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RICCATI EQUATION AND THE FIEDLER-PTÁK SPECTRAL GEOMETRIC MEAN

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Abstract. For positive invertible operators on a Hilbert space, Fiedler and Pták introduced the spectral geometric mean which is a modification of the geometric mean $A\#B$ of operators. In this note, we show that it is characterized by Riccati equations, which shows its basic properties easily.

In the field of operator theory, simple Riccati equation $XBX = A$ has been discussed by Pedersen-Takesaki [11] and Nakamoto [10]. For positive (invertible) operators A and B on a Hilbert space, the geometric (operator) mean

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} = B^{1/2}(B^{-1/2}AB^{-1/2})^{1/2}B^{1/2}$$

is introduced by Pusz-Woronowicz [12] and Ando [2]. According to these results, Anderson-Trapp [1] gave the following view:

Theorem (Anderson-Trapp). *There exists a unique positive solution $A\#B$ for a Riccati equation $XB^{-1}X = A$.*

Remark 1. Carlin-Noble [4] introduced the geometric mean $A\#_C B$ by

$$A(A^{-1}B)^{1/2} = B(B^{-1}A)^{1/2}.$$

But their square root is not always positive and they did not determine it as an explicit form. As pointed out in [8], this square root is rationalized as

$$(A^{-1}B)^{1/2} = A^{-1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2},$$

since $A^{-1}B$ is weakly positive and hence the square root is uniquely determined. From this viewpoint, the definition of geometric operator mean might be introduced by Calkin-Noble [4].

Afterwards, Kubo-Ando [9] established a general theory of operator means: Only non-negative operator monotone functions f on $(0, \infty)$ with $f(1) = I$ can define operator means m_f by

$$Am_fB = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}A^{1/2}.$$

(Note that $f(x) = 1\ m_f x$.) One of the operations among operator means, the *dual* $f^\perp$ is defined by

$$f^\perp(x) = \frac{x}{f(x)}.$$

The above geometric mean $A\#B$ is only a self-dual one.

In this context, we essentially generalized the Anderson-Trapp theorem in [6], which we reformulate here:

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Theorem 1. *An operator mean g is the geometric one $\#$ if and only if*

$$(AgB)(Am^{\perp}B)^{-1}(AgB) = AmB \tag{1}$$

for all operator means m and all positive invertible operators A and B.

Proof. Putting $g = \#$, we have

$$\begin{aligned}(A\#B)(Am^{\perp}B)^{-1}(A\#B) &= (A\#B)(B^{-1}mA^{-1})(A\#B)\\ &= \left[(A\#B)B^{-1}(A\#B)\right] m \left[(A\#B)A^{-1}(A\#B)\right]\\ &= B^{1/2}\sqrt{B^{-1/2}AB^{-1/2}}^{2}B^{1/2}\ m\ A^{1/2}\sqrt{A^{-1/2}BA^{-1/2}}^{2}A^{1/2}\\ &= AmB.\end{aligned}$$

Conversely suppose (1). Then, putting $m = g^{\perp}$ and $f(x) = 1\ g\ x = 1\ m^{\perp}x$, we have

$$\frac{x}{f(x)} = f(x)^{\perp} = f(x)(f(x)^{-1})f(x) = f(x),$$

which shows $f(x) = \sqrt{x}$, that is, g is the geometric mean. □

The *spectral geometric mean* $A\tilde{\#}B$ by Fiedler-Pták [5] for positive invertible operators A and B is defined as

$$A\tilde{\#}B = (A^{-1}\#\ B)^{1/2}A(A^{-1}\#B)^{1/2}.$$

Then we give a characterization which is implicitly mentioned in [5]:

Theorem 2. *The spectral geometric mean $Y = A\tilde{\#}B$ is characterized by the following Riccati equations:*

$$Y = XAX = X^{-1}BX^{-1} \tag{2}$$

for some positive invertible operator X which should be $(A^{-1}\#B)^{1/2}$.

Proof. Suppose (2) holds. Then $X^2AX^2 = B$ and hence the Anderson-Trapp theorem shows $X^2 = A^{-1}\#B$, that is, $X = (A^{-1}\#B)^{1/2}$. Thus $Y = XAX = A\tilde{\#}B$. Conversely, suppose $Y = A\tilde{\#}B$. Putting $X = (A^{-1}\#B)^{1/2}$, we have

$$\begin{aligned}X^2AX^2 &= (A^{-1}\#B)A(A^{-1}\#B) = A^{-1/2}(1\#A^{1/2}BA^{1/2})^2A^{-1/2}\\ &= A^{-1/2}(A^{1/2}BA^{1/2})A^{-1/2} = B\end{aligned}$$

which shows (2). □

This theorem shows that the spectral geometric mean is symmetric; $A\tilde{\#}B = B\tilde{\#}A$. Moreover, we obtain easily its various properties:

Corollary 3. $(A\tilde{\#}B)^{-1} = A^{-1}\tilde{\#}B^{-1}$.

Proof. Taking inverse for (2) and putting $Z = X^{-1}$, we have

$$Y^{-1} = ZA^{-1}Z = Z^{-1}B^{-1}Z^{-1},$$

which implies $Y^{-1} = A^{-1}\tilde{\#}B^{-1}$. □

The following result is the reason why it is called a *spectral* geometric mean:

Corollary 4. *$(A\tilde{\#}B)^2$ is positively similar to AB and $\sigma((A\tilde{\#}B)^2) = \sigma(AB)$.*

Proof. The required results follows from

$$(A\tilde{\#}B)^2 = (XAX)(X^{-1}BX^{-1}) = XABX^{-1}$$

for some positive X. □

Corollary 5. *The following equivalences are hold:*

(i) *$A \geq B$ if and only if $(A^{-1}\tilde{\#}B) \leq I$.*

(ii) *$A\tilde{\#}B \leq A$ if and only if $B \leq A\tilde{\#}B$.*

Proof. The condition $(A^{-1}\tilde{\#}B) \leq I$ means

$$XA^{-1}X = X^{-1}BX^{-1} \leq I,$$

that is,

$$A \geq X^2 \quad \text{and} \quad B \leq X^2.$$

Thus $A \geq B$. Conversely suppose $A \geq B$. Then $A \geq X^2 = A\#B \geq B$ implies the above, which shows (i). Next, $A\tilde{\#}B \leq A$ is equivalent to

$$X^{-1}BX^{-1} \leq A, \qquad \text{namely} \qquad B \leq XAX \leq A\tilde{\#}B,$$

which shows (ii). □

Considering above properties, we easily have the following equivalence ([5]):

$$B^{-1} \leq A \iff A^{-1}\tilde{\#}B^{-1} \geq I \iff \sigma(AB) \geq 1.$$

Finally we observe the characterization of chaotic order by Furuta-Seo [7] where the chaotic order $A \gg B$ means $\log A \geq \log B$:

Theorem (Furuta-Seo). *For positive invertible operators A and B, the chaotic order $B \ll A$ holds if and only if there exists a positive invertible contraction T_p with*

$$B^p = T_p A^p T_p \tag{3}$$

for all $p > 0$.

In the above theorem, there is little information for T_p. But Riccati equation (3) implies T_p is uniquely determined as a solution

$$T_p = A^{-p}\#B^p.$$

The contractivity corresponds with Ando's characterization [3] of chaotic order (though it is not expressed in this context):

Theorem (Ando). *For positive invertible operators A and B, the chaotic order $B \ll A$ holds if and only if $A^{-p}\#B^p$ is decreasing for $[0, \infty)$.*

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