_M(\alpha;x) \Big)^q dx \right]$$

Noticing that

$$b_M(\alpha;\alpha) = 0, \qquad \frac{\partial}{\partial\alpha} b_M(\alpha;x) = -\, b_{M-1}(\alpha)$$

we have

$$g'(\alpha) = (-1)^m \, 2\, q\, b_{M-1}(\alpha) \left[\int_0^\alpha \Big((-1)^{m-1} \, b_M(\alpha;x) \Big)^{q-1} dx - \right.$$
$$\left. \int_\alpha^{1/2} \Big((-1)^m \, b_M(\alpha;x) \Big)^{q-1} dx \right] \tag{5.12}$$

Note that

$$(-1)^m b_{M-1}(\alpha) > 0 \qquad (0<\alpha<1/2)$$

from Lemma 5.2 (3). Differentiating

$$h(\alpha) = \int_0^\alpha \Big((-1)^{m-1} \, b_M(\alpha;x) \Big)^{q-1} dx - \int_\alpha^{1/2} \Big((-1)^m \, b_M(\alpha;x) \Big)^{q-1} dx \tag{5.13}$$

with respect to α, noticing the fact $b_M(\alpha;\alpha) = 0$ we have

$$h'(\alpha) = (q-1)\,(-1)^m\, b_{M-1}(\alpha) \left[\int_0^\alpha \Big((-1)^{m-1}\, b_M(\alpha;x) \Big)^{q-2} dx + \int_\alpha^{1/2} \Big((-1)^m\, b_M(\alpha;x) \Big)^{q-2} dx \right] > 0 \qquad (0<\alpha<1/2) \tag{5.14}$$

Thus we showed that the equation $g'(\alpha) = 0$ that is

$$\int_0^\alpha \Big((-1)^{m-1}\, b_M(\alpha;x) \Big)^{q-1} dx - \int_\alpha^{1/2} \Big((-1)^m\, b_M(\alpha;x) \Big)^{q-1} dx = 0$$

has only one solution $\alpha = \alpha_0$ in an interval $0<\alpha<1/2$. We showed that

$$\min_{0\le\alpha\le 1/2} \| \, b_M(\alpha;x) \, \|_q = \| \, b_M(\alpha_0;x) \, \|_q \tag{5.15}$$

and

$$\int_0^{\alpha_0} \Big((-1)^{m-1}\, b_M(\alpha_0;x) \Big)^{q-1} dx - \int_{\alpha_0}^{1/2} \Big((-1)^m\, b_M(\alpha_0;x) \Big)^{q-1} dx = 0 \tag{5.16}$$

This is equivalent to

$$\int_0^1 \mathrm{sgn}\big((-1)^{m-1}\, b_M(\alpha_0;x) \big) \, \big| (-1)^{m-1}\, b_M(\alpha_0;x) \big|^{q-1} dx = 0 \tag{5.17}$$

Now we introduce a new function

$$f(x) = \mathrm{sgn}\big((-1)^{m-1}\, b_M(\alpha_0;x) \big) \, \big| (-1)^{m-1}\, b_M(\alpha_0;x) \big|^{q-1} \qquad (0<x<1) \tag{5.18}$$

The equality (5.17) means that

$$\int_0^1 f(y)\, dy = 0$$

For this function $f(x)$ the solution $u(x)$ to the boundary value problem

$$\begin{cases} (-1)^m\, u^{(M)} = f(x) & (0<x<1) \\ u^{(i)}(1) - u^{(i)}(0) = 0 & (0 \le i \le M-1) \\ \displaystyle\int_0^1 u(x)\, dx = 0 & \end{cases}$$

is given by

$$u(x) = \int_0^1 (-1)^{m-1}\, b_M(\, |x-y| \,)\, f(y)\, dy \qquad (0 \le x \le 1) \tag{5.19}$$

Note that $u(x) \in H_M$. Interchanging x and y we have

$$u(y) = \int_0^1 f(x)\, (-1)^{m-1}\, b_M(\, |x-y| \,)\, dx \qquad (0 \le y \le 1)$$

Since the solvability condition $\displaystyle\int_0^1 f(y)\,dy = 0$ holds, we have

$$u(y) = \int_0^1 f(x)\,(-1)^{m-1}\,b_M(\,\alpha_0; |x-y|\,)\,dx \qquad (0 \le y \le 1)$$

Putting $y = 0$, we have

$$u(0) = \int_0^1 f(x)\,(-1)^{m-1}\,b_M(\alpha_0; x)\,dx$$

Noticing (5.18), we have

$$u(0) = \|\,b_M(\alpha_0; x)\,\|_q^q = \|\,b_M(\alpha_0; x)\,\|_q\,\|\,b_M(\alpha_0; x)\,\|_q^{q/p} =$$
$$\|\,b_M(\alpha_0; x)\,\|_q\,\|\,f(x)\,\|_p = \|\,b_M(\alpha_0; x)\,\|_q\,\left\|\,u^{(M)}(x)\,\right\|_p \tag{5.20}$$

From (5.10) we have

$$\sup_{0\le y\le 1}|\,u(y)\,| \le \|\,b_M(\alpha_0; x)\,\|_q\,\left\|\,u^{(M)}(x)\,\right\|_p \tag{5.21}$$

Combining this with (5.20), we have

$$\|\,b_M(\alpha_0; x)\,\|_q\,\left\|\,u^{(M)}(x)\,\right\|_p = u(0) \le$$
$$\sup_{0\le y\le 1}|\,u(y)\,| \le \|\,b_M(\alpha_0; x)\,\|_q\,\left\|\,u^{(M)}(x)\,\right\|_p \tag{5.22}$$

This shows

$$\sup_{0\le y\le 1}|\,u(y)\,| = \|\,b_M(\alpha_0; x)\,\|_q\,\left\|\,u^{(M)}(x)\,\right\|_p \tag{5.23}$$

This completes the proof of Theorem 1.1 (2). ■

6 The special cases We show the first three best constants.

Theorem 6.1 (1) *In the case of $M = 1$.*

$$C^q(1) = \|\,b_1(x)\,\|_q^q = \frac{1}{q+1}\left(\frac{1}{2}\right)^q \tag{6.1}$$

Especially if $p = q = 2$ then we have

$$C^2(1) = \|\,b_1(x)\,\|_2^2 = \frac{1}{12} \tag{6.2}$$

(2) *In the case of $M = 2$.*

$$C^q(2) = \min_{0\le\alpha\le 1/2}\|\,b_2(\alpha; x)\,\|_q^q = \|\,b_2(\alpha_0; x)\,\|_q^q = \frac{1}{2q+1}\left(\frac{1}{2}\,\alpha_0\,(1-\alpha_0)\right)^q \tag{6.3}$$

α_0 $(0 \le \alpha_0 \le 1/2)$ *is the unique solution to the equation*

$$\int_0^{\frac{\alpha_0}{1-2\alpha_0}} x^{q-1}(1+x)^{q-1}dx = \frac{1}{2}\,\mathrm{B}(q, q) \tag{6.4}$$

Especially if $p = q = 2$ *then we have*

$$C^2(2) \;=\; \min_{0\le\alpha\le 1/2} \|\, b_2(\alpha;x)\,\|_2^2 \;=\; \|\, b_2(\alpha_0;x)\,\|_2^2 \;=\; \frac{1}{20}\,\alpha_0^2\,(1-\alpha_0)^2 \;=\; \frac{1}{720} \tag{6.5}$$

$\alpha_0\ (0 \le \alpha_0 \le 1/2)$ *is the unique solution to the equation*

$$6\,\alpha_0\,(1-\alpha_0) \;=\; 1 \qquad \text{that is} \qquad \alpha_0 \;=\; \frac{3-\sqrt{3}}{6} \tag{6.6}$$

(3) *In the case of* $M = 3$.

$$C^q(3) \;=\; \|\, b_3(x)\,\|_q^q \;=\; \frac{1}{2}\left(\frac{1}{48}\right)^q \frac{\Gamma(q+1)\,\Gamma\left(\dfrac{1}{2}\,(q+1)\right)}{\Gamma\left(\dfrac{3}{2}(q+1)\right)} \tag{6.7}$$

Especially if $p = q = 2$ *then we have*

$$C^2(3) \;=\; \|\, b_3(x)\,\|_2^2 \;=\; \frac{1}{30240} \tag{6.8}$$

We show the first best function.

Theorem 6.2 *In the case of* $M = 1$.

$$u(x) \;=\; \left\{\begin{array}{ll} \dfrac{1}{q}\left(\dfrac{1}{2} - x\right)^q & (0 < x < 1/2) \\ \dfrac{1}{q}\left(x - \dfrac{1}{2}\right)^q & (1/2 < x < 1) \end{array}\right\} \;-\; \frac{1}{q\,(q+1)}\left(\frac{1}{2}\right)^q \tag{6.9}$$

$$u(0) \;=\; u(1) \;=\; \frac{1}{q+1}\left(\frac{1}{2}\right)^q, \quad u\left(\frac{1}{2} - 0\right) \;=\; u\left(\frac{1}{2} + 0\right) \;=\; -\,\frac{1}{q\,(q+1)}\left(\frac{1}{2}\right)^q \tag{6.10}$$

$$u'(x) \;=\; \left\{\begin{array}{lll} -\left(\dfrac{1}{2} - x\right)^{q-1} < 0 & & (0 < x < 1/2) \\ \left(x - \dfrac{1}{2}\right)^{q-1} > 0 & & (1/2 < x < 1) \end{array}\right. \tag{6.11}$$

$$u'(0) \;=\; -\,u'(1) \;=\; -\left(\frac{1}{2}\right)^{q-1}, \qquad u'\left(\frac{1}{2} - 0\right) \;=\; u'\left(\frac{1}{2} + 0\right) \;=\; 0 \tag{6.12}$$

Especially if $p = q = 2$ *then we have*

$$u(x) \;=\; b_2(x) \qquad (0 < x < 1) \tag{6.13}$$

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