THEORIES OF ORDERED COMMUTATIVE MONOIDS

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ABSTRACT. In this paper, we study some theories of lexicographic products of ordered commutative monoids. In particular we show that the lexicographic product of the ordered commutative monoid of nonnegative integers and the ordered commutative monoid of nonnegative rational numbers admits elimination of quantifiers in some expansive language of the language of ordered monoids.

1. INTRODUCTION

Let $L_{og} := \{0, +, <\}$, $L_0 := L_{og} \cup \{n \mid x : n > 0, n \in \mathbb{N}\} \cup \{1\}$ and $\mathbb{Q}_{\geq 0} := \{a \in \mathbb{Q} : a \geq 0\}$.

It is well-known that the ordered commutative monoid $\mathbb{N}$ and the ordered abelian group $\mathbb{Z}$ admit elimination of quantifiers in the language L_0, where $n \mid x$ means 'n divides x'; see for example [4] or [10]. In [3] and [11], Komori and Weispfenning independently showed that the lexicographically ordered abelian group $\mathcal{M} := \mathbb{Z} \times \mathbb{Q}$ admits elimination of quantifiers in the same language L_0; here $0^{\mathcal{M}} := \langle 0, 0\rangle$ and $1^{\mathcal{M}} := \langle 1, 0\rangle$. In [8], the author showed the converse of them.

Let $\mathcal{L}$ be an expansion of L_{og}. Suppose that H is an $\mathcal{L}$-structure whose reduct to the language L_{og} is an ordered abelian group and K is an ordered divisible abelian group. Then, extending the result of Komori and Weispfenning, Suzuki [7] showed that if H admits elimination of quantifiers in $\mathcal{L}$ and the set $\{0\} \times K$ is defined by some quantifier-free $\mathcal{L}$-formula in the lexicographic product $G := H \times K$, then G admits elimination of quantifiers in $\mathcal{L}$. In [9], the author and Yokoyama showed the converse of it. However, the lexicographically ordered commutative monoid $\mathcal{N} := \mathbb{N} \times \mathbb{Q}_{\geq 0}$ dose not admit elimination of quantifiers in L_0, where $0^{\mathcal{N}} := \langle 0, 0\rangle$ and $1^{\mathcal{N}} := \langle 1, 0\rangle$ (Lemma 3.1).

In section 2, we give some axioms for ordered commutative monoids.

In section 3, we show that the lexicographically ordered commutative monoid $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ admits elimination of quantifiers in the language L, where the language L is the union of L_0, a unary relation symbol $R(x)$ and binary relation symbols $E_1(x, y), E_2(x, y)$. By Definition 2.1 we notice that the language L is a definable expansion of L_0. We also show the converse of it.

In [3] and [11], Komori and Weispfenning studied model completions of theories of ordered abelian groups. In section 4, we study model completions of theories of ordered commutative monoids.

In [2], Belegradek, Verbovskiy and Wagner showed that the algebraic closure in $\mathrm{Th}_{L_0}(\mathbb{Z} \times \mathbb{Q})$ satisfies the Exchange Principle. However, in section 5, we show that the algebraic closure in $\mathrm{Th}_L(\mathbb{N} \times \mathbb{Q}_{\geq 0})$ does not satisfy the Exchange Principle.

In [1], Belegradek, Peterzil and Wagner showed that the L_0-structure $\mathbb{Z} \times \mathbb{Q}$ is quasi-o-minimal, that is, in any structure elementarily equivalent to $\mathbb{Z} \times \mathbb{Q}$ the definable subsets

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are exactly the Boolean combinations of $\emptyset$-definable subsets and intervals. However, in section 6, we show that the L-structure $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ is not quasi-o-minimal.

The reader is assumed to be familiar with model theory; see for example [5] or [6].

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2. Some axioms for ordered commutative monoids

Let $\mathbb{N}$ be the ordered commutative monoid of nonnegative integers. Let $\mathbb{Z}$ be the ordered abelian group of integers. Let $\mathbb{Q}$ be the ordered abelian group of rational numbers and $\mathbb{Q}_{\geq 0}$ the ordered commutative monoid of nonnegative rational numbers.

We denote $L_0 := \{0, +, <\} \cup \{n \mid x : n > 0, n \in \mathbb{N}\} \cup \{1\}$, where $n \mid x$ is a unary relation symbol for each positive integer n. We denote $L := L_0 \cup \{R(x), E_1(x, y), E_2(x, y)\}$, where $R(x)$ is a unary relation symbol and $E_1(x, y), E_2(x, y)$ are binary relation symbols. The terms $t + \cdots + t$ and $1 + \cdots + 1$ (t and 1 repeated n times) are written as nt and n, respectively. The formulas $s < t \wedge t < u$ and $s < t \vee s = t$ are written as $s < t < u$ and $s \leq t$, respectively.

We now give some axioms for ordered commutative monoids.

Definition 2.1.

1. The axioms for commutative monoids:
   $\forall x \forall y \forall z((x + y) + z = x + (y + z))$;
   $\forall x(x + 0 = x)$;
   $\forall x \forall y(x + y = y + x)$.
2. The axioms for a linear ordering compatible with monoid structures:
   $\forall x \forall y(x < y \vee x = y \vee y < x)$;
   $\forall x(\neg(x < x))$;
   $\forall x \forall y \forall z(x < y \to x + z < y + z)$;
   $0 < 1$;
   $\forall x(0 \leq x)$.
3. The axioms for $n \mid x$:
   $\forall x(n \mid x \leftrightarrow \exists y \exists z(z < 1 \wedge x = ny + z))$ for each integer $n > 0$;
   $\forall x \left(\bigvee_{0 \leq r < n} n \mid x + r\right)$ for each integer $n > 1$.
4. The axioms for infinitesimals:
   $\forall x(x < 1 \to nx < 1)$ for each integer $n > 1$.
5. The axiom for $R(x)$:
   $\forall x(R(x) \leftrightarrow \forall y \forall z(y < x \wedge z < 1 \to y + z < x))$.
6. The axioms for $E_1(x, y)$ and $E_2(x, y)$:
   $\forall x \forall y(E_1(x, y) \leftrightarrow \exists z(x + z = y \wedge R(z)))$;
   $\forall x \forall y(E_2(x, y) \leftrightarrow \exists z(x + z = y \wedge \neg R(z)))$.
7. $\forall x \exists y \exists z(x = y + z \wedge R(y) \wedge z < 1)$.
8. The axioms for difference:
   $\forall x \forall y(x < y \wedge y < 1 \to \exists z(x + z = y))$;
   $\forall x \forall y(x < y \wedge R(x) \wedge R(y) \to \exists z(x + z = y))$.
9. The axioms for divisible infinitesimals:
   $\forall x(x < 1 \to \exists y(x = ny))$ for each integer $n > 1$.
10. The axiom for discrete ordering:
    $\forall x(\neg(0 < x < 1))$.
11. The axiom for existence of infinitesimals:
    $\exists x(0 < x < 1)$.

Note that the language L is a definable expansion of the language L_0.

Definition 2.2. We denote $\mathrm{NSS} := (1) \cup (2) \cup (3) \cup (4) \cup (5) \cup (6) \cup (7) \cup (8)$. We denote $\mathrm{NDC} := \mathrm{NSS} \cup (10)$ and $\mathrm{NSC} := \mathrm{NSS} \cup (9) \cup (11)$.

We consider the lexicographic order from left to right on the ordered commutative monoid $\mathbb{N} \times \mathbb{Q}_{\geq 0}$. Then $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ is a model of the theory NSC, where 0 and 1 are interpreted as $\langle 0, 0 \rangle$ and $\langle 1, 0 \rangle$, respectively. It is well-known that $\mathrm{Th}_{L_0}(\mathbb{N})$ admits elimination of quantifiers and $\mathrm{Th}_{L_0}(\mathbb{N}) = (1) \cup (2) \cup (3) \cup (10) \cup \{\forall x \forall y (x < y \to \exists z (x + z = y))\}$. Thus, it follows that the theory NDC admits elimination of quantifiers in the language L and $\mathrm{NDC} = \mathrm{Th}_L(\mathbb{N})$.

3. Quantifier eliminable ordered commutative monoids

In this section, we show that the theory NSC admits elimination of quantifiers in the language L and $\mathrm{NSC} = \mathrm{Th}_L(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. We also show that if M is a model of the theory NSS and $\mathrm{Th}_L(M)$ admits elimination of quantifiers, then M is a model of either the theory NDC or the theory NSC.

Lemma 3.1. *The theory* $\mathrm{Th}_{L_0}(\mathbb{N} \times \mathbb{Q}_{\geq 0})$ *dose not admit elimination of quantifiers.*

Proof. Let $\varphi(x) :\equiv \forall y \forall z (y < x \wedge z < 1 \to y + z < x)$. Then we have $\mathbb{N} \times \{0\} = \varphi(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. Thus, any quantifier-free L_0-formula is not equivalent to the formula $\varphi(x)$ in the theory $\mathrm{Th}_{L_0}(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. □

To show that the theory NSC admits elimination of quantifiers in the language L, we first prove some lemmas needed later.

Lemma 3.2. *We have that* $\mathrm{NSS} \models \forall x \forall y (x < 1 \wedge y < 1 \to x + y < 1)$.

Proof. Suppose for a contradiction that there exists a model M of NSS and $x, y \in M$ such that $x < 1$, $y < 1$ and $x + y \geq 1$. By Axiom (4), we have $2x < 1$ and $2y < 1$. Thus, we obtain $2(x + y) < 2$, a contradiction. □

Lemma 3.3. *We have that* $\mathrm{NSS} \models \forall x \forall y (R(x) \wedge R(y) \leftrightarrow R(x + y))$.

Proof. Let M be a model of NSS. Let $x, y \in M$.

Suppose that $R(x)$ and $R(y)$. By Axiom (7), there exist $z_1, z_2 \in M$ with $R(z_1)$ and $z_2 < 1$ such that $x + y = z_1 + z_2$. Since $R(x)$ and $R(y)$, we obtain $x \leq z_1$ and $y \leq z_1$. By Axiom (8), there exists $u \in M$ with $y + u = z_1$. Since $x + y = z_1 + z_2$, we have $x = u + z_2$. Because $R(x)$ and $z_2 < 1$, we get $x = u$. Thus, we have $z_2 = 0$. It follows $x + y = z_1$, as desired.

Suppose that $R(x + y)$. By Axiom (7), there exist $x_1, x_2 \in M$ with $R(x_1)$ and $x_2 < 1$ such that $x = x_1 + x_2$. Since $x + y = (x_1 + y) + x_2$ and $R(x + y)$, we have $x + y = x_1 + y$. Hence $x = x_1$, and therefore $R(x)$. Similarly, we get $R(y)$. □

By Axiom (6), the following lemma holds.

Lemma 3.4. *Let* $i \in \{1, 2\}$. *Then, we have* $\mathrm{NSS} \models \forall x \forall y \forall z (E_i(x, y) \leftrightarrow E_i(x + z, y + z))$.

Lemma 3.5. *Let p be a positive integer and* $i \in \{1, 2\}$. *Then, we have that* $\mathrm{NSS} \models \forall x \forall y (E_i(x, y) \leftrightarrow E_i(px, py))$.

Proof. We only show that $\mathrm{NSS} \models \forall x \forall y (E_1(x, y) \leftrightarrow E_1(px, py))$. The other case is similar. Let M be a model of NSS and $x, y \in M$.

Suppose that $E_1(x, y)$. There exists $z \in M$ with $R(z)$ such that $x + z = y$. Then $px + pz = py$. By Lemma 3.3, we get $R(pz)$. Thus, it follows $E_1(px, py)$.

Suppose that $E_1(px, py)$. There exists $u \in M$ with $R(u)$ such that $px + u = py$. By Axiom (7), there exist $x_1, x_2, y_1, y_2 \in M$ with $R(x_1), x_2 < 1, R(y_1), y_2 < 1$ such that

$x = x_1 + x_2$ and $y = y_1 + y_2$. By Lemma 3.3, we obtain $R(px_1 + u)$ and $R(py_1)$. Since $(px_1 + u) + px_2 = py_1 + py_2$, we have $px_1 + u = py_1$ and $px_2 = py_2$. Thus, we get $x_1 \leq y_1$ and $x_2 = y_2$. By Axiom (8), there exists $w \in M$ with $R(w)$ such that $x_1 + w = y_1$. It follows $E_1(x, y)$. □

Using the lemmas above, we show the following.

Theorem 3.6. *The theory* NSC *admits elimination of quantifiers in the language* L.

Proof. Let $\exists x\varphi$ be a formula, where φ is a quantifier-free formula in L. We may assume that φ is of the form $\psi_1 \wedge \cdots \wedge \psi_m$, where each ψ_i is an atomic formula or the negation of an atomic formula. In addition, each ψ_i is of one of the forms $t = s$, $\neg(t = s)$, $t < s$, $\neg(t < s)$, $n \mid t$, $\neg(n \mid t)$, $R(t)$, $\neg R(t)$, $E_1(t, s)$, $\neg E_1(t, s)$, $E_2(t, s)$ or $\neg E_2(t, s)$, where t and s are terms and n is a positive integer. Moreover $\neg(t = s)$, $\neg(t < s)$ and $\neg(n \mid t)$ are equivalent to $t < s \vee s < t$, $t = s \vee s < t$ and $n \mid t + 1 \vee \cdots \vee n \mid t + n - 1$, respectively.

Now, each term t can be written in the form $px + s$ with $p \in \mathbb{N}$ and s a term which does not contain x. Therefore $\exists x\varphi$ can be written as

$$\exists x\Big(\bigwedge_{i\in A} p_i x + t_i = s_i \wedge \bigwedge_{i\in B} u'_i < q_i x + u_i \wedge \bigwedge_{i\in B'} r_i x + v_i < v'_i \\ \wedge \bigwedge_{i\in C} n_i \mid m_i x + w_i \wedge \psi\Big),$$

where each p_i, q_i, r_i, m_i, n_i are positive integer, each s_i, t_i, u_i, u'_i, v_i, v'_i, w_i are terms which do not contain x, the sets A, B, B', C may be empty, and the formula ψ is a finite conjunction of formulas of the forms R, E_1, E_2 or the negation of these. By Axiom (9), for each positive integer p the formula $p \mid x$ is equivalent to $\exists y(x = py)$. Thus, by Lemmas 3.3 and 3.5, we may assume that the formula $\exists x\varphi$ is equivalent to

$$\exists x\Big(\bigwedge_{i\in A} x + t_i = s_i \wedge \bigwedge_{i\in B} u'_i < x + u_i \wedge \bigwedge_{i\in B'} x + v_i < v'_i \\ \wedge \bigwedge_{i\in C} n_i \mid x + w_i \wedge \psi\Big).$$

Let $\theta(x)$ be the formula

$$\bigwedge_{i\in I_1} E_1(x, \alpha_i) \wedge \bigwedge_{i\in I_2} \neg E_1(x, \alpha'_i) \wedge \bigwedge_{i\in I_3} E_1(\beta_i, x) \wedge \bigwedge_{i\in I_4} \neg E_1(\beta'_i, x) \\ \wedge \bigwedge_{i\in J_1} E_2(x, \gamma_i) \wedge \bigwedge_{i\in J_2} \neg E_2(x, \gamma'_i) \wedge \bigwedge_{i\in J_3} E_2(\delta_i, x) \wedge \bigwedge_{i\in J_4} \neg E_2(\delta'_i, x),$$

where each $\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i, \gamma'_i, \delta_i, \delta'_i$ are terms which do not contain x and each I_i, J_i may be empty. By Lemmas 3.3 and 3.4, we may assume that the formula $\exists x\varphi$ is equivalent to

$$\exists x\Big(\bigwedge_{i\in A} x + t = s_i \wedge \bigwedge_{i\in B} u_i < x + t \wedge \bigwedge_{i\in B'} x + t < v_i \wedge \bigwedge_{i\in C} n_i \mid x + t + w_i \\ \wedge \bigwedge_{i\in D} R(x) \wedge \bigwedge_{i\in D'} \neg R(x) \wedge \theta(x + t)\Big),$$

where t is term which does not contain x and the sets D, D' may be empty. If D is not empty and D' is empty, then we may assume that the formula $\exists x\varphi$ is equivalent to

$$\exists y\Big(\bigwedge_{i\in A} y = s_i \wedge \bigwedge_{i\in B} u_i < y \wedge \bigwedge_{i\in B'} y < v_i \wedge \bigwedge_{i\in C} n_i \mid y + w_i \wedge \theta(y) \wedge E_1(t, y)\Big).$$

If D' is not empty and D is empty, then we may assume that the formula $\exists x\varphi$ is equivalent to

$$\exists y\Big(\bigwedge_{i\in A} y=s_i\wedge\bigwedge_{i\in B} u_i<y\wedge\bigwedge_{i\in B'} y<v_i\wedge\bigwedge_{i\in C} n_i\mid y+w_i\wedge\theta(y)\wedge E_2(t,y)\Big).$$

Hence, without loss of generality we may assume that the formula $\exists x\varphi$ is equivalent to $\exists y\left(u<y<v\wedge\bigwedge_{i\in C} n_i\mid y+w_i\wedge\theta(y)\right)$, where u,v are terms which do not contain x. By Axiom (3), this formula is equivalent to $\bigvee_{0\le r<n}(\bigwedge_{i\in C} n_i\mid r+w_i\wedge\exists z(u<nz+r<v \wedge\theta(nz+r)))$, where n is the least common multiple of n_i $(i\in C)$. By Lemma 3.4, the formula $\exists z(u<nz+r<v\wedge\theta(nz+r))$ is equivalent to $\exists z(u+(n-r)<n(z+1)<v+(n-r)\wedge\theta(n(z+1)))$ for each integer r with $0\le r<n$. Hence, we may assume that the formula $\exists x\varphi$ is equivalent to $\exists z(u<nz<v\wedge\theta(nz))$. Moreover, we may assume that the formula $\exists x\varphi$ is equivalent to

$$\begin{aligned}&\exists z(u<nz<v\wedge E_1(nz,\alpha)\wedge\neg E_1(nz,\alpha')\wedge E_1(\beta,nz)\wedge\neg E_1(\beta',nz)\\ &\quad\wedge E_2(nz,\gamma)\wedge\neg E_2(nz,\gamma')\wedge E_2(\delta,nz)\wedge\neg E_2(\delta',nz)),\end{aligned}$$

where $\alpha,\alpha',\beta,\beta',\gamma,\gamma',\delta,\delta'$ are terms which do not contain x. Then, this formula is equivalent to some quantifier-free formula. For example, the formula $\exists z(u<nz\wedge E_1(nz,\alpha))$ is equivalent to

$$\begin{aligned}&u<\alpha\wedge(n\mid\alpha\vee[n\mid u\to E_2(u,\alpha)\vee u+n<\alpha]\\ &\qquad\qquad\vee[\bigvee_{1\le r<n}(n\mid u+r\to u+r<\alpha)]).\end{aligned}$$

Hence, the formula $\exists x\varphi$ is equivalent to some quantifier-free formula. Therefore, the theory NSC admits elimination of quantifiers in the language L. □

Fact 3.7 ([5, Proposition 1.1.8])**.** *Let M be an L-structure and N a substructure of M. Suppose that φ is a quantifier-free L-sentence. Then, $M\models\varphi$ if and only if $N\models\varphi$.*

Theorem 3.8. *The theory* NSC *is complete. Namely, we have* $\mathrm{NSC}=\mathrm{Th}_L(\mathbb{N}\times\mathbb{Q}_{\ge 0})$.

Proof. Let M be a model of NSC. Suppose that $f:\mathbb{N}\to M$ by $f(n)=n^M$. Then f is an embedding. Thus, by Theorem 3.6 and Fact 3.7, the theory NSC is complete. □

Lemma 3.9. *Let $\psi(x)$ be a quantifier-free L-formula with one free variable x. Suppose that $M\models$ NSS. Then, either $M\models\psi(a)$ for each a with $0<a<1$, or $M\models\neg\psi(a)$ for each a with $0<a<1$.*

Proof. Let $\psi(x)$ be a quantifier-free L-formula with one free variable x. The formula $\psi(x)$ is equivalent to a Boolean combination of formulas which is of the forms $px+q=0$, $px=q$, $px+q<0$, $px<q$, $0<px+q$, $q<px$, $n\mid px+q$, $R(px+q)$, $E_1(px+q,0)$, $E_1(px,q)$, $E_1(0,px+q)$, $E_1(q,px)$ $E_2(px+q,0)$, $E_2(px,q)$, $E_2(0,px+q)$, $E_2(q,px)$, where $n,p\in\mathbb{N}\setminus\{0\}$ and $q\in\mathbb{N}$.

Let $M\models pa=q$ for some $a\in M$ with $0<a<1$. Then, by Axiom (4), we have $0<pa<1$, a contradiction.

Let $M\models pa<q$ for some $a\in M$ with $0<a<1$. Then, by $0<pa$, we have $1\le q$. Thus, $M\models px<q$ for each $x\in M$ with $0<x<1$.

Let $M\models q<pa$ for some $a\in M$ with $0<a<1$. Then, by $pa<1$, we have $q=0$. Thus, $M\models q<px$ for each $x\in M$ with $0<x<1$.

Let $M\models n\mid pa+q$ for some $a\in M$ with $0<a<1$. Then, by $0<pa<1$, there exists $m\in\mathbb{N}$ such that $q=mn$. Thus, we have $M\models n\mid px+q$ for each $x\in M$ with $0<x<1$.

Let $M\models E_1(0,pa+q)$ for some $a\in M$ with $0<a<1$. Then, by $0<pa<1$, we get $M\models\neg R(pa+q)$, a contradiction.

The other cases are similar. This completes the proof of the lemma. □

We show the converse of Theorem 3.6.

Theorem 3.10. *Let M be a model of* NSS*. Suppose that* $\mathrm{Th}_L(M)$ *admits elimination of quantifiers. Then M is a model of either* NDC *or* NSC*. Namely, we have either $M \equiv_L \mathbb{N}$ or $M \equiv_L \mathbb{N} \times \mathbb{Q}_{\geq 0}$.*

Proof. First, suppose that Axiom (10) holds in M. Then, the structure M is a model of NDC.

Secondly, suppose that Axiom (11) holds in M. Let n be a positive integer with $n > 1$. Because $\mathrm{Th}(M)$ admits elimination of quantifiers, there exists a quantifier-free L-formula $\psi_n(x)$ such that

$$\mathrm{Th}(M) \models \forall x[(x < 1 \to \exists y(x = ny)) \leftrightarrow \psi_n(x)].$$

Let $a \in M$. Now $M \models \psi_n(a)$ if $a = 0$ or $1 \leq a$. Assume that $0 < a < 1$. Then $M \models \psi_n(na)$. By Lemma 3.9, we obtain $M \models \psi(a)$. It follows that $M \models \psi(a)$ for each $a \in M$. Thus, Axiom (9) holds in M. Therefore, the structure M is a model of NSC. □

Remark 3.11. *By Theorem* 3.10*, the lexicographically ordered commutative monoid $\mathbb{N} \times \mathbb{N}$ dose not admit elimination of quantifiers in the language L, where 0 and 1 are interpreted as $\langle 0, 0\rangle$ and $\langle 1, 0\rangle$, respectively. However, in a similar way to Theorem* 3.6 *we show that the lexicographically ordered commutative monoid $\mathbb{N} \times \mathbb{N}$ admits elimination of quantifiers in the language $L \cup \{1'\}$, where $1'$ is interpreted as $\langle 0, 1\rangle$.*

4. Model completion

In this section, we show that the theory NSC is a model completion of the theory NSS.

Recall the notion of the model companion and the model completion from [5].

Definition 4.1. Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure. Suppose that $\mathrm{Diag}(M) := \{\varphi(m_1, \ldots, m_n) : \varphi(x_1, \ldots, x_n)$ is either an atomic $\mathcal{L}$-formula or the negation of an atomic $\mathcal{L}$-formula, $m_1, \ldots, m_n \in M$ and $M \models \varphi(m_1, \ldots, m_n)\}$. Suppose that T and T' are $\mathcal{L}$-theories. We say that T' is a *model companion* of T if

(i) T' is model-complete;
(ii) every model of T has an extension that is a model of T';
(iii) every model of T' has an extension that is a model of T.

Moreover, if T' is a model companion of T and $T' \cup \mathrm{Diag}(M)$ is a complete $L(M)$-theory for any $M \models T$, then T' is called a *model completion* of T.

Fact 4.2 ([3])**.** $\mathrm{Th}_{L_0}(\mathbb{Z} \times \mathbb{Q})$ *is a model completion of the L_0-theory* SS*, where the L_0-theory* SS *is defined by* [3]*.*

Lemma 4.3. *Any model of* NSS *can be embedded in a model of* NSC*.*

Proof. Let M be a model of NSS.

Suppose that Axiom (10) holds in M. We now consider the lexicographic order on $M \times \mathbb{Q}_{\geq 0}$. Then, the lexicographically ordered commutative monoid $M \times \mathbb{Q}_{\geq 0}$ is a model of NSC, where 0 and 1 are interpreted as $\langle 0, 0\rangle$ and $\langle 1, 0\rangle$, respectively. Moreover, $f : M \to M \times \mathbb{Q}_{\geq 0}$ by $f(a) = \langle a, 0\rangle$ is an embedding.

On the other hand, suppose that Axiom (11) holds in M. Let $N := \{\langle a, n\rangle : a \in M, n > 0, n \in \mathbb{N}, n \mid a\}$. We define an equivalence relation $\sim$ on N by $\langle a, m\rangle \sim \langle b, n\rangle$ if $na = mb$. Let $[\langle a, m\rangle]$ denote the $\sim$-class of $\langle a, m\rangle \in N$ and $N' := \{[\langle a, m\rangle] : \langle a, m\rangle \in N\}$. We define $+$ on N' by $[\langle a, m\rangle] + [\langle b, n\rangle] := [\langle na + mb, mn\rangle]$. We also define an order on N' by

$[\langle a, m\rangle] < [\langle b, n\rangle]$ if $na < mb$. Then, the ordered commutative monoid N' is a model of NSC. Moreover, $g : M \to N'$ by $g(a) = [\langle a, 1\rangle]$ is an embedding. ☐

Fact 4.4 ([6, Theorem 9.2.2])**.** *Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Then, the following conditions are equivalent*:

(i) *T admits elimination of quantifiers in the $\mathcal{L}$*;
(ii) *For every model M of T and every substructure A of M, we have that $T \cup \mathrm{Diag}(A)$ is a complete $L(A)$-theory.*

By Theorem 3.6, Fact 4.4 and Lemma 4.3, we have the following.

Theorem 4.5. *The theory* NSC *is a model completion of the theory* NSS*.*

Recall that the L_0-theory SC is the set of the following axioms ([3]).

- The axioms for ordered abelian groups.
- The axioms for a semi-discrete ordering:
  $0 < 1$, $\forall x(2x < 1 \vee 1 < 2x)$.
- The axioms for infinitesimals:
  $\forall x(2x < 1 \to nx < 1)$ for each integer $n > 2$.
- The axioms for $n \mid x$:
  $\forall x(n \mid x \leftrightarrow \exists y \exists z(0 < 2z + 1 \wedge 2z < 1 \wedge x = ny + z))$ for each integer $n > 0$,
  $\forall x \left(\bigvee_{0 \le r < n} n \mid x + r\right)$ for each integer $n > 0$.
- The axioms for divisible infinitesimals:
  $\forall x(0 < 2x + 1 \wedge 2x < 1 \to \exists y(x = ny))$ for each integer $n > 1$.
- The axioms for existence of infinitesimals:
  $\exists x(0 < x < 1)$.

Fact 4.6 ([3, Theorem 1.1])**.** *The L_0-theory* SC *admits elimination of quantifiers and is complete. In particular, we have* $\mathrm{SC} = \mathrm{Th}_{L_0}(\mathbb{Z} \times \mathbb{Q})$, *where we interpret* 0 *and* 1 *as* $\langle 0, 0\rangle$ *and* $\langle 1, 0\rangle$, *respectively.*

Theorem 4.7. *Any model of* NSC *can be embedded in a model of* SC*.*

Proof. Let M be a model of NSC. We define an equivalence relation $\sim$ on $M \times M$ by $\langle a, b\rangle \sim \langle a', b'\rangle$ if $a + b' = a' + b$. Let $[\langle a, b\rangle]$ denote the $\sim$-class of $\langle a, b\rangle \in M \times M$ and $N := \{[\langle a, b\rangle] : \langle a, b\rangle \in M \times M\}$. We define $+$ on N by $[\langle a, b\rangle] + [\langle a', b'\rangle] := [\langle a + a', b + b'\rangle]$. We also define an order on N by $[\langle a, b\rangle] < [\langle a', b'\rangle]$ if $a + b' < a' + b$.

Then, the structure N is a model of SC. Moreover, $f : M \to N$ by $f(a) = [\langle a, 0\rangle]$ is an embedding. This completes the proof of the theorem. ☐

5. Exchange principle

In this section, we show that the algebraic closure in the theory NSS does not satisfy the Exchange Principle.

Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure. Let A be a subset of M. We say that $a \in M$ is *algebraic* over A if there exists an $\mathcal{L}$-formula $\varphi(x, y_1, \dots, y_n)$ and $b_1, \dots, b_n \in A$ such that $M \models \varphi(a, b_1, \dots, b_n)$ and $\{c \in M : M \models \varphi(c, b_1, \dots, b_n)\}$ is finite. The *algebraic closure* of A in M, denoted $\mathrm{acl}(A)$, is given by $\{a \in M : a \text{ is algebraic over } A\}$.

Definition 5.1. Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure. We say that the algebraic closure in M satisfies the *Exchange Principle* if $A \subseteq M$, $a, b \in M$ and $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$, then $b \in \mathrm{acl}(A \cup \{a\})$. The algebraic closure in an $\mathcal{L}$-theory T is said to satisfy the *Exchange Principle* if the algebraic closure in any model M of T satisfies the Exchange Principle.

Fact 5.2 ([2, Corollary 50]). *The algebraic closure in the theory* $\mathrm{Th}_{L_0}(\mathbb{Z} \times \mathbb{Q})$ *satisfies the Exchange Principle.*

Theorem 5.3. *The algebraic closure in the theory* NSC *does not satisfy the Exchange Principle.*

Proof. Let $\mathbb{Q}_{>0} := \{a \in \mathbb{Q} : a > 0\}$ and $\mathbb{R}_{\geq 0} := \{a \in \mathbb{R} : a \geq 0\}$. Let M_1 be the lexicographically ordered structure $\{0\} \times \mathbb{N} \times \mathbb{R}_{\geq 0}$ and M_2 the lexicographically ordered structure $\mathbb{Q}_{>0} \times \mathbb{Z} \times \mathbb{R}_{\geq 0}$. In $M := M_1 \sqcup M_2$ we define

- $+$ is defined coordinatewise;
- $a < b$ whenever $a \in M_1$ and $b \in M_2$;
- 0 and 1 are interpreted as $\langle 0, 0, 0\rangle$ and $\langle 0, 1, 0\rangle$, respectively.

Then M is a model of the theory NSC.

Let $a = \langle 1, 3, 2\rangle$, $b = \langle 2, 6, 4\sqrt{2}\rangle$ and $c = \langle 0, 0, 4\sqrt{2}\rangle$. Let $\varphi(x) :\equiv 2a < x < 2a + 1 \wedge E_1(c, x)$. Then $\{x \in M : M \models \varphi(x)\} = \{b\}$. Thus, we get $b \in \mathrm{acl}(\{a, c\})$. There exists no $n \in \mathbb{N}$ with $\langle 2, 6, 0\rangle = n\langle 0, 1, 0\rangle$. Hence, by Theorem 3.6, we have $b \notin \mathrm{acl}(\{c\})$. Now, there exists no $p \in \mathbb{Q}$ with $2 = 4p\sqrt{2}$. Thus, by Theorem 3.6, we obtain $a \notin \mathrm{acl}(\{b, c\})$. Therefore, the algebraic closure in M does not satisfy the Exchange Principle. This finishes the proof. □

Remark 5.4. *The algebraic closure in the* L*-structure* $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ *satisfies the Exchange Principle.*

6. Non-quasi-o-minimality

In this section, we show that the L-structure $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ is not quasi-o-minimal.

Recall the notion of quasi-o-minimal structures from [1].

Definition 6.1. A structure $(M, <, \dots)$, where $<$ is a linear ordering of M, is said to be *quasi-o-minimal* if in any structure elementarily equivalent to it the definable subsets are exactly the Boolean combinations of $\emptyset$-definable subsets and intervals.

The following fact is known.

Fact 6.2 ([1]). *The* L_0*-structures* $\mathbb{N}$, $\mathbb{Z}$ *and* $\mathbb{Z} \times \mathbb{Q}$ *are quasi-o-minimal.*

Theorem 6.3. *The* L*-structure* $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ *is not quasi-o-minimal.*

Proof. Suppose for a contradiction that the L-structure $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ is quasi-o-minimal. Let $\varphi(x) :\equiv E_1(\langle 2, 1\rangle, x)$ and $A := \varphi(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. Then $A = \{\langle n, 1\rangle : n \geq 2, n \in \mathbb{N}\}$, that is, A is infinite. By quasi-o-minimality of $\mathbb{N} \times \mathbb{Q}_{\geq 0}$, there exist $a, b \in \mathbb{N} \times \mathbb{Q}_{\geq 0}$ and infinite $\emptyset$-definable set B such that $a < b$ and A contains $(a, b) \cap B$. By Theorem 3.6, there exists some quantifier-free formula $\psi(x)$ without parameters such that $B = \psi(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. Since B is infinite, there exists $n \in \mathbb{N}$ and $d \in \mathbb{Q}_{\geq 0}$ such that $d \neq 1$ and $\langle n, d\rangle \in (a, b) \cap B$. It follows $(a, b) \cap B \nsubseteq A$, a contradiction. □

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