PROPERTIES OF THE COMPLEX MATRIX VARIATE DIRICHLET DISTRIBUTION

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Abstract. In this paper, several properties of the complex matrix variate Dirichlet type I distribution are studied. Also, the asymptotic expansion of the probability density function of the complex matrix variate Dirichlet distribution is derived.

1 Introduction. Let X be an $m \times m$ random Hermitian positive definite matrix such that all its eigenvalues are in the open interval $(0,1)$. Then, X is said to have a complex matrix variate beta type I distribution with parameters (a_1, a_2), denoted as $X \sim \mathbb{C}B^I_m(a_1, a_2)$, if its p.d.f. is given by

$$\{\tilde{B}_m(a_1, a_2)\}^{-1} \det(X)^{a_1-m} \det(I_m - X)^{a_2-m}, \tag{1}$$

where $a_1 > m-1$, $a_2 > m-1$, $\tilde{B}_m(a_1,a_2) = \tilde{\Gamma}_m(a_1)\tilde{\Gamma}_m(a_2)/\tilde{\Gamma}_m(a_1+a_2)$ and

$$\tilde{\Gamma}(a) = \pi^{m(m-1)/2} \prod_{i=1}^{m} \Gamma(a-i+1), \mathrm{Re}(a) > m-1. \tag{2}$$

As an n matrix variate generalization of the density in (1), we define the complex matrix variate Dirichlet type I distribution as follows:

The $m \times m$ random Hermitian positive definite matrices $X_1, \ldots, X_n$ are said to have a complex matrix variate Dirichlet type I distribution with parameters $(a_1, \ldots, a_n;\, a_{n+1})$, denoted by $(X_1, \ldots, X_n) \sim \mathbb{C}D^I_m(a_1, \ldots, a_n; a_{n+1})$, if their joint p.d.f. is given by

$$\{\tilde{B}_m(a_1, \ldots, a_n, a_{n+1})\}^{-1} \prod_{i=1}^{n} \det(X_i)^{a_i-m} \det\left(I_m - \sum_{i=1}^{n} X_i\right)^{a_{n+1}-m} \tag{3}$$

where $I_m - \sum_{i=1}^n X_i$ is Hermitian positive definite, $a_i > m-1$, for $i = 1, \ldots, n+1$ and

$$\tilde{B}_m(a_1, \ldots, a_n, a_{n+1}) = \frac{\prod_{i=1}^{n+1} \tilde{\Gamma}_m(a_i)}{\tilde{\Gamma}_m(\sum_{i=1}^{n+1} a_i)}.$$

The complex matrix variate Dirichlet distributions have been defined and studied by several authors (see, for example, Troskie [5], Tan [4], Gupta and Nagar [2], and Cui, Gupta and Nagar [1]). An extensive review on the matrix variate Dirichlet distributions is available in Gupta and Nagar [3].

In this article, we derive certain properties including the asymptotic expansion of the complex matrix variate Dirichlet type I distribution.

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2 Properties. In this section we give various properties of the complex matrix variate Dirichlet type I distribution. First, we state the following notations and results that will be used in this and subsequent sections. Let $A = (a_{ij})$ be an $m \times m$ matrix of complex numbers. Then, A' denotes the transpose of A; $\bar{A}$ denotes conjugate of A; A^H denotes conjugate transpose of A; $\text{tr}(A) = a_{11} + \cdots + a_{mm}$; $\text{etr}(A) = \exp(\text{tr}(A))$; $\det(A) =$ determinant of A; $A = A^H > 0$ means that A is Hermitian positive definite and $A^{\frac{1}{2}}$ denotes the unique Hermitian positive definite square root of $A = A^H > 0$. Further, for the partition $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $\det(A_{11}) \neq 0$, the Schur complement of A_{11} is defined as $A_{22\cdot 1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Now, we derive several results on the complex matrix variate Dirichlet type I distribution.

Theorem 2.1 *Let $(X_1, \ldots, X_n) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$ and A be an $m \times m$ constant nonsingular complex matrix. Define $Z_i = AX_iA^H$, $i = 1, \ldots, n$. Then, the p.d.f. of $(Z_1, \ldots, Z_n)$ is given by*

$$\frac{\prod_{i=1}^n \det(Z_i)^{a_i - m} \det(AA^H - \sum_{i=1}^n Z_i)^{a_{n+1}-m}}{\tilde{B}_m(a_1, \ldots, a_n, a_{n+1}) \det(AA^H)^{\sum_{i=1}^{n+1} a_i - m}}, \tag{4}$$

where $Z_i = Z_i^H > 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^n Z_i < AA^H$.

Proof: Making the transformation $Z_i = AX_iA^H$, $i = 1, \ldots, n$ with the Jacobian

$$J(X_1, \ldots, X_n \to Z_1, \ldots, Z_n) = \det(AA^H)^{-mn}$$

in (3), we get the desired result. ■

The above distribution will be denoted by

$$(AX_1A^H, \ldots, AX_nA^H) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1}; AA^H).$$

Note that $\mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1}; I_m) \equiv \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$. Also, it is straightforward to show that if $(W_1, \ldots, W_n) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1}; B)$, then

$$(B^{-\frac{1}{2}}W_1B^{-\frac{1}{2}}, \ldots, B^{-\frac{1}{2}}W_nB^{-\frac{1}{2}}) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1}).$$

In the next theorem, it is shown that the complex matrix variate Dirichlet type I distribution is unitary invariant.

Theorem 2.2 *Let $(X_1, \ldots, X_n) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$ and U be an $m \times m$ unitary matrix, whose elements are either constants or random variables distributed independently of $(X_1, \ldots, X_n)$. Then, the distribution of $(X_1, \ldots, X_n)$ is unitary invariant under the transformation $X_i \to UX_iU^H$, $i = 1, \ldots, n$ and is independent of U in the latter case.*

Proof: First, let U be a constant unitary matrix. Then, using Theorem 2.1, we have $(UX_1U^H, \ldots, UX_nU^H) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$ since $UU^H = I_m$. If, however, U is a random unitary matrix, then $(UX_1U^H, \ldots, UX_nU^H)|U \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$. Since this distribution does not depend on U it is also the unconditional distribution, and the proof is complete. ■

Theorem 2.3 *If $(X_1, \ldots, X_n) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$, then, for $1 \le i \le n$,*

$$\left(X_1, \ldots, X_{i-1}, I_m - \sum_{r=1}^n X_r, X_{i+1}, \ldots, X_n\right) \sim \mathbb{C}D_m^I(a_1, \ldots, a_{i-1}, a_{n+1}, a_{i+1}, \ldots, a_n; a_i).$$

Proof: The transformation $Y_k = X_k$, $k = 1, \ldots, i-1, i+1, \ldots, n$ and $Y_i = I_m - \sum_{r=1}^{n} X_r$ with the Jacobian $J(X_1, \ldots, X_n \to Y_1, \ldots, Y_n) = 1$ in the density (3) yields the desired result. ■

Theorem 2.4 *Let* $(X_1, \ldots, X_n) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$ *and define*

$$W_j = \left(I_m - \sum_{i=s+1}^{n} X_i\right)^{-\frac{1}{2}} X_j \left(I_m - \sum_{i=s+1}^{n} X_i\right)^{-\frac{1}{2}}, \; j = 1, \ldots, s.$$

Then, $(W_1, \ldots, W_s)$ *and* $(X_{s+1}, \ldots, X_n)$ *are independent,* $(W_1, \ldots, W_s) \sim \mathbb{C}D_m^I(a_1, \ldots, a_s; a_{n+1})$*, and* $(X_{s+1}, \ldots, X_n) \sim \mathbb{C}D_m^I(a_{s+1}, \ldots, a_n; \sum_{i=1}^{s} a_i + a_{n+1})$.

Proof: Transforming $W_i = \left(I_m - \sum_{i=s+1}^{n} X_i\right)^{-\frac{1}{2}} X_i \left(I_m - \sum_{i=s+1}^{n} X_i\right)^{-\frac{1}{2}}, i = 1, \ldots, s$ with the Jacobian $J(X_1, \ldots, X_s \to W_1, \ldots, W_s) = \det(I_m - \sum_{i=s+1}^{n} X_i)^{ms}$, in the density of $(X_1, \ldots, X_n)$, we get the desired result. ■

Corollary 2.4.1 *Let* $(X_1, \ldots, X_n) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$ *and define*

$$Z_r = \left(I_m - \sum_{i=r+1}^{n} X_i\right)^{-\frac{1}{2}} X_r \left(I_m - \sum_{i=r+1}^{n} X_i\right)^{-\frac{1}{2}}, \; r = 1, \ldots, n-1.$$

Then, $Z_1, \ldots, Z_{n-1}$ *and* X_n *are mutually independent,* $Z_i \sim \mathbb{C}B_m^I(a_i, a_{n+1})$, $i = 1, \ldots, n-1$ *and* $X_n \sim \mathbb{C}B_m^I(a_n, \sum_{i=1}^{n-1} a_i + a_{n+1})$.

Using (3), the $(h_1, \ldots, h_n)^{\text{th}}$ mixed "moment" is derived as

$$E[\det(X_1)^{h_1} \cdots \det(X_n)^{h_n}] = \frac{\tilde{\Gamma}_m(\sum_{i=1}^{n+1} a_i)}{\tilde{\Gamma}_m(\sum_{i=1}^{n+1} a_i + \sum_{i=1}^{n} h_i)} \prod_{i=1}^{n} \frac{\tilde{\Gamma}_m(a_i + h_i)}{\tilde{\Gamma}_m(a_i)}.$$

if $a_i + h_i > m - 1$, $i = 1, \ldots, n$, and does not exist otherwise. The means, variances and the covariances are obtained as

$$E[\det(X_i)] = \prod_{r=1}^{m} \frac{(a_i - r + 1)}{(\sum_{i=1}^{n+1} a_i - r + 1)}, i = 1, \ldots, n,$$

$$\text{Var}[\det(X_i)] = \frac{m \sum_{k=1(\neq i)}^{n} a_k}{(\sum_{i=1}^{n+1} a_i + 1)(a_i - m + 1)} \prod_{r=1}^{m} \frac{(a_i - r + 1)^2}{(\sum_{i=1}^{n+1} a_i - r + 1)^2}, i = 1, \ldots, n,$$

$$\text{Cov}[\det(X_i), \det(X_j)] = -\frac{m}{\sum_{i=1}^{n+1} a_i + 1} \prod_{r=1}^{m} \frac{(a_i - r + 1)(a_j - r + 1)}{(\sum_{i=1}^{n+1} a_i - r + 1)^2}, \quad i \neq j, i, j = 1, \ldots, n.$$

In the next theorem, we derive the joint p.d.f. of partial sums of random matrices distributed jointly as complex matrix variate Dirichlet type I.

Theorem 2.5 *Let* $(X_1, \ldots, X_n) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$ *and define*

$$X_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} X_j, \; a_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} a_j, \; n_0^* = 0, \; n_i^* = \sum_{j=1}^{i} n_j,$$

$$W_j = X_{(i)}^{-\frac{1}{2}} X_j X_{(i)}^{-\frac{1}{2}},\ j = n_{i-1}^*+1,\dots,n_i^*-1,\ i=1,\dots,\ell.$$

Then, $(W_{n_{i-1}^*+1},\dots,W_{n_i^*-1})$, $i=1,\dots,\ell$ *and* $(X_{(1)},\dots,X_{(\ell)})$ *are independently distributed. Further, for* $i=1,\dots,\ell$, $(W_{n_{i-1}^*+1},\dots,W_{n_i^*-1}) \sim \mathbb{C}D_m^I(a_{n_{i-1}^*+1},\dots,a_{n_i^*-1};a_{n_i^*})$, *and*
$(X_{(1)},\dots,X_{(\ell)}) \sim \mathbb{C}D_m^I(a_{(1)},\dots,a_{(\ell)};a_{n+1})$.

Proof: Make the transformation

$$X_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} X_j,\ W_j = X_{(i)}^{-\frac{1}{2}} X_j X_{(i)}^{-\frac{1}{2}},\ j = n_{i-1}^*+1,\dots,n_i^*-1,\ i=1,\dots,\ell. \tag{5}$$

The Jacobian of this transformation is given by

$$\begin{aligned} J(X_1,\dots,X_n &\to W_1,\dots,W_{n_1-1},X_{(1)},\dots,W_{n_{\ell-1}^*+1},\dots,W_{n-1},X_{(\ell)}) \\ &= \prod_{i=1}^{\ell} J(X_{n_{i-1}^*+1},\dots,X_{n_i^*} \to W_{n_{i-1}^*+1},\dots,W_{n_i^*-1},X_{(i)}) \\ &= \prod_{i=1}^{\ell} \det(X_{(i)})^{m(n_i-1)}. \end{aligned} \tag{6}$$

Now, substituting from (5) and (6) in the joint density of $(X_1,\dots,X_n)$ given by (3), we get the joint density of $W_{n_{i-1}^*+1},\dots,W_{n_i^*-1},X_{(i)}$, $i=1,\dots,\ell$ as

$$\begin{aligned} &\{\tilde{B}_m(a_1,\dots,a_n,a_{n+1})\}^{-1} \prod_{i=1}^{\ell} \det(X_{(i)})^{a_{(i)}-m} \det\left(I_m - \sum_{i=1}^{\ell} X_{(i)}\right)^{a_{n+1}-m} \\ &\times \prod_{i=1}^{\ell}\left[\prod_{j=n_{i-1}^*+1}^{n_i^*-1} \det(W_j)^{a_j-m} \det\left(I_m - \sum_{j=n_{i-1}^*+1}^{n_i^*-1} W_j\right)^{a_{n_i^*}-m}\right], \end{aligned} \tag{7}$$

where $X_{(i)} = X_{(i)}^H > 0$, $\sum_{i=1}^{\ell} X_{(i)} < I_m$, $W_j = W_j^H > 0$, $j = n_{i-1}^*+1,\dots,n_i^*-1$, $\sum_{j=n_{i-1}^*+1}^{n_i^*-1} W_j < I_m$, $i=1,\dots,\ell$. From (7), it is straightforward to see that $(X_{(1)},\dots,X_{(\ell)})$ and $(W_{n_{i-1}^*+1},\dots,W_{n_i^*-1})$, $i=1,\dots,\ell$, are independently distributed. Further, $(X_{(1)},\dots,X_{(\ell)}) \sim \mathbb{C}D_m^I(a_{(1)},\dots,a_{(\ell)};a_{n+1})$ and for $i=1,\dots,\ell$, $(W_{n_{i-1}^*+1},\dots,W_{n_i^*-1}) \sim \mathbb{C}D_m^I(a_{n_{i-1}^*+1},\dots,a_{n_i^*-1};a_{n_i^*})$. ■

When $\ell=1$, $\sum_{i=1}^n X_i \sim \mathbb{C}B_m^I(\sum_{i=1}^n a_i, a_{n+1})$.

Next, we state results on marginal and conditional distributions of Dirichlet type I random matrices that will be used to obtain several distributional results.

Theorem 2.6 *Let* $(X_1,\dots,X_n) \sim \mathbb{C}D_{m_1+m_2}^I(a_1,\dots,a_n,a_{n+1})$ *and* X_i *be partitioned as*

$$X_i = \begin{pmatrix} X_{11(i)} & X_{12(i)} \\ X_{12(i)}^H & X_{22(i)} \end{pmatrix}, X_{11(i)}\,(m_1\times m_1), i=1,\dots,n.$$

Then, (i) $(X_{11(1)},\dots,X_{11(n)})$ *and* $(X_{22\cdot1(1)},\dots,X_{22\cdot1(n)})$ *are distributed independently. Further,*

$$(X_{11(1)},\dots,X_{11(n)}) \sim \mathbb{C}D_{m_1}^I(a_1,\dots,a_n;a_{n+1}),$$

and

$$(X_{22\cdot 1(1)},\dots,X_{22\cdot 1(n)}) \sim \mathbb{C}D^I_{m_2}(a_1-m_1,\dots,a_n-m_1;a_{n+1}+(n-1)m_1).$$

(ii) $(X_{22(1)},\dots,X_{22(n)})$ *and* $(X_{11\cdot 2(1)},\dots,X_{11\cdot 2(n)})$ *are distributed independently. Further,*

$$(X_{22(1)},\dots,X_{22(n)}) \sim \mathbb{C}D^I_{m_2}(a_1,\dots,a_n;a_{n+1}),$$

and

$$(X_{11\cdot 2(1)},\dots,X_{11\cdot 2(n)}) \sim \mathbb{C}D^I_{m_1}(a_1-m_2,\dots,,a_n-m_2;a_{n+1}+(n-1)m_2).$$

Proof: See Tan [4] ■

The distributions of $(AX_1A^H,\dots,AX_nA^H)$ and $((AX_1^{-1}A^H)^{-1},\dots,(AX_n^{-1}A^H)^{-1})$ where $A\,(q\times m)$ is a constant matrix of rank $q\,(\le m)$, are now derived.

Theorem 2.7 *Let* $(X_1,\dots,X_n)\sim \mathbb{C}D^I_m(a_1,\dots,a_n;a_{n+1})$. *Then, for a complex constant matrix* $A\,(q\times m)$ *of rank* $q\,(\le m)$, $(AX_1A^H,\dots,AX_nA^H)\sim \mathbb{C}D^I_q(a_1,\dots,a_n;a_{n+1};AA^H)$.

Proof: Write $A=M\left(I_q\ 0\right)G$, where $M\,(q\times q)$ and $G\,(m\times m)$ are complex nonsingular and unitary matrices, respectively. Now, for $i=1,\dots,n$,

$$AX_iA^H = M\left(I_q\ 0\right)GX_iG^H\left(I_q\ 0\right)^H M^H = MZ_{11(i)}M^H,$$

where $Z_i=GX_iG^H$ and $Z_{11(i)}\,(q\times q)$ is the first principal diagonal block of Z_i. From Theorem 2.2 and Theorem 2.6, we know that $(Z_1,\dots,Z_n)\sim\mathbb{C}D^I_m(a_1,\dots,a_n;a_{n+1})$ and $(Z_{11(1)},\dots,Z_{11(n)})\sim\mathbb{C}D^I_q(a_1,\dots,a_n;a_{n+1})$. Hence, using Theorem 2.1,

$$(MZ_{11(1)}M^H,\dots,MZ_{11(n)}M^H)\sim\mathbb{C}D^I_q(a_1,\dots,a_n;a_{n+1};MM^H)$$

and the result follows by noting that $AX_iA^H=MZ_{11(i)}M^H$, $i=1,\dots,n$ and $MM^H=AA^H$. ■

Corollary 2.7.1 *Let* $(X_1,\dots,X_n)\sim\mathbb{C}D^I_m(a_1,\dots,a_n;a_{n+1})$ *and* $\mathbf{c}\in\mathbb{C}^m$, $\mathbf{c}\neq\mathbf{0}$, *then*

$$\left(\frac{\mathbf{c}^HX_1\mathbf{c}}{\mathbf{c}^H\mathbf{c}},\dots,\frac{\mathbf{c}^HX_n\mathbf{c}}{\mathbf{c}^H\mathbf{c}}\right)\sim D^I(a_1,\dots,a_n;a_{n+1}).$$

Proof: Take $q=1$ in Theorem 2.7. ■

The Dirichlet type I distribution designated by $D^I(a_1,\dots,a_n;a_{n+1})$ used in the above corollary is defined by the p.d.f.

$$\frac{\Gamma(\sum_{i=1}^{n+1}a_i)}{\prod_{i=1}^{n+1}\Gamma(a_i)}\prod_{i=1}^n x_i^{a_i-1}\left(1-\sum_{i=1}^n x_i\right)^{a_{n+1}-1},$$

where $x_i>0$, $i=1,\dots,n,\sum_{i=1}^n x_i<1$ and $a_i>0$, $i=1,\dots,n+1$.

In Corollary 2.7.1 the distribution of $\left(\frac{\mathbf{c}^HX_1\mathbf{c}}{\mathbf{c}^H\mathbf{c}},\dots,\frac{\mathbf{c}^HX_n\mathbf{c}}{\mathbf{c}^H\mathbf{c}}\right)$ does not depend on $\mathbf{c}$. Thus if $\mathbf{z}\,(m\times 1)$ is a complex random vector, independent of $(X_1,\dots,X_n)$, and $P(\mathbf{z}\neq\mathbf{0})=1$, then it follows that

$$\left(\frac{\mathbf{z}^HX_1\mathbf{z}}{\mathbf{z}^H\mathbf{z}},\dots,\frac{\mathbf{z}^HX_n\mathbf{z}}{\mathbf{z}^H\mathbf{z}}\right)\sim D^I(a_1,\dots,a_n;a_{n+1}).$$

Theorem 2.8 *Let $A\,(q\times m)$ be a complex constant matrix of rank $q\,(\le m)$. If $(X_1,\dots,X_n) \sim \mathbb{C}D_m^I(a_1,\dots,a_n;a_{n+1})$, then*

$$((AX_1^{-1}A^H)^{-1},\dots,(AX_n^{-1}A^H)^{-1}) \sim \mathbb{C}D_q^I(a_1-m+q,\dots,a_n-m+q;\; a_{n+1}+(n-1)(m-q);(AA^H)^{-1}).$$

Proof: Write $A = M\begin{pmatrix}I_q & 0\end{pmatrix}G$, where $M\,(q\times q)$ is complex nonsingular and $G\,(m\times m)$ is unitary. Now, for $i=1,\dots,n$,

$$\begin{aligned}(AX_i^{-1}A^H)^{-1} &= [M\begin{pmatrix}I_q & 0\end{pmatrix}GX_i^{-1}G^H\begin{pmatrix}I_q & 0\end{pmatrix}^H M^H]^{-1}\\ &= (M^H)^{-1}\left[\begin{pmatrix}I_q & 0\end{pmatrix}Z_i^{-1}\begin{pmatrix}I_q\\ 0\end{pmatrix}\right]^{-1}M^{-1}\\ &= (M^H)^{-1}(Z_i^{11})^{-1}M^{-1},\end{aligned}$$

where $Z_i = GX_iG^H = \begin{pmatrix}Z_{11(i)} & Z_{12(i)}\\ Z_{21(i)} & Z_{22(i)}\end{pmatrix}$, $Z_{11(i)}\,(q\times q)$, and $Z_i^{11} = Z_{11\cdot2(i)}^{-1}$, $i=1,\dots,n$. Note that $(Z_1,\dots,Z_n)\sim\mathbb{C}D_m^I(a_1,\dots,a_n;a_{n+1})$. Hence, from Theorem 2.6,

$$(Z_{11\cdot2(1)},\dots,Z_{11\cdot2(n)})\sim\mathbb{C}D_q^I(a_1-m+q,\dots,a_n-m+q;a_{n+1}+(n-1)(m-q))$$

and from Theorem 2.1,

$$((M^H)^{-1}Z_{11\cdot2(1)}M^{-1},\dots,(M^H)^{-1}Z_{11\cdot2(n)}M^{-1}) \sim\mathbb{C}D_q^I(a_1-m+q,\dots,a_n-m+q;a_{n+1}+(n-1)(m-q);(MM^H)^{-1}).$$

The proof of is now completed by observing that $(AX_i^{-1}A^H)^{-1} = (M^H)^{-1}Z_{11\cdot2(i)}M^{-1}$, $i=1,\dots,n$ and $MM^H = AA^H$. ∎

From the above theorem, when $\mathbf{c}\in\mathbb{C}^m$, $\mathbf{c}\ne\mathbf{0}$, it follows that

$$\left(\frac{\mathbf{c}^H\mathbf{c}}{\mathbf{c}^HX_1^{-1}\mathbf{c}},\dots,\frac{\mathbf{c}^H\mathbf{c}}{\mathbf{c}^HX_n^{-1}\mathbf{c}}\right)\sim D^I(a_1-m+1,\dots,a_n-m+1;a_{n+1}+(n-1)(m-1)).$$

Further, if $\mathbf{z}\,(m\times1)$ is a complex random vector independent of $(X_1,\dots,X_n)$, and $P(\mathbf{z}\ne\mathbf{0})=1$, then

$$\left(\frac{\mathbf{z}^H\mathbf{z}}{\mathbf{z}^HX_1^{-1}\mathbf{z}},\dots,\frac{\mathbf{z}^H\mathbf{z}}{\mathbf{z}^HX_n^{-1}\mathbf{z}}\right)\sim D^I(a_1-m+1,\dots,a_n-m+1;a_{n+1}+(n-1)(m-1)).$$

Finally, substituting $n=1$ in the results derived for the complex matrix variate Dirichlet type I distribution, we obtain following interesting properties of the complex matrix variate beta type I distribution. We assume that $X\sim\mathbb{C}B_m^I(a_1,a_2)$.

1. Let A be an $m\times m$ constant nonsingular complex matrix. Then, the distribution of $Z = AXA^H$, denoted by $Z\sim\mathbb{C}B_m^I(a_1,a_2;AA^H)$, is given by the following p.d.f.

$$\frac{\det(Z)^{a_1-m}\det(AA^H-Z)^{a_2-m}}{\tilde{B}_m(a_1,a_2)\det(AA^H)^{a_1+a_2-m}}, 0<Z=Z^H<AA^H.$$

2. If $W\sim\mathbb{C}B_m^I(a_1,a_2;B)$, then $B^{-\frac12}WB^{-\frac12}\sim\mathbb{C}B_m^I(a_1,a_2)$.

3. Let $U\,(m \times m)$ be an unitary matrix, whose elements are either constants or random variables distributed independently of X. Then, the distribution of X is unitary invariant under the transformation $X \rightarrow UXU^H$, and is independent of U in the latter case.

4. For a complex constant matrix $A\,(q \times m)$ of rank $q\,(\leq m)$, $AXA^H \sim \mathbb{C}B_q^I(a_1, a_2; AA^H)$ and $(AX^{-1}A^H)^{-1} \sim \mathbb{C}B_q^I(a_1 - m + q, a_2; (AA^H)^{-1})$.

5. If $\mathbf{c} \in \mathbb{C}^m$, $\mathbf{c} \neq \mathbf{0}$, then $\frac{\mathbf{c}^H X \mathbf{c}}{\mathbf{c}^H \mathbf{c}} \sim B^I(a_1, a_2)$ and $\frac{\mathbf{c}^H \mathbf{c}}{\mathbf{c}^H X^{-1} \mathbf{c}} \sim B^I(a_1 - m + 1, a_2)$. Further, if $\mathbf{z}\,(m \times 1)$ is a complex random vector independent of X and $P(\mathbf{z} \neq \mathbf{0}) = 1$, then $\frac{\mathbf{z}^H X \mathbf{z}}{\mathbf{z}^H \mathbf{z}} \sim B^I(a_1, a_2)$ and $\frac{\mathbf{z}^H \mathbf{z}}{\mathbf{z}^H X^{-1} \mathbf{z}} \sim B^I(a_1 - m + 1, a_2)$.

The univariate beta type I distribution denoted by $B^I(a_1, a_2)$ is defined by the p.d.f.

$$\frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} x^{a_1 - 1}(1 - x)^{a_2 - 1}, 0 < x < 1.$$

The expectations of X and X^{-1} can easily be obtained using above results. For any fixed $\mathbf{c} \in \mathbb{C}^{m \times 1}$, $\mathbf{c} \neq \mathbf{0}$, we know that $\frac{\mathbf{c}^H X \mathbf{c}}{\mathbf{c}^H \mathbf{c}} \sim B^I(a_1, a_2)$ and $\frac{\mathbf{c}^H \mathbf{c}}{\mathbf{c}^H X^{-1} \mathbf{c}} \sim B^I(a_1 - m + 1, a_2)$ so that

$$E(\mathbf{c}^H X \mathbf{c}) = E(u_1)\mathbf{c}^H \mathbf{c} \text{ and } E(\mathbf{c}^H X^{-1} \mathbf{c}) = E\left(\frac{1}{u_2}\right)\mathbf{c}^H \mathbf{c}$$

where $u_1 \sim B^I(a_1, a_2), u_2 \sim B^I(a_1 - m + 1, a_2)$. Hence, for all $\mathbf{c} \in \mathbb{C}^{m \times 1}$,

$$\mathbf{c}^H E(X)\mathbf{c} = \left(\frac{a_1}{a_1 + a_2}\right)\mathbf{c}^H \mathbf{c}, a_1 > m - 1, a_2 > m - 1,$$

and

$$\mathbf{c}^H E(X^{-1})\mathbf{c} = \left(\frac{a_1 + a_2 - m}{a_1 - m}\right)\mathbf{c}^H \mathbf{c}, a_1 > m, a_2 > m - 1,$$

which imply that

$$E(X) = \left(\frac{a_1}{a_1 + a_2}\right) I_m, a_1 > m - 1, a_2 > m - 1, \tag{8}$$

and

$$E(X^{-1}) = \left(\frac{a_1 + a_2 - m}{a_1 - m}\right) I_m, a_1 > m, a_2 > m - 1. \tag{9}$$

Since, for a complex constant matrix $A\,(q \times m)$ of rank $q\,(\leq m)$,

$$(AA^H)^{-\frac{1}{2}} AXA^H (AA^H)^{-\frac{1}{2}} \sim \mathbb{C}B_q^I(a_1, a_2)$$

and

$$(AA^H)^{\frac{1}{2}} (AX^{-1}A^H)^{-1} (AA^H)^{\frac{1}{2}} \sim \mathbb{C}B_q^I(a_1 - m + q, a_2)$$

we obtain from (8) and (9),

$$E(AXA^H) = \left(\frac{a_1}{a_1 + a_2}\right) AA^H, a_1 > m - 1, a_2 > m - 1,$$

$$E(AXA^H)^{-1} = \left(\frac{a_1 + a_2 - q}{a_1 - q}\right) (AA^H)^{-1}, a_1 > \max\{m - 1, q\}, a_2 > m - 1,$$

$$E(AX^{-1}A^H)^{-1} = \left(\frac{a_1 - m + q}{a_1 + a_2 - m + q}\right)(AA^H)^{-1}, a_1 > m-1, a_2 > m-1,$$

and

$$E(AX^{-1}A^H) = \left(\frac{a_1 + a_2 - m}{a_1 - m}\right) AA^H, a_1 > m, a_2 > m-1.$$

3 Asymptotic Expansion. In this section we derive the asymptotic expansion of the probability density function of the complex Dirichlet type I random matrices. We first give three lemmas needed to derive the final result.

Lemma 3.1 *For $|\arg(z)| \le \pi - \epsilon, \epsilon > 0$, the logarithm of $\Gamma(z+c)$ can be expanded as*

$$\begin{aligned}\ln\Gamma(z+c) \,=\, & (z + c - .5)\ln z - z + \ln\sqrt{2\pi} \\ & + \sum_{s=1}^{r} \frac{(-1)^{s+1} B_{s+1}(c)}{s(s+1)} z^{-s} + O(z^{-r-1}),\end{aligned}$$

where $B_k(x)$ is the Bernoulli polynomial of degree k and order unity.

Lemma 3.2 *For c_1, c_2 scalars, we have*

$$\begin{aligned}\ln\left[\frac{\tilde{\Gamma}_m(z+c_1)}{\tilde{\Gamma}_m(z+c_2)}\right] \,=\, & (c_1 - c_2)m\ln z \\ & + \sum_{i=1}^{m}\sum_{s=1}^{r} \frac{(-1)^{s+1}}{s(s+1)} \left[B_{s+1}\left(c_1 - i + 1\right) - B_{s+1}\left(c_2 - i + 1\right)\right] z^{-s} \\ & + O(z^{-r-1}), |\arg(z)| \le \pi - \epsilon, \epsilon > 0\end{aligned}$$

where $B_k(x)$ is the Bernoulli polynomial of degree k and order unity.

Proof: Writing complex multivariate gamma functions in terms of ordinary gamma functions using (2), one obtains

$$\frac{\tilde{\Gamma}_m(z+c_1)}{\tilde{\Gamma}_m(z+c_2)} = \prod_{i=1}^{m} \frac{\Gamma(z + c_1 - i + 1)}{\Gamma(z + c_2 - i + 1)}. \tag{10}$$

Now, taking logarithm of the above expression and using Lemma 3.1, one gets the desired result. ■

Lemma 3.3 *For $\max_{1\le i\le n}|\lambda_i| < 1$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the matrix Z/n,*

$$-\ln\det\left(I_m - \frac{Z}{n}\right) = \sum_{s=1}^{r} \frac{n^{-s}\operatorname{tr}(Z^s)}{s} + O(n^{-r-1}).$$

Theorem 3.1 *Let $(X_1, \ldots, X_n) \sim \mathbb{C}D_m^I(a_1, \ldots, a_n; a_{n+1})$ and define $W_i = a_{n+1}X_i$, $i = 1, \ldots, n$. Then, the p.d.f. of $(W_1, \ldots, W_n)$ can be expressed as*

$$\left[\prod_{i=1}^{n} \frac{\det(W_i)^{a_i - m}}{\tilde{\Gamma}_m(a_i)}\right] \operatorname{etr}\left(-\sum_{i=1}^{n} W_i\right) \left[1 + \frac{\tilde{d}_1}{2a_{n+1}} + \frac{3\tilde{d}_1^2 + 4\tilde{d}_2}{24a_{n+1}^2} + O(a_{n+1}^{-3})\right], \tag{11}$$

where $W_i = W_i^H > 0$, $i = 1, \ldots, n$, $\tilde{d}_1 = -\operatorname{tr}(\sum_{i=1}^n W_i)^2 + 2m\operatorname{tr}(\sum_{i=1}^n W_i) + am(a-m)$, $\tilde{d}_2 = -2\operatorname{tr}\left(-\sum_{i=1}^n W_i\right)^3 + 3m\operatorname{tr}\left(\sum_{i=1}^n W_i\right)^2 - (1/2)am(2a^2 - 3am + 2m^2 - 1)$ and $a = \sum_{i=1}^n a_i$.

Proof: Substituting $W_i = a_{n+1}X_i$, $i = 1, 2, \dots, n$, with $J(X_1, \dots, X_n \to W_1, \dots, W_n) = a_{n+1}^{-nm^2}$ in (3), we obtain the p.d.f. of $(W_1, \dots, W_n)$ as

$$\left[\prod_{i=1}^{n} \frac{\det(W_i)^{a_i-m}}{\tilde{\Gamma}_m(a_i)}\right] \mathcal{I}_1\mathcal{I}_2,\ W_i = W_i^H > 0,\ i = 1, \dots, n, \tag{12}$$

where

$$\mathcal{I}_1 = \frac{\tilde{\Gamma}_m(\sum_{i=1}^{n+1} a_i)}{\tilde{\Gamma}_m(a_{n+1})} a_{n+1}^{-m\sum_{i=1}^{n} a_i},$$

$$\mathcal{I}_2 = \det\left(I_m - \frac{W}{a_{n+1}}\right)^{a_{n+1}-m} \text{ with } W = \sum_{i=1}^{n} W_i.$$

Now, using Lemma 3.2 with $r = 2, z = a_{n+1}, c_1 = a$ and $c_2 = 0$, we obtain

$$\begin{aligned}\ln \mathcal{I}_1 \ =\ & \frac{1}{2a_{n+1}} \sum_{i=1}^{m} [B_2\,(a - i + 1) - B_2\,(1 - i)] \\ & - \frac{1}{6a_{n+1}^2} \sum_{i=1}^{m} [B_3\,(a - i + 1) - B_3\,(1 - i)] + O(a_{n+1}^{-3})\end{aligned}$$

where $B_2(x) = x^2 - x + 1/6$ and $B_3(x) = x^3 - 3x^2/2 + x/2$. Now, substituting for $B_2(\cdot)$ and $B_3(\cdot)$ in the above expression and simplifying, the above expression is re-written as

$$\ln \mathcal{I}_1 \ =\ \frac{am(a-m)}{2a_{n+1}} - \frac{am(2a^2 - 3am + 2m^2 - 1)}{12\,a_{n+1}^2} + O(a_{n+1}^{-3}). \tag{13}$$

Further, application of Lemma 3.3 yields

$$\begin{aligned}\ln \mathcal{I}_2 \ =\ & \operatorname{tr}(-W) + \frac{1}{2a_{n+1}}[2m \operatorname{tr}(W) - \operatorname{tr}(W^2)] \\ & + \frac{1}{6a_{n+1}^2}[3m \operatorname{tr}(W^2) - 2\operatorname{tr}(W^3)] + O(a_{n+1}^{-3}).\end{aligned} \tag{14}$$

Therefore, using (13) and (14) we obtain

$$\ln \mathcal{I}_1 + \ln \mathcal{I}_2 = \operatorname{tr}(-W) + \frac{\tilde{d}_1}{2a_{n+1}} + \frac{\tilde{d}_2}{6a_{n+1}^2} + O(a_{n+1}^{-3})$$

where $\tilde{d}_1$ and $\tilde{d}_2$ are given in the Theorem 3.1. Hence we get

$$\mathcal{I}_1\mathcal{I}_2 = \operatorname{etr}(-W)\left[1 + \frac{\tilde{d}_1}{2a_{n+1}} + \frac{3\tilde{d}_1^2 + 4\tilde{d}_2}{24a_{n+1}^2} + O(a_{n+1}^{-3})\right]. \tag{15}$$

Finally, substituting from (15) in (12) we get the desired result. ■

The expression (11) may be used to yield a corresponding asymptotic formula for the c.d.f. of $(X_1, \dots, X_n)$, *i.e.*,

$$P_n(A_1, \dots, A_n; a_1, \dots, a_n; a_{n+1}) = P_n(0 < X_1 < A_1, \dots, 0 < X_n < A_n)$$

where $A_1, \dots, A_n$ are Hermitian positive definite matrices. Writing $B_i = a_{n+1}A_i$, $i = 1, 2, \dots, n$ we have

$$
\begin{aligned}
(16)\qquad & P_n(A_1, \dots, A_n; a_1, \dots, a_n; a_{n+1}) \\
&= P_n(0 < W_1 < B_1, \dots, 0 < W_n < B_n) \\
&= \int_{0<W_1<B_1} \cdots \int_{0<W_n<B_n} \left[\prod_{i=1}^{n} \frac{\det(W_i)^{a_i-m}}{\tilde{\Gamma}_m(a_i)}\right] \exp\left(-\sum_{i=1}^{n} W_i\right) \\
&\quad \times \left[1 + \frac{\tilde{d}_1}{2a_{n+1}} + \frac{3\tilde{d}_1^2 + 4\tilde{d}_2}{24a_{n+1}^2} + O(a_{n+1}^{-3})\right] dW_1 \cdots dW_n
\end{aligned}
$$

where $B_1, \dots, B_n$ are Hermitian positive definite matrices. It is seen that each term in (16) is a combination of the functions

$$
\begin{aligned}
(17)\qquad & \tilde{G}_{\alpha,K_1,K_2}(B_1, \dots, B_n) \\
&= \int_{0<W_1<B_1} \cdots \int_{0<W_n<B_n} \left[\prod_{i=1}^{n} \frac{\det(W_i)^{a_i-m}}{\tilde{\Gamma}_m(a_i)}\right] \exp\left(-\sum_{i=1}^{n} W_i\right) \\
&\quad \times \left[\operatorname{tr}\left(-\sum_{i=1}^{n} W_i\right)^{\alpha}\right]^{K_1} \left[\operatorname{tr}\left(-\sum_{i=1}^{n} W_i\right)\right]^{K_2} dW_1 \cdots dW_n.
\end{aligned}
$$

The integral on the right-hand side of (17) does not seem to be easy to evaluate. Further work on this will be reported elsewhere.

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