ON A STOCHASTIC INVENTORY SYSTEM WITH PARTIALLY OBSERVED LEVEL AND DEMAND

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ABSTRACT. The vast majority of work done on inventory system is based on the critical assumption of fully observed inventory inventory level dynamics and demand. Modern technology, like the internet, offers a tremendous number of opportunities to businesses to collect imperfect but useful information which helps them planning efficiently to meet future demand. A good example is the internet. Visits to commercial web sites constitute a source of partial information on future demands of one or more of the commodities (or services) offered by companies. Many factors contribute to make inventories hard to be fully observed by the management. Using Hidden Markov Models techniques we exploit partial information on inventory systems to estimate the current inventory level as well as future demands. The parameters of the model are updated via the EM algorithm.

1. INTRODUCTION

Modern technology, like the internet, offers a tremendous number of opportunities to businesses to collect useful information which helps them planning efficiently to meet future demand. Visits to commercial web sites constitute a source of partial information on future demands of one or more of the commodities (or services) offered by companies. Warnings by e-mail (or by some other means such as mobile phone short message service) of customers on change in the price of a commodities provide a source of potential sales.

Another way of acquiring partial information on future demand is provided by a company that uses sales representatives to market its products. Each contact of a sales representative with a customer yields a potential demand. Sometimes sales representatives prepare sales vouchers as means for quoting the customers showing willingness to buy. Since it usually takes some time for a potential sale to be materialized, the collection of sales representatives' information as to the number of customers interested in a product (such as the number of outstanding sales vouchers) can generate an indication about the future sales of that product [21].

Treharne and Sox [22] discuss a non-stationary demand situation where the demand is partially observed. They model the demand as a composite-state, partially observed Markov Decision Process.

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Another example is provided by DeCroix and Mookerjee [15] who consider a periodic review problem in which there is an option of purchasing demand information at the beginning of each period. They consider two levels of demand information: Perfect information allows the decision maker to know the exact demand of the coming period, whereas the imperfect one identifies a particular posterior demand distribution.

Karaesmen, Buzacott, and Dallery [17] consider a capacitated problem under partial information on demand and stochastic lead times. They model the problem via a discrete time make-to-stock queue.

Many factors contribute to make inventories hard to be fully observed by the management. Among these factors are thefts, shoplifting, damaged or misplaced items, low production yield processes [23], perished items [20] etc.

An earlier literature review on partially observed systems can be found in Monahan [19]. Since then, there have not been much research activity in the study of partially observed inventories.

Bensousan et al. [9, 10, 11, 12, 13], Treharne and Sox [22] study partially observed demands in the context of discrete time optimal control. In their studies, the demand is Markov modulated but the underlying demand state is unobserved. Another example of a Markov modulated model is discussed in Beyer et al. [14].

Models discussing filtering and parameter estimation using hidden Markov models techniques are consider by Aggoun et al. [2, 3, 4, 5] and Aggoun [1].

In this article we extend the model discussed in Aggoun [1]. We consider a discrete-time, discrete state inventory model with unobserved inventory level and perished items where the demand is a partially observed finite-state process modulated by a Markov chain. This information is made available at the beginning of each period. These two processes, in turn modulate a replenishment process. In other words the amount to be ordered and stocked to satisfy the (estimated) demand which must be met, say in the next period, relies on the partial information on futures sales collected and made available in the current period.

For the sake of simplicity and to be dealing with only finite state processes, we assume that information does not accumulate without bound. That is, information on potential sales from earlier periods are discarded.

This article is divided into six sections and is organized as follows. In §2 we define the model. In §3 we describe the reference probability method used in computing our filters. In §4 and §5 we derive filters for various quantities of interest. The parameters of the model are re-estimated via the EM algorithm in §6.

2. Model Description

For the benefit of the reader we briefly recall few facts about finite-state-homogeneous Markov chains.

Definition 1. *A discrete-time stochastic process* $\{\eta_n\}$*, with finite-state space* $S = \{s_1, s_2, \dots, s_L\}$*, defined on a probability space* $(\Omega, \mathcal{F}, P)$ *is a Markov chain if*

$$P(\eta_{n+1} = s_{i_{n+1}} \mid \eta_0 = s_{i_0}, \dots, \eta_n = s_{i_n}) = P(\eta_{n+1} = s_{i_{n+1}} \mid \eta_n = s_{i_n}),$$

for all $n \geq 0$ *and all states* $s_{i_0}, \dots, s_{i_n}, s_{i_{n+1}} \in S$.

$\{\eta_n\}$ *is a homogeneous Markov chain if*

$$P(\eta_{n+1} = s_j \mid \eta_n = s_i) \stackrel{\triangle}{=} \pi_{ji}$$

is independent of n.

The matrix $\Pi = \{\pi_{ji}\}$ *is called the probability transition matrix of the homogeneous Markov chain and it satisfies the property* $\sum_{j=1}^{N} \pi_{ji} = 1$.

Note that our transition matrix Π is the transpose of the traditional transition matrix defined elsewhere. The convenience of this choice will be apparent later.

Consider the filtration $\{\mathcal{F}_n\} = \sigma\{\eta_0, \eta_1, \dots, \eta_n\}$.

Write $Y_n = (I_{(\eta_n = s_1)}, I_{(\eta_n = s_2)}, \dots, I_{(\eta_n = s_L)})$, where $I_{(A)}$ is the usual indicator function of a set A.

Since at each time n only one entry of the vector Y_n is not 0 and equal to 1, then Y is a discrete-time Markov chain with state space the set of unit vectors $e_1 = (1, 0, \dots, 0)', \dots, e_N = (0, \dots, 1)'$ of $\mathbb{R}^L$. However, the probability transitions matrix of Y is Π. We can write:

$$E[Y_n \mid \mathcal{F}_{n-1}] = E[Y_n \mid Y_{n-1}] = \Pi Y_{n-1},$$

from which we conclude that ΠY_{n-1} is the predictable part of Y_n, given the history of Y up to time $n-1$ and the non-predictable or 'noise' part of Y_n must be $M_n \stackrel{\triangle}{=} Y_n - \Pi Y_{n-1}$. In fact it can be easily shown that $M_n \in \mathbb{R}^N$ is a mean 0, $\mathcal{F}_n$-vector martingale and we have the semimartingale (or Doob decomposition) representation of the Markov chain $\{Y_n\}$ (see [6, 16]):

$$Y_n = \Pi Y_{n-1} + M_n. \tag{2.1}$$

With Y one of the unit (column) vectors e_i, $1 \leq i \leq N$, prime denoting transpose, and using the inner product notation $\langle a, b \rangle = a'b$, this idempotent property allows us to write the square YY' as $\sum_{i=1}^{N} \langle Y, e_i \rangle e_i e_i'$ and so obtain closed (finite-dimensional), recursive filters in Section 6.

More generally, any real function $f(Y)$ can be expressed as a linear functional $f(Y) = \langle f, Y \rangle$ where $\langle f, e_i \rangle = f(e_i) = f_i$ and $f = (f_1, \dots, f_N)$. Thus with $Y^i = \langle Y, e_i \rangle$,

$$f(Y) = \sum_{i=1}^{N} f(e_i) Y^i = \sum_{i=1}^{N} f_i Y^i. \tag{2.2}$$

Our model consists of the following components.

1. Let Y_n be the number of potential demands available at the beginning of period n.

   Let $I_1, \dots, I_L$ be a partition of the set of natural numbers $\{0, 1, 2, \dots\}$.

   For $1 \le i, j \le L$, write

$$p_{ji} = P(Y_n \in I_j \mid Y_{n-1} \in I_i),$$

   and $P = (p_{ji})$, $\sum_{j=1}^{L} p_{ji} = 1$. Again the unusual notation p_{ji} instead of p_{ji} is used for convenience. Therefore the process Y is an L-state discrete-time Markov process $Y = \{Y_k, 1 \le k\}$.

   Without any loss of generality, as explained above, we shall identify the state space of Y with the canonical basis $\mathcal{L} = \{e_1, e_2, \dots, e_L\}$. Again the essential point of this canonical representation of a Markov chain, is that the state dynamics can be written down in the form

$$Y_n = A\,Y_{n-1} + V_n. \tag{2.3}$$

   Here V is a $(P, \sigma\{Y_1, Y_2, \dots, Y_n\})$-martingale increment and $A \in \mathbb{R}^{L \times L}$ is a matrix of state transition probabilities such that $P(Y_n = j \mid Y_{n-1} = i) \triangleq a_{ji}$.

   The physical interpretation of this representation is that we are assuming that at each period n, the number of potential demands Y_n is not completely independent of the past. However the dependence we assume (dependence on Y_{n-1}) is the simplest mathematical scenario.

2. Similarly we assume that the actual demand process D is a finite-state process with N states $\{d_1, \dots, d_N\}$. Without loss of generality, we identify the state space $\{d_1, \dots, d_N\}$ with the sets of standard unit vectors $\{f_1, f_2, \dots, f_N\}$ of $\mathbb{R}^N$. We shall assume that:

$$P(D_n = f_m \mid D_1, \dots, D_{n-1}, Y_0, Y_1, \dots, Y_{n-1}) = P(D_n = f_m \mid D_{n-1}, Y_{n-1}).$$

   Write $b_{m\ell i} = P(D_n = f_m \mid D_{n-1} = f_\ell, Y_{n-1} = e_i)$ and
   $B = \{b_{m\ell i}\}$, $m, \ell = 1, \dots, N;\ i = 1, \dots, L$.

   Therefore $\sum_{m=1}^{N} b_{m\ell i} = 1$ and we have the semimartingale representation

$$D_n = BD_{n-1} \otimes Y_{n-1} + W_n. \tag{2.4}$$

   Here W_n is a sequence of martingale increments. For (column) vectors $x \in \mathbb{R}^L$, $y \in \mathbb{R}^N$ their tensor or Kronecker product $x \otimes y$ is the vector $xy' \in \mathbb{R}^{LN}$.

   Note here that the actual demand D_n, at time n, is related to the actual demands $\{D_k, k < n\}$ as well as potential demands $\{Y_k, k < n\}$ observed in the past but, again, we are assuming the mathematically simplest kind of dependence between these stochastic processes.

3. Each item in the stock at the beginning of the n-th period is assumed to be perished (damaged, stolen etc.) with probability $(1-\alpha)$ independently of the other items, where $0<\alpha<1$ or is intact with probability α.

   We shall be using the Binomial thining operator " $\circ$ " which is well-known in Time Series Analysis [7], [18]. This operator is defined as follows.

   For any nonnegative integer-valued random variable X and $\alpha \in (0,1)$, $\alpha \circ X = \sum_{j=1}^{X} \mathfrak{Y}_j$, where $\mathfrak{Y}_1, \mathfrak{Y}_2, \dots$ is a sequence of of i.i.d. random variables independent of X, such that $P(\mathfrak{Y}_j = 1) = 1 - P(\mathfrak{Y}_j = 0) = \alpha$. Now let X_n be an integer-valued random variable representing the number of items in stock at the beginning of period n in the inventory with dynamics

$$X_n = \alpha \circ X_{n-1} + U_{n-1} - \sum_{\ell=1}^{N} d_\ell \langle D_{n-1}, f_\ell \rangle, \tag{2.5}$$

   with X_0 constant (integer) or its distribution known. If X_{n-1} is negative then $\alpha \circ X_{n-1} = 0$. Note that a negative X_n is interpreted as shortage.

   Equation (2.5) simply means that the inventory level available at the beginning of the n-th period is whatever survived from earlier periods plus a certain replenishment minus one of the possible demands $\{d_1, \dots, d_N\}$ which incurred in the previous period.
4. A replenishment process U such that for $n \geq 1$,
   $P(U_n = u \mid \sigma\{U_k, X_k, Y_k, D_k, k \leq n-1\}) = \xi(u, U_{n-1}, X_{n-1}, Y_{n-1}, D_{n-1})$.

   It is natural to assume that the replenishment at time n, U_n, depends on the information available at time $n-1$.
5. As in [9] we suppose there is a finite storage capacity a, the inventory can take values in the interval [0, a]. This interval can be partitioned into M + 2 disjoint intervals, namely,
   $I_0 := [a_0, a_0]; I_1 := (a_0, a_1]; I_2 := (a_1, a_2]; \dots ; I_M := (a_{M-1}, a_M];$
   $I_{M+1} := [a_M, a_M]$, for $0 = a_0 < a_1 < \cdots < a_M = a$. The observations process is then $Z_n = z,$ if $X_n \in I_z, 0 \leq z \leq M+1$.
   In practice, the interval observations as defined above would happen when the inventory is stored in modules, e.g., bins, shelves or different locations. The manager can see empty and full bins by simply walking in the storage area. In a typical case, the bins may be prioritized in such a way that items in bin i are not used until items in bin $i+1$ are finished. Then a_1 would be the first bin's capacity, a_2 would be the first and second bins' cumulative capacity, etc. If three bins are full, the fourth is semi-full and the others are empty, the manager would conclude $I_n \in I_4 = (a_3, a_4]$ and observe the signal $Z_n = 4$.

   Process Z is a discrete-time Markov Chain with the state space $\{0, 1, \dots, M+1\}$ which we identify with the sets of standard unit vectors $\{g_0, \dots, g_{M+1}\}$ of $\mathbb{R}^{M+2}$.

Write $P[Z_n = g_s \mid Z_{n-1} = g_r] = c_{rs}$, and $C = \{c_{rs}\}$, $r, s = 0, \ldots, M+1$.

Write the following complete filtration $\mathcal{Y}_n = \sigma\{Z_k, U_k, Y_k, k \leq n\}$,
$\mathcal{G}_n = \sigma\{X_k, Z_k, U_k, Y_k, D_k,\, k \leq n\}$.

3. Reference Probability

In our context, the objective of the method of reference probability is to choose a measure $\overline{P}$, on the measurable space $(\Omega, \mathcal{F})$, under which:

(i) Process D is a sequence of i.i.d. random variables uniformly distributed on the set $\{f_1, f_2, \ldots, f_N\}$, that is $P(D_n \mid \mathcal{G}_{n-1}) = \dfrac{1}{N}$.

(ii) Process Y is a sequence of i.i.d. random variables uniformly distributed on the set $\{e_1, e_2, \ldots, e_L\}$, that is $P(Y_n \mid \mathcal{G}_{n-1}) = \dfrac{1}{L}$.

(iii) Process Z is a sequence of i.i.d. random variables uniformly distributed on the set $\{g_0, g_2, \ldots, g_{M+1}\}$, that is $P(Z_n = g_s \mid \mathcal{G}_{n-1}) = \dfrac{1}{M+2}$.

(iv) Process U is a sequence of independent random variables such that $P(U_n = u \mid \mathcal{G}_{n-1}) = \xi_n(u)$.

Further, under the measure $\overline{P}$, the dynamics for X are unchanged.

This mathematical trick is the key to the derivation of our results.

The probability measure P is referred to as the 'real world' probability measure, that is, under this probability measure

$$P \quad \begin{cases} Y_n = AY_{n-1} + V_n, \\ D_n = BD_{n-1} \otimes Y_{n-1} + W_n, \\ Z_n = CZ_{n-1} + M_n, \\ P(U_n = u \mid \mathcal{G}_{n-1}) = \xi(u, U_{n-1}, X_{n-1}, Y_{n-1}, D_{n-1}). \end{cases} \tag{3.1}$$

Definition 2. *Denote by $\Gamma = \{\Gamma_k, 0 \leq k\}$ the stochastic process whose value at n is given by*

$$\Gamma_n = \prod_{k=0}^{n} \gamma_n, \tag{3.2}$$

where $\gamma_0 = 1$ and

$$\begin{aligned} \gamma_k = & \frac{\xi_k(U_k, U_{k-1}, X_{k-1}, Y_{k-1}, D_{k-1})}{\xi_k(U_k)} \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_k, e_j\rangle \langle Y_{k-1}, e_i\rangle} \\ & \prod_{m,\ell=1}^{N} \prod_{i=1}^{L} (Nb_{m\ell i})^{\langle D_n, f_m\rangle \langle D_{k-1}, f_\ell\rangle \langle Y_{k-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{k-1}, g_r\rangle \langle Z_k, g_s\rangle}. \end{aligned} \tag{3.3}$$

*Note that the notation $\langle Y_k, e_i\rangle$ is equivalent to the indicator function
$I(Y_k = e_i)$.*

We define the 'real world' measure P in terms of $\overline{P}$, by setting $\left.\frac{dP}{d\overline{P}}\right|_{\mathcal{G}_n} \triangleq \Gamma_n$*. The existence of P follows from Kolmogorov Extension Theorem.*

Lemma 1. *Under the reference probability measure $\overline{P}$, $\overline{E}[\gamma_k \mid \mathcal{G}_{k-1}] = 1$, where $\overline{E}$ denotes expectation under $\overline{P}$.*

Proof. In view of (3.3) and the distributions assumptions under $\overline{E}$

$$\overline{E}[\gamma_k \mid \mathcal{G}_{k-1}]$$
$$= \sum_{i,j=1}^{L} \sum_{m,\ell=1}^{N} NLa_{ji} b_{m\ell i} \langle D_{k-1}, f_\ell \rangle \langle Y_{k-1}, e_i \rangle$$
$$\sum_{r,s=0}^{M+1} (M+2) c_{sri} \langle Z_{k-1}, g_r \rangle \langle Z_k, g_s \rangle$$
$$\overline{E}[\sum_u \frac{\xi_k(u, U_{k-1}, X_{k-1}, e_i, f_\ell)}{\xi_k(u)} \xi_k(u) \langle Y_n, e_j \rangle \langle D_n, f_m \rangle \langle Z_k, g_s \rangle \mid \mathcal{G}_{k-1}]$$
$$= \sum_{i,j=1}^{L} a_{ji} \sum_{m,\ell=1}^{N} \sum_{i=1}^{L} b_{m\ell i} \langle D_{k-1}, f_\ell \rangle \langle Y_{k-1}, e_i \rangle$$
$$\sum_{r,s=0}^{M+1} c_{sri} \langle Z_{k-1}, g_r \rangle \langle Z_k, g_s \rangle = 1$$

□

4. Recursive Estimation

What we wish to do now is to derive, under the 'ideal' reference probability measure $\overline{P}$ a recursive formula for the unnormalized conditional joint distribution of the demand and the inventory given the observed data at each epoch n. Using Bayes' Theorem [16, 6]

$$E\Big[\langle D_n, f_v \rangle I(X_n = x) \mid \mathcal{Y}_n\Big] = \frac{\overline{E}\Big[\Gamma_n \langle D_n, f_v \rangle I(X_n = x) \mid \mathcal{Y}_n\Big]}{\overline{E}\Big[\Gamma_n \mid \mathcal{Y}_n\Big]}.$$

Write $\overline{E}\Big[\Gamma_n \langle D_n, f_v \rangle I(X_n = x) \mid \mathcal{Y}_n\Big] = \rho_n(v, x)$.

Theorem 1. *Denote by $\rho_0(v, x)$, the initial probability distribution of (D_0, X_0). The unnormalised probability $\rho_n(v, x)$, satisfies the recursion*

$$\rho_n(v, x)$$
$$= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j \rangle \langle Y_{n-1}, e_i \rangle} \prod_{r,s=0}^{M+1} ((M+2) c_{sr})^{\langle Z_{n-1}, g_r \rangle \langle Z_n, g_s \rangle} I(a_{s-1} < x \leq a_s)$$
$$\sum_{\ell=1}^{N} \sum_{i=1}^{L} b_{v\ell i} \langle Y_{n-1}, e_i \rangle \sum_{\substack{\mathfrak{z} \geq x + d_\ell - U_{n-1} \\ a_{r-1} < \mathfrak{z} \leq a_r}} \mathrm{Bin}(\mathfrak{z}, \alpha, x + d_\ell - U_{n-1})$$

$$\times \frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_i, f_\ell)}{\xi_n(U_n)} \rho_{n-1}(\ell, \mathfrak{z}),$$

where

$$\mathrm{Bin}(\mathfrak{z}, \alpha, x + d_\ell - U_{n-1}) = \binom{\mathfrak{z}}{x + d_\ell - U_{n-1}} (\alpha)^{x+d_\ell-U_{n-1}} (1-\alpha)^{\mathfrak{z}-x-d_\ell+U_{n-1}}.$$

Proof. In view of (3.3), (3.2) and de independence assumptions under $\overline{P}$

$$\rho_n(v, x)$$

$$= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$

$$\overline{E}\Big[\Gamma_{n-1}\langle D_n, f_v\rangle \prod_{m,\ell=1}^{N} \prod_{i=1}^{L} (Nb_{m\ell i})^{\langle D_n, f_m\rangle\langle D_{n-1}, f_\ell\rangle\langle Y_{n-1}, e_i\rangle}$$

$$\frac{\xi_n(U_n, U_{n-1}, X_{n-1}, Y_{n-1}, D_{n-1})}{\xi_n(U_n)} I(X_n = x) \mid \mathcal{Y}_n\Big]$$

$$= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$

$$\sum_{\ell=1}^{N} \sum_{i=1}^{L} b_{v\ell i}\langle Y_{n-1}, e_i\rangle \overline{E}\Big[\Gamma_{n-1}\langle D_{n-1}, f_\ell\rangle \frac{\xi_n(U_n, U_{n-1}, X_{n-1}, e_i, f_\ell)}{\xi_n(U_n)}$$

$$I(\alpha \circ X_{n-1} + U_{n-1} - \sum_{\ell=1}^{N} d_\ell \langle D_{n-1}, f_\ell\rangle = x)$$

$$I(a_{s-1} < x \le a_s) I(a_{r-1} < X_{n-1} \le a_r) \mid \mathcal{Y}_n\Big]$$

$$= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$

$$\sum_{\ell=1}^{N} \sum_{i=1}^{L} b_{v\ell i}\langle Y_{n-1}, e_i\rangle \overline{E}\Big[\Gamma_{n-1}\langle D_{n-1}, f_\ell\rangle \frac{\xi_n(U_n, U_{n-1}, X_{n-1}, e_i, f_\ell)}{\xi_n(U_n)}$$

$$I(\alpha \circ X_{n-1} = -U_{n-1} + d_\ell + x)$$

$$I(a_{s-1} < x \le a_s) I(a_{r-1} < X_{n-1} \le a_r) \mid \mathcal{Y}_{n-1}\Big]$$

$$= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$

$$\sum_{\ell=1}^{N} \sum_{i=1}^{L} b_{v\ell i}\langle Y_{n-1}, e_i\rangle I(a_{s-1} < x \le a_s) \sum_{\substack{\mathfrak{z} \ge x + d_\ell - U_{n-1} \\ a_{r-1} < \mathfrak{z} \le a_r}}$$

$$\mathrm{Bin}(\mathfrak{z}, \alpha, x + d_\ell - U_{n-1}) \frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_i, f_\ell)}{\xi_n(U_n)}$$

$$\overline{E}\Big[\Gamma_{n-1}\langle D_{n-1}, f_\ell\rangle I(X_{n-1} = \mathfrak{z}) \mid \mathcal{Y}_{n-1}\Big]$$

$$= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$

$$\sum_{\ell=1}^{N}\sum_{i=1}^{L} b_{v\ell i}\langle Y_{n-1}, e_i\rangle I(a_{s-1} < x \le a_s) \sum_{\substack{\mathfrak{z}\ge x+d_\ell-U_{n-1}\\ a_{r-1}<\mathfrak{z}\le a_r}}$$
$$\mathrm{Bin}(\mathfrak{z}, \alpha, x+d_\ell-U_{n-1})\frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_i, f_\ell)}{\xi_n(U_n)}\rho_{n-1}(\ell, \mathfrak{z}).$$

Here
$$\mathrm{Bin}(\mathfrak{z}, \alpha, x+d_\ell-U_{n-1}) = \binom{\mathfrak{z}}{x+d_\ell-U_{n-1}}(\alpha)^{x+d_\ell-U_{n-1}}(1-\alpha)^{\mathfrak{z}-x-d_\ell+U_{n-1}}. \qquad \square$$

5. Parameter Updating

In this section we show how, using the expectation maximization (EM) algorithm, the parameters of the model can be estimated. In fact, it is a conditional pseudo log-likelihood that is maximized, and the new parameters are expressed in terms of the recursive estimates obtained in Section 6. We begin by describing the EM algorithm.

The basic idea behind the EM algorithm is as follows [8]. Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability measures on a measurable space $(\Omega, \mathcal{G})$ all absolutely continuous with respect to a fixed probability measure P_0 and let $\mathcal{Y} \subset \mathcal{G}$. The likelihood function for computing an estimate of the parameter θ based on the information available in $\mathcal{Y}$ is $L(\theta) = E_0\left[\frac{dP_\theta}{dP_0} \mid \mathcal{Y}\right]$, and the maximum likelihood estimate (MLE) is defined by $\hat{\theta} \in \mathrm{argmax}_{\theta\in\Theta} L(\theta)$.

The reasoning is that the most likely value of the parameter θ is the one that maximizes this conditional expectation of the density.

In general, the MLE is difficult to compute directly, and the EM algorithm provides an iterative approximation method:

1. Set $p = 0$ and choose $\hat{\theta}_0$.
2. (E-step) Set $\theta^* = \hat{\theta}_p$ and compute $Q(\cdot, \theta^*)$, where $Q(\theta, \theta^*) = E_{\theta^*}\left[\log \frac{dP_\theta}{dP_{\theta^*}} \,\middle|\, \mathcal{Y}\right]$.
3. (M-step) Find $\hat{\theta}_{p+1} \in \mathrm{argmax}_{\theta\in\Theta} Q(\theta, \theta^*)$.
4. Replace p by $p+1$ and repeat beginning with Step 2 until a stopping criterion is satisfied.

The sequence generated $\left\{\hat{\theta}_p, p \ge 0\right\}$ gives non decreasing values of the likelihood function to a local maximum of the likelihood function: it follows from Jensen's Inequality that

$$\log L\left(\hat{\theta}_{p+1}\right) - \log L\left(\hat{\theta}_p\right) \ge Q\left(\hat{\theta}_{p+1}, \hat{\theta}_p\right),$$

with equality if $\hat{\theta}_{p+1} = \hat{\theta}_p$. We call $Q(\theta, \theta^*)$ a conditional pseudo-log-likelihood.

Our model is determined by the set of parameters

$$\theta := (a_{ji},\ 1 \le i, j \le L,\ c_{sr},\ 0 \le r, s \le M+1,\ b_{m\ell i},\ 1 \le m, \ell \le N)$$

Suppose our model is determined by such a set θ and we wish to determine a new set

$$\hat{\theta} = \left(\hat{a}_{ji}(n),\ 1 \le i, j \le L,\ \hat{c}_{sr}(n),\ 0 \le r, s \le M+1,\ \hat{b}_{m\ell i}(n),\ 1 \le m, \ell \le N\right)$$

which maximizes the conditional pseudo-log-likelihoods defined below. To replace the parameters a_{ji} by $\hat{a}_{ji}(n)$ in the Markov chain Y and the parameters c_{sr} by $\hat{c}_{sr}(n)$ in the Markov chain Z we define

$$\Gamma_n = \prod_{k=1}^{n} \prod_{i,j=1}^{L} \left[\frac{\hat{a}_{ji}(n)}{a_{ji}}\right]^{\langle Y_k,e_j\rangle\langle Y_{k-1},e_i\rangle} \prod_{r,s=0}^{M+1} \left[\frac{\hat{c}_{sr}(n)}{c_{sr}}\right]^{\langle Z_k,g_s\rangle\langle Z_{k-1},g_r\rangle}.$$

In case $a_{ji} = 0$, take $\hat{a}_{ji}(n) = 0$ and $\dfrac{\hat{a}_{ji}(n)}{a_{ji}} = 1$. The same thing holds for the parameter c_{sr}. Set $\left.\dfrac{dP_{\hat{\theta}}}{dP_\theta}\right|_{\mathcal{G}_n} = \Gamma_n$.

Theorem 2. *The new estimates of the parameters $\hat{a}_{sr}(n)$ and $\hat{c}_{sr}(n)$ given the observations up to time n are given, when defined, by*

$$\hat{a}_{ji}(n) = \frac{\mathfrak{T}_n^{ij}}{\mathfrak{J}_n^i}, \quad \hat{c}_{sr}(n) = \frac{\mathcal{T}_n^{rs}}{\mathcal{J}_n^r},$$

where $\mathfrak{T}_n^{(\mathrm{j},\mathrm{i})}$, is a discrete time counting process for the transitions $e_\mathrm{i} \to e_\mathrm{j}$ of the (observed) markov chain Y, where $i \neq j$,

$$\mathfrak{T}_n^{(\mathrm{j},\mathrm{i})} = \sum_{k=1}^{n} \langle Y_{k-1}, e_\mathrm{i}\rangle\langle Y_k, e_\mathrm{j}\rangle,$$

$\mathfrak{J}_n^\mathrm{i}$ is the cumulative sojourn time spent by the Markov chain Y in state e_i,

$$\mathfrak{J}_n^\mathrm{i} = \sum_{k=1}^{n} \langle Y_{k-1}, e_\mathrm{i}\rangle,$$

$\mathcal{T}_n^{(\mathfrak{s},\mathfrak{r})}$ is a discrete time counting process for the transitions $g_\mathfrak{r} \to g_\mathfrak{s}$ of the process Z, where $r \neq s$,

$$\mathcal{T}_n^{(\mathfrak{s},\mathfrak{r})} = \sum_{k=1}^{n} \langle Z_{k-1}, g_\mathfrak{r}\rangle\langle Z_k, g_\mathfrak{s}\rangle,$$

and $\mathcal{J}_n^\mathfrak{r}$, the cumulative sojourn time spent by the process Z in state $g_\mathfrak{r}$, is

$$\mathcal{J}_n^\mathfrak{r} = \sum_{k=1}^{n} \langle Z_{k-1}, g_\mathfrak{r}\rangle.$$

Proof. Consider the parameter a_{ji}.

$$\begin{aligned} \log \Gamma_n &= \sum_{i,j=1}^{L} \sum_{k=1}^{n} \langle Y_n, e_j\rangle \langle Y_{\ell-1}, e_i\rangle [\log \hat{a}_{ji}(n) - \log a_{ji}] \\ &= \sum_{i,j=1}^{L} \mathfrak{T}_n^{ij} \log \hat{a}_{ji}(n) + R(a) \end{aligned} \tag{5.1}$$

where $R(a)$ is independent of $\hat{a}$.

Now the $\hat{a}_{ji}(n)$ must satisfy

$$\sum_{j=1}^{L} \hat{a}_{ji}(n) = 1. \tag{5.2}$$

Observe that

$$\sum_{j=1}^{L} \mathfrak{T}_n^{ij} = \mathfrak{J}_n^i \tag{5.3}$$

We wish, therefore, to choose the $\hat{a}_{ji}(n)$ to maximize (5.1) subject to the constraint (5.2). Write λ for the Lagrange multiplier and put

$$L(\hat{a}, \lambda) = \sum_{i,j=1}^{L} \mathfrak{T}_n^{ij} \log \hat{a}_{ji}(n) + \hat{R}(a) + \lambda \left(\sum_{s=1}^{L} \hat{a}_{ji}(n) - 1 \right).$$

Differentiating in λ and $\hat{a}_{ji}(n)$, and equating the derivatives to 0, we have the optimum choice of $\hat{a}_{ji}(n)$ is given by the equations

$$\frac{1}{\hat{a}_{ji}(n)} \mathfrak{T}_n^{ij} + \lambda = 0 \tag{5.4}$$

$$\sum_{s=1}^{L} \hat{a}_{ji}(n) = 1. \tag{5.5}$$

From (5.3)–(5.5) we see that $\lambda = -\mathfrak{J}_n^r$ so the optimum choice of $\hat{a}_{ji}(n)$, $1 \leq i, j \leq L$, is $\hat{a}_{ji}(n) = \dfrac{\mathfrak{T}_n^{ij}}{\mathfrak{J}_n^i}$. □

Remark 1. *Since each of the Markov chains Y and Z jumps at most n times up to time n, we have:*
$0 \leq \mathfrak{T}_n^{ij} \leq \mathfrak{J}_n^i \leq n$, $0 \leq \mathcal{T}_n^{rs} \leq \mathcal{J}_n^r \leq n$.

Notation 1. *For any process ϕ_n, $n \in \mathbb{N}$, write $\widehat{\phi}_n = E[\phi_n \mid \mathcal{Y}_n]$ for its $\mathcal{Y}$-optional projection.*

Consider now the parameters $b_{m\ell i}$ in the matrix B. To replace the parameters $b_{m\ell i}$ by $\hat{b}_{m\ell i}(n)$ we must now consider the Radon-Nikodym derivative

$$\widetilde{\Gamma}_n = \prod_{k=1}^{n} \prod_{m,\ell=1}^{N} \prod_{i=1}^{L} \left[\frac{\hat{b}_{m\ell i}(n)}{b_{m\ell i}} \right]^{\langle D_n, f_m \rangle \langle D_{n-1}, f_\ell \rangle \langle Y_{n-1}, e_i \rangle}.$$

Now we introduce a new probability by setting $\left.\dfrac{dP_{\hat{\theta}}}{dP_\theta}\right|_{\mathcal{G}_n} = \widetilde{\Gamma}_n$. Then

$$E\left[\log \widetilde{\Gamma}_n \mid \mathcal{Y}_n\right] = \sum_{r=1}^{L} \sum_{s=1}^{n} \widehat{G}_n^{m\ell i} \log \hat{b}_{m\ell i}(n) + R(b) \tag{5.6}$$

where $R(b)$ is independent of $\hat{b}$. Now the $\hat{b}_{m\ell i}(n)$ must also satisfy

$$\sum_{m=1}^{N} \hat{b}_{m\ell i}(n) = 1. \tag{5.7}$$

Observe that $\sum_{m=1}^{N} G_n^{m\ell i} = S_n^{\ell i}$ and conditional form $\sum_{s=1}^{N} \widehat{G}_n^{m\ell i} = \widehat{S}_n^{\ell i}$

We wish, therefore, to choose the $\hat{b}_{m\ell i}(n)$ to maximize (5.6) subject to the constraint (5.7). Following the same procedure as above we obtain:

Theorem 3. *The maximum log likelihood estimates of the parameters* $\hat{b}_{m\ell i}(n)$ *given the observation up to time n are given, when defined, by* $\hat{b}_{m\ell i}(n) = \dfrac{\mathfrak{q}_n(G_n^{m\ell i})}{\mathfrak{q}_n(S_n^{\ell i})}$,
where $G_n^{\mathfrak{m}l\mathfrak{i}}$ *is the number of times the process D jumps from state* f_l *to state* f_m *while the Markov chain Y is in state* $e_\mathfrak{i}$,

$$G_n^{\mathfrak{m}l\mathfrak{i}} = \sum_{k=1}^{n} \langle D_{k-1}, f_l\rangle\langle D_k, f_\mathfrak{m}\rangle\langle Y_{k-1}, e_\mathfrak{i}\rangle,$$

and $S_n^{l\mathfrak{i}}$ *is the number of times the process D is in state* f_l *while the Markov chain Y is in state* $e_\mathfrak{i}$,

$$S_n^{l\mathfrak{i}} = \sum_{k=1}^{n} \langle D_{k-1}, f_l\rangle\langle Y_{k-1}, e_\mathfrak{i}\rangle. \tag{5.8}$$

Remark 2. *The revised parameters give new probability measures for the model. The sequences of densities* Γ_n *and* $\tilde{\Gamma}_n$ *are improved by construction, and the model parameters are updated or tuned to the observations.*

6. Finite-Dimensional Filters for $G_n^{ml\mathfrak{i}}$ and $S_n^{l\mathfrak{i}}$

Rather than directly estimating the quantities, $G_n^{\mathfrak{m}l\mathfrak{i}}$, and $S_n^{l\mathfrak{i}}$ recursive forms can be found by estimating the related product-quantities, $G_n^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x) \in \mathbb{R}^N$ and $S_n^{l\mathfrak{i}} D_n I(X_n = x) \in \mathbb{R}^N$. The outputs of these filters can then be manipulated to marginalise out the process (X, D), resulting in filtered estimates of the quantities of primary interest.

Write $\mathfrak{q}_n(G_n^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x)) \stackrel{\triangle}{=} \overline{E}\big[\Gamma_n G_n^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x) \mid \mathcal{Y}_n\big]$.

Lemma 2. *The process* $\mathfrak{q}_n(G_n^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x))$ *is computed recursively by the dynamics*

$$\begin{aligned}
&\mathfrak{q}_n(G_n^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x)) \\
&= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle} \\
&\sum_{\ell,m=1}^{N}\sum_{i=1}^{L} b_{m\ell i}\langle Y_{n-1}, e_i\rangle I(a_{s-1} < x \le a_s) \sum_{\substack{\mathfrak{z} \ge x + d_\ell - U_{n-1} \\ a_{r-1} < \mathfrak{z} \le a_r}} \\
&\mathrm{Bin}(\mathfrak{z}, \alpha, x + d_\ell - U_{n-1}) \frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_i, f_\ell)}{\xi_n(U_n)} \\
&\langle \mathfrak{q}_n(G_{n-1}^{\mathfrak{m}l\mathfrak{i}} D_{n-1} I(X_{n-1} = \mathfrak{z})), f_\ell\rangle f_m \\
&+ \prod_{j=1}^{L} (La_{j\mathfrak{i}})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_\mathfrak{i}\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle} \\
&b_{\mathfrak{m}l\mathfrak{i}}\langle Y_{n-1}, e_\mathfrak{i}\rangle I(a_{s-1} < x \le a_s) \sum_{\substack{\mathfrak{z} \ge x + d_l - U_{n-1} \\ a_{r-1} < \mathfrak{z} \le a_r}} \\
&\mathrm{Bin}(\mathfrak{z}, \alpha, x + d_l - U_{n-1}) \frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_\mathfrak{i}, f_l)}{\xi_n(U_n)} \rho_{n-1}(l, \mathfrak{z}) f_\mathfrak{m}
\end{aligned}$$

Proof. First note that $G_n^{\mathfrak{m}l\mathfrak{i}} = G_{n-1}^{\mathfrak{m}l\mathfrak{i}} + \langle D_{n-1}, f_l\rangle\langle D_n, f_\mathfrak{m}\rangle\langle Y_{n-1}, e_\mathfrak{i}\rangle$. Therefore
$\mathfrak{q}_n(G_n^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x)) = \overline{E}\big[\Gamma_n G_{n-1}^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x) \mid \mathcal{Y}_n\big]$
$+ \overline{E}\big[\Gamma_n \langle D_{n-1}, f_l\rangle\langle D_n, f_\mathfrak{m}\rangle\langle Y_{n-1}, e_\mathfrak{i}\rangle D_n I(X_n = x) \mid \mathcal{Y}_n\big]$

The first expectation yields:

$$\overline{E}\big[\Gamma_n G_{n-1}^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x) \mid \mathcal{Y}_n\big]$$
$$= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$
$$\sum_{\ell,m=1}^{N} \sum_{i=1}^{L} b_{m\ell i}\langle Y_{n-1}, e_i\rangle I(a_{s-1} < x \le a_s) \sum_{\substack{\mathfrak{z} \ge x + d_\ell - U_{n-1} \\ a_{r-1} < \mathfrak{z} \le a_r}}$$
$$\mathrm{Bin}(\mathfrak{z}, \alpha, x + d_\ell - U_{n-1}) \frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_i, f_\ell)}{\xi_n(U_n)}$$
$$\langle \mathfrak{q}_n(G_{n-1}^{\mathfrak{m}l\mathfrak{i}} D_{n-1} I(X_{n-1} = \mathfrak{z})), f_\ell\rangle f_m$$

The second expectation is simply

$$\overline{E}\big[\Gamma_n \langle D_{n-1}, f_l\rangle\langle D_n, f_\mathfrak{m}\rangle\langle Y_{n-1}, e_\mathfrak{i}\rangle D_n I(X_n = x) \mid \mathcal{Y}_n\big]$$
$$= \prod_{j=1}^{L} (La_{j\mathfrak{i}})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_\mathfrak{i}\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$
$$b_{\mathfrak{m}l\mathfrak{i}}\langle Y_{n-1}, e_\mathfrak{i}\rangle I(a_{s-1} < x \le a_s) \sum_{\substack{\mathfrak{z} \ge x + d_l - U_{n-1} \\ a_{r-1} < \mathfrak{z} \le a_r}}$$
$$\mathrm{Bin}(\mathfrak{z}, \alpha, x + d_l - U_{n-1}) \frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_\mathfrak{i}, f_l)}{\xi_n(U_n)} \rho_{n-1}(l, \mathfrak{z}) f_\mathfrak{m}. \qquad \square$$

Write $\mathfrak{q}_n(S_n^{l\mathfrak{i}} D_n I(X_n = x)) \triangleq \overline{E}\big[\Gamma_n S_n^{l\mathfrak{i}} D_n \mid \mathcal{Y}_n\big]$.

A similar argument yields the following results.

Lemma 3. *The process $\mathfrak{q}_n(S_n^{l\mathfrak{i}} D_n I(X_n = x))$ is computed recursively by the dynamics*

$$\mathfrak{q}_n(S_n^{l\mathfrak{i}} D_n I(X_n = x))$$
$$= \prod_{i,j=1}^{L} (La_{ji})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_i\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$
$$\sum_{\ell,m=1}^{N} \sum_{i=1}^{L} b_{m\ell i}\langle Y_{n-1}, e_i\rangle I(a_{s-1} < x \le a_s) \sum_{\substack{\mathfrak{z} \ge x + d_\ell - U_{n-1} \\ a_{r-1} < \mathfrak{z} \le a_r}}$$
$$\mathrm{Bin}(\mathfrak{z}, \alpha, x + d_\ell - U_{n-1}) \frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_i, f_\ell)}{\xi_n(U_n)}$$
$$\langle \mathfrak{q}_n(S_{n-1}^{l\mathfrak{i}} D_{n-1} I(X_{n-1} = \mathfrak{z})), f_\ell\rangle f_m$$
$$+ \prod_{j=1}^{L} (La_{j\mathfrak{i}})^{\langle Y_n, e_j\rangle\langle Y_{n-1}, e_\mathfrak{i}\rangle} \prod_{r,s=0}^{M+1} ((M+2)c_{sr})^{\langle Z_{n-1}, g_r\rangle\langle Z_n, g_s\rangle}$$

$$\sum_{m=1}^{N} b_{ml\mathfrak{i}} \langle Y_{n-1}, e_{\mathfrak{i}} \rangle I(a_{s-1} < x \leq a_s) \sum_{\substack{\mathfrak{z} \geq x+d_l-U_{n-1} \\ a_{r-1} < \mathfrak{z} \leq a_r}}$$

$$\mathrm{Bin}(\mathfrak{z}, \alpha, x + d_l - U_{n-1}) \frac{\xi_n(U_n, U_{n-1}, \mathfrak{z}, e_{\mathfrak{i}}, f_l)}{\xi_n(U_n)} \rho_{n-1}(l, \mathfrak{z}) f_m.$$

The filter recursions given above provide updates to estimate product processes, each involving the process (X, D). What we would like to do, is manipulate these filters so as to remove the dependence upon the process (X, D). Let Φ_n be any of our processes. Then with $\mathbf{1} = (1, 1, \ldots, 1)$

$$\sum_x \langle \mathfrak{q}_n(\Phi_n D_n I(X_n = x)), \mathbf{1} \rangle = \sum_x \langle \overline{E}\big[\Gamma_n \Phi_n D_n I(X_n = x) \mid \mathcal{Y}_n\big], \mathbf{1} \rangle$$

$$= \sum_x \overline{E}\big[\Gamma_n \Phi_n I(X_n = x) \langle D_n, \mathbf{1} \rangle \mid \mathcal{Y}_n\big]$$

$$= \overline{E}\big[\Gamma_n \Phi_n \sum_x I(X_n = x) \mid \mathcal{Y}_n\big] = \mathfrak{q}_n(\Phi_n).$$

It follows that our quantities of interest are computed by

$$\mathfrak{q}_n(G_n^{\mathfrak{m}l\mathfrak{i}}) = \sum_x \langle \mathfrak{q}_n(G_n^{\mathfrak{m}l\mathfrak{i}} D_n I(X_n = x)), \mathbf{1} \rangle,$$

$$\mathfrak{q}_n(S_n^{l\mathfrak{i}}) = \textstyle\sum_x \langle \mathfrak{q}_n(S_n^{l\mathfrak{i}} D_n I(X_n = x)), \mathbf{1} \rangle.$$

References

[1] Aggoun, L., (2006) Filtering of a Partially Observed Inventory System in *Hidden Markov Models in Finance*, Eds. Rogemar Mamon and Robert J. Elliott, Springer's International Series in Operations Research and Management Science, Accepted.

[2] Aggoun, L., Benkherouf, L. and Tadj, L. (1997) A Hidden Markov Model For An Inventory System With Perishable Items. *Journal of Applied Mathematics and Stochastic Analysis*, Vol. 10, Number 4 423–430.

[3] Aggoun, L., Benkherouf, L. and Tadj, L. (2000) A Stochastic Jump Inventory Model with Deteriorating items. *Stochastic Analysis and Appl*, Vol. 18(1), 1–10.

[4] Aggoun, L., Benkherouf, L. (2002) M-ary Detection Of Markov Modulated Poisson Processes in Inventory Models. *Journal of Applied Mathematics and Computation* **132** pp.315-324.

[5] Aggoun, L., Benkherouf, L. and Benmerzouga, A. (2002) Recursive Estimation of Inventory Quality Classes Using Sampling. *Journal of Applied Mathematics and Decisions Sciences* Vol. 7, No. 4, 249–263.

[6] Aggoun, L., Elliott, R.J. (2004) *Measure Theory and Filtering: Introduction and Applications*, Cambridge University Press, Cambridge, UK.

[7] Al-Osh, M.N and Alzaid, A.A (1987) First order integer-valued autoregressive (INAR(1)) process. *J. Time Series Anal*, **8**, 261-275.

[8] Baum, L.E., Petrie, T. (1966), Statistical Inference for Probabilistic Functions of Finite State Markov Chains, *Annals of the Institute of Statistical Mathematics* 37 pp. 1554-1563.

[9] Bensoussan Alain, Metin Çakanyildirim, Suresh P. Sethi (2005) Partially Observed Inventory Systems: The case of Zero Balance Walk. Working paper SOM 200548, School of Management, University of Texas at Dallas,

[10] Bensoussan Alain, Metin Çakanyildirim, Suresh P. Sethi (2005) Optimal ordering Policies for Inventory Problems with Dynamic Information Delays. Working paper, School of Management, University of Texas at Dallas, TX.

[11] Bensoussan Alain, Metin Çakanyildirim, Suresh P. Sethi (2005). On the optimal control of partially observed inventory systems. To appear in the Comptes Rendus de l'Academie des Sciences.

[12] Bensoussan Alain, Metin Çakanyildirim, Suresh P. Sethi (2005). Optimality of base stock and (s; S) policies for inventory problems with information delays. Working paper SOM 200547, School of Management, University of Texas at Dallas, TX.

[13] Bensoussan Alain, Metin Çakanyildirim, Suresh P. Sethi (2005). A multiperiod newsvendor problem with partially observed demand. Working paper SOM 200550, School of Management, University of Texas at Dallas, TX.

[14] Beyer D. , Sethi S.P. and Taksar M.I. (1998). Inventory models with Markovian demands and cost functions of polynomial growth. Journal of Optimization Theory and Applications, Vol.98, No.2: 281-323.

[15] DeCroix, G. A., and Mookerjee, V. S. (1997), Purchasing Demand Information in a Stochastic-Demand Inventory, *European Journal of Operational Research* 102, 36-57

[16] Elliott, R. J., Aggoun, L., Moore, J. B. (1995) *Hidden Markov Models, Estimation and Control*, Springer Verlag.

[17] Karaesmen F., Buzacott J.A., and Dallery Y. (2002), Integrating Advance Order Information in Make-to-stock Production Systems, *IIE Transactions* 34, 649-662.

[18] McKenzie, E (1985) Some simple models for discrete variate time series. *Water Res Bull*, **21**, 645-650.

[19] Monahan G.E. (1982). A survey of partially observable markov decision processes. *Management Science* Vol.28 No.1: 1-16.

[20] Nahmias, S. (1982). Perishable inventory theory: *A review. Operations Research*, Vol.30, No.4: 680-708.

[21] Tarkan Tan, Refik Gullu, Nesim Erkip (2003) Optimal Inventory Policies under Imperfect Advance Demand Information.

[22] Treharne, J.T., and Sox, C.R. (2002), Adaptive Inventory Control for Non-Stationary Demand and Partial Information, *Management Science* 48, 607-624.

[23] Yano, C.A. and Lee, H.L. (1995). Lot sizing with random yields: A review. *Operations Research* Vol.43, No.2: 311-333.

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