WEAK-OPEN MAPS AND SEQUENCE-COVERING MAPS

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ABSTRACT. Recently some results on spaces with a special weak-base were obtained in terms of weak-open maps and sequence-covering maps of metric spaces. In this paper, we give further results on weak-open maps and sequence-covering maps. Moreover, a question of Lin and Yan is answered.

1. INTRODUCTION

We assume that all spaces are regular T_1, all maps are continuous and onto. For $A \subset X$, $Int_X A$ is the interior of A in X. The letter $\mathbb{N}$ is the set of natural numbers.

Definition 1.1. Let X be a space. For $x \in X$, let $\mathcal{B}_x$ be a family of subsets of X. Then $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ is called a *weak-base* for X [2] if it satisfies the following conditions: (1) every element of $\mathcal{B}_x$ contains x, (2) for $B_0, B_1 \in \mathcal{B}_x$, there exists $B \in \mathcal{B}_x$ such that $B \subset B_0 \cap B_1$ and (3) $G \subset X$ is open iff for each $x \in G$ there exists $B \in \mathcal{B}_x$ with $B \subset G$. A space X is called *g-first countable* [12] if it has a weak-base $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ such that each $\mathcal{B}_x$ is countable. A space X is called *g-second countable* [12] if it has a countable weak-base.

Every g-first countable space is sequential, and a space is first countable iff it is g-first countable and Fréchet [2].

Definition 1.2. Let $f : X \to Y$ be a map.

(1) f is *weak-open* [15] if there exist a weak-base $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$ for Y and a point $x_y \in f^{-1}(y)$ for each $y \in Y$ such that for each open neighborhood U of x_y, $f(U)$ contains some element of $\mathcal{B}_y$.
(2) f is *sequence-covering* [11] if whenever $\{y_n\}_{n\in\omega}$ is a sequence in Y converging to $y \in Y$, there exists a sequence $\{x_n\}_{n\in\omega}$ in X converging to a point $x \in f^{-1}(y)$ such that $x_n \in f^{-1}(y_n)$.
(3) f is *1-sequence-covering* [6] if for each $y \in Y$, there exists a point $x_y \in f^{-1}(y)$ such that whenever $\{y_n\}_{n\in\omega}$ is a sequence in Y converging to a point $y \in Y$, there exists a sequence $\{x_n\}_{n\in\omega}$ in X converging to the point x_y with $x_n \in f^{-1}(y_n)$.

A sequence-covering map was introduced by F. Siwiec to characterize sequential spaces, Fréchet spaces and strongly Fréchet spaces in terms of maps. Note that a weak-open map is quotient. A weak-open map was introduced by S. Xia to characterize certain g-first countable spaces.

Theorem 1.3. (1) *a space is g-first countable iff it is a weak-open image of a metric space* [16]*;*
(2) *a space is g-second countable iff it is a weak-open image of a separable metric space* [15, Theorem 2.4]*;*

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(3) *every weak-open map of a first-countable space is 1-sequence-covering* [15, Proposition 3.3]*;*

(4) *every 1-sequence-covering map onto a sequential space is weak-open* [15, Proposition 3.4].

The following proposition will be used in the second section.

Proposition 1.4. *For a map $f : X \to Y$, the following are equivalent.*

(1) *f is weak-open;*

(2) *for each $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that for $G \subset Y$, $Int_X f^{-1}(G) \supset \{x_y;\, y \in G\}$ implies that G is open in Y.*

Proof. (1) → (2): Take a weak-base $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$ for Y and a point $x_y \in f^{-1}(y)$ for each $y \in Y$ such that for each open neighborhood U of x_y, $f(U)$ contains some element of $\mathcal{B}_y$. Let G be a subset of Y satisfying $Int_X f^{-1}(G) \supset \{x_y;\, y \in G\}$. For each $y \in G$, $Int_X f^{-1}(G)$ is an open neighborhood of x_y. Hence there exists $B \in \mathcal{B}_y$ with $y \in B \subset f(Int_X f^{-1}(G))$. Then $y \in B \subset G$. Thus G is open in Y.

(2) → (1): For each $y \in Y$, take a point $x_y \in f^{-1}(y)$ such that for $G \subset Y$, $Int_X f^{-1}(G) \supset \{x_y;\, y \in G\}$ implies that G is open in Y. For each $y \in Y$, let $\mathcal{B}_y = \{f(U) : x_y \in U, U$ is open in $X\}$ and $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$. We have only to see that $\mathcal{B}$ is a weak-base for Y. Let $G \subset Y$, and assume $G = \bigcup_{y\in G} B_y$, where $B_y \in \mathcal{B}_y$. Let $B_y = f(U_y)$, where U_y is an open neighborhood of x_y. Then $Int_X f^{-1}(G) \supset \bigcup_{y\in G} U_y \supset \{x_y;\, y \in G\}$. Hence G is open in Y. Other conditions on a weak-base are easy to show. □

2. Characterizations of spaces in terms of maps

Siwiec proved in [11, Theorem 4.1] that a space Y is sequential iff every sequence-covering map onto Y is quotient. As described in Theorem 1.3 (4), if a space Y is sequential, then every 1-sequence-covering map onto Y is weak-open. We note that the converse is true.

Proposition 2.1. *For a space Y, the following are equivalent.*

(1) *Y is sequential;*

(2) *every 1-sequence-covering map onto Y is weak-open;*

(3) *every 1-sequence-covering map onto Y is quotient.*

Proof. (1) → (2): This is due to Xia, see Theorem 1.3 (4).

(2) → (3): This is trivial, because every weak-open map is quotient.

(3) → (1): For each $y \in Y$, let $\mathcal{B}_y$ be the family of all sequential neighborhood of y in Y. Let $X(y) = Y$, and we give a topology on $X(y)$ as follows: all points but y are isolated in $X(y)$, and take $\mathcal{B}_y$ as a neighborhood base of y. Obviously each $X(y)$ is regular T_1. Note that every sequence in Y converging to y is a convergent sequence in $X(y)$. This means that every sequentially open set of $X(y)$ containing y is an element of $\mathcal{B}_y$, hence open in $X(y)$. Thus each $X(y)$ is sequential. Consider the natural map f of the topological sum $\bigoplus_{y\in Y} X(y)$ onto Y. The map is obviously continuous and 1-sequence-covering. Hence f is quotient. Since $\bigoplus_{y\in Y} X(y)$ is sequential, the quotient image Y is also sequential. □

In view of this proposition, it is natural to investigate a space Y satisfying that every sequence-covering map onto Y is weak-open.

Theorem 2.2. *For a space Y, the following are equivalent.*

(1) *every sequence-covering map onto Y is weak-open;*

(2) *Y has a weak-base $\mathcal{B} = \cup\{\mathcal{B}_y : y \in Y\}$ such that each $\mathcal{B}_y$ is a countable and decreasing family, and each element of $\mathcal{B}_y$ is a convergent sequence with the limit y;*

(3) *Y is a weak-open image of the topological sum of some convergent sequences;*
(4) *Y is sequential and for each $y \in Y$, there exists a sequence L_y converging to y such that for any sequence L converging to y, $L - L_y$ is finite.*

Proof. (1) → (2): Let $\{S_\alpha : \alpha < \kappa\}$ be the family of all convergent sequences in Y. Let f be the natural map of the topological sum $X = \bigoplus_{\alpha<\kappa} S_\alpha$ onto Y. The map is obviously sequence-covering, hence it is weak-open by our assumption. By Proposition 1.4, for each $y \in Y$, there exists $x_y \in f^{-1}(y)$ satisfying that $G \subset Y$ is open iff $\mathrm{Int}_X f^{-1}(G) \supset \{x_y; y \in G\}$. Let $x_y \in S_{\alpha_y}$, take a decreasing open neighborhood base $\{U_n(x_y) : n \in \omega\}$ of x_y in S_{α_y}. If x_y is isolated in S_{α_y}, $U_n(x_y) = \{x_y\}$ for all $n \in \omega$. Let

$$\mathcal{B} = \bigcup_{y \in Y} \mathcal{B}_y, \text{ where } \mathcal{B}_y = \{f(U_n(x_y)) : n \in \omega\}.$$

We observe that $\mathcal{B}$ is a weak-base for Y. Let $G \subset Y$ and assume that for each $y \in G$, there exists $n_y \in \omega$ such that $f(U_{n_y}(x_y)) \subset G$. Then

$$f^{-1}(G) \supset \bigcup_{y \in G} U_{n_y}(x_y) \supset \{x_y;\ y \in G\}.$$

Hence G is open in Y. Other conditions on a weak-base are obviously satisfied.

(2) → (3): Let $\mathcal{B} = \bigcup_{y \in Y} \mathcal{B}_y$ be a weak-base for Y satisfying the condition (2). Let $\mathcal{B}_y = \{B_n(y) : n \in \omega\}$. Consider the natural map f of the topological sum $X = \bigoplus_{y \in Y} B_0(y)$ onto Y. We see that f is weak-open. If $y \in Y$ is isolated, then take any point $x_y \in f^{-1}(y)$. If $y \in Y$ is not isolated, then take the point $x_y = y \in f^{-1}(y) \cap B_0(y)$. Assume that G is a subset of Y satisfying $\mathrm{Int}_X f^{-1}(G) \supset \{x_y : y \in G\}$. Then, for each $y \in G$, there exists $n_y \in \omega$ such that $G = \bigcup_{y \in G} B_{n_y}(y)$. Since $\mathcal{B}$ is a weak-base for Y, G is open in Y.

(3) → (4): Let f be a weak-open map of the topological sum $X = \bigoplus_{\alpha<\kappa} S_\alpha$ of some convergent sequences onto Y. Since a weak-open map is quotient, Y is sequential. By Theorem 1.3 (3), f is 1-sequence-covering. For each $y \in Y$, take $x_y \in f^{-1}(y)$ such that whenever $\{y_n\}_{n\in\omega}$ is a sequence converging to y, there exists a sequence $\{x_n\}_{n\in\omega}$ converging to x_y with $x_n \in f^{-1}(y_n)$. If $y \in Y$ is isolated, then let $L_y = \{y\}$. If $y \in Y$ is not isolated, since there exists a non-trivial sequence in Y converging to y, x_y is the limit point of some non-trivial convergent sequence S_{α_y}. Let $L_y = f(S_{\alpha_y}) - \{y\}$. The family $\{L_y : y \in Y\}$ is a desired one.

(4) → (1): Let $\{L_y : y \in Y\}$ be a family in the condition (4). Let f be a sequence-covering map of X onto Y. For each convergent sequence $\{y\} \cup L_y$, there exist $x_y \in f^{-1}(y)$ and a sequence C_y converging to x_y with $f(C_y) = L_y$. We see that f is weak-open. Suppose that $G \subset Y$ satisfies $\mathrm{Int}_X f^{-1}(G) \supset \{x_y : y \in G\}$. Since Y is sequential, we have only to see that G is sequentially open in Y. Let $\{y_n\}_{n\in\omega}$ be a sequence converging to $y \in G$. Since $\{y_n\}_{n\in\omega} - L_y$ is finite, there exists $k \in \omega$ such that $\{y_n : n \geq k\}$ is an image of a subsequence of C_y. By $x_y \in \mathrm{Int}_X f^{-1}(G)$, $\{y_n\}_{n\in\omega}$ is eventually in G. □

Remark 2.3. By the previous theorem, every sequence-covering map onto Arens' space S_2 is always weak-open, but not every sequence-covering map onto the sequential fan S_ω is weak-open. While, every 1-sequence-covering map onto the sequential fan is always weak-open by Proposition 2.1. Arens' space S_2 is the quotient space of the topological sum of countably many non-trivial convergent sequences $\{C_n : n \in \omega\}$, obtained by identifying, the limit point of C_n with the nth term of C_0 for all $n > 0$ [3, Example 1.6.19]. The sequential fan S_ω is the space obtained by identifying the limits of countably many convergent sequences.

3. Questions of Lin and Yan

In [7, Question 3.11], Lin and Yan asked whether every separable space which is a sequence-covering, quotient and s-image of a metric space is a local $\aleph_0$-space. A map is called an *s-map* if each fiber of the map is separable, and a space is called an *$\aleph_0$-space* [8] if it has a countable k-network. We give a counterexample of this question.

Example 3.1. Let M and Y be the spaces, and $f : M \to Y$ be the map constructed in [4, Example 9.3]. By the construction, f is a sequence-covering, quotient and compact-map (in fact, two-to-one map) of the metric space M, where a map is called *compact* if each fiber of the map is compact. The space Y is separable and a local $\aleph_0$-space which is not meta-Lindelöf. A family $\mathcal{F}$ of subsets of a space X is called *point-regular* [1] if for each $x \in X$ and each open neighborhood U of x, the set $\{F \in \mathcal{F} : x \in F, F \cap (X - U) \neq \emptyset\}$ is finite. Ikeda, Liu and Tanaka proved in [5, Theorem 9] that a space X has a point-regular weak-base iff it is a sequence-covering, quotient and compact-image of a metric space. Hence Y has a point-regular weak-base. Now, let $Y_n = Y$ for $n \in \omega$, let $\mathcal{B}(n)$ be a point-regular weak-base for Y_n. Consider the topological sum $\bigoplus_{n\in\omega} Y_n$, and let

$$Z = \{p\} \cup (\bigoplus_{n\in\omega} Y_n).$$

We give a topology on Z as follows: $\bigoplus_{n\in\omega} Y_n$ is an open subspace of Z and take the family

$$\mathcal{B}_p = \{\{p\} \cup (\bigoplus\{Y_n : n \geq k\}) : k \in \omega\}$$

as a neighborhood base of p. Then it is not difficult to see that Z is separable and the family

$$\mathcal{B}_p \cup (\cup\{\mathcal{B}(n) : n \in \omega\})$$

is a point-regular weak-base for Z. Hence, by the above theorem due to Ikeda, Liu and Tanaka, Z is a sequence-covering, quotient and compact-image of a metric space. Note that a compact-image of a metric space is an s-image. But each neighborhood of p is not meta-Lindelöf, hence Z is not a local $\aleph_0$-space.

In [7, Question 4.10], the authors asked whether a Fréchet space with a countable *cs*-network is a closed and sequence-covering image of a separable metric space. For the question, Yan, Lin and Jiang proved [17, Theorem 1] that every closed and sequence-covering image of a metric space is metrizable, and showed that the sequential fan S_ω is a counterexample.

A space X is said to be *strongly Fréchet* [11] if for each decreasing sequence $\{A_n : n \in \omega\}$ of subsets of X, $x \in \bigcap_{n\in\omega} \overline{A_n}$ implies that there exists a sequence $\{x_n\}_{n\in\omega}$ converging to x with $x_n \in A_n$ for each $n \in \omega$. It is known that a strongly Fréchet space which is a closed image of a metric space is metrizable [9, Corollary 9.10]. Therefore it is natural to ask whether the strong Fréchet property is preserved by a closed and sequence-covering map. We show that it is true under a weak condition.

A space is said to have *property wD* [14] if every infinite closed discrete subset has an infinite subset A such that there exists a discrete open family $\{U_x : x \in A\}$ with $U_x \cap A = \{x\}$ for each $x \in A$. Normal spaces, countably paracompact spaces and realcompact spaces have this property, see [14].

Theorem 3.2. *Let X be a strongly Fréchet space with property wD. If $f : X \to Y$ is a closed and sequence-covering map, then Y is strongly Fréchet.*

Proof. Since a closed image of a Fréchet space is Fréchet, Y is Fréchet. Assume that Y is not strongly Fréchet. Then Y contains a homeomorphic copy of the sequential fan S_ω [13,

(b) of (16), p. 31], and the copy can be closed in Y [10]. Hence let $S_\omega \subset Y$ as a closed set. We put

$$S_\omega = \{\infty\} \cup \{y_{m,n} : m, n \in \omega\},$$

where each $L_m = \{y_{m,n}\}_{n\in\omega}$ is a convergent sequence to ∞.

For each $n \in \mathbb{N}$, since $\{\infty\} \cup L_0 \cup L_n$ is a convergent sequence, there exist $x_n \in f^{-1}(\infty)$ and a sequence C_n converging to x_n such that $f(C_n) = L_0 \cup L_n$. For each $k \in \omega$, let

$$A_k = \bigcup\{f^{-1}(L_n) : n \geq k\}.$$

Suppose that there exists $z \in X$ such that for every open neighborhood U of z, $\{n \in \mathbb{N} : x_n \in U\}$ is infinite. Then $z \in \bigcap_{k\in\mathbb{N}} \overline{A}_k$. Take $z_k \in A_k$ such that $\{z_k\}_{k\in\mathbb{N}}$ converges to z. But $\{f(z_k)\}_{k\in\mathbb{N}}$ does not converge to ∞, which is a contradiction. Therefore the set $\{x_n\}_{n\in\mathbb{N}}$ is infinite, closed and discrete in X.

By property wD of X, there exist an infinite subset $\{x_{n_j}\}_{j\in\omega}$ of $\{x_n\}_{n\in\mathbb{N}}$ and a discrete open family $\{U_j\}_{j\in\omega}$ such that $U_j \cap \{x_{n_j}\}_{j\in\omega} = \{x_{n_j}\}$. Recall that C_{n_j} converges to x_{n_j} and $f(C_{n_j}) = L_0 \cup L_{n_j}$. Therefore we can take $u_j \in U_j \cap C_{n_j}$ such that $\{f(u_j)\}_{j\in\omega}$ is infinite and contained in L_0. Since $\{u_j\}_{j\in\omega}$ is closed in X, $\{f(u_j)\}_{j\in\omega}$ is closed in S_ω. This is a contradiction. Thus Y is strongly Fréchet. □

Corollary 3.3 ([17]). *Every closed and sequence-covering image of a metric space is metrizable.*

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