FISHER INFORMATIONS FROM CONTINUOUS AND DISCRETE OBSERVATIONS OF THE DRIFT OF THE ORNSTEIN-UHLENBECK PROCESS

Takayuki Fujii and Nobuo Inagaki

Received September 26, 2005; revised March 25, 2006

Abstract. When we consider maximum likelihood estimators for the drift coefficient of the Ornstein-Uhlenbeck process from both the continuous observations and the discrete ones, their asymptotic variances are related to each of Fisher informations. However, it is important to see that discrete observations are more applicable than continuous observations from the practical points of view. After delicate calculations, we show that the Fisher information from discrete observations is, of course, less than the one from continuous observations but almost equal to it, if the discretizing time-interval is sufficiently small.

1 Introduction Statistical estimation of the parameters of diffusion processes has been well studied. Küchler and Sørensen (1997) study asymptotic properties of the maximum likelihood estimators of drift parameters obtained from continuous observations. For discrete observations of diffusion processes, Dachnha-Castelle and Florens-Zmirou (1986) discuss asymptotic properties of the estimator due to a quasi-likelihood function and Kessler (2000) treats with the estimator due to more general estimating functions. We could say that discrete observations are more applicable than continuous observations from the practical points of view. We focus on Fisher informations for the continuous observations and the discrete ones, because Fisher information relates to the efficiency of maximum likelihood estimators.

In the present paper, we treat with the Ornstein-Uhlenbeck process which is the simplest diffusion process and discuss the estimation of the drift parameter from its continuous and discrete observations. Then, we call the Fisher informations of its drift parameter obtained from the continuous observations and the discrete ones the continuous and discrete Fisher informations, respectively. Our main aim is to compare the continuous and discrete Fisher informations and to show that the discrete one is slightly less than the continuous one if the time interval of observation is sufficiently small. Of course, we see the discrete Fisher information is less than the continuous one.

In section 2, we see the likelihood function and the Fisher information from continuous observations of the Ornstein-Uhlenbeck process. In section 3, we have them for the discrete case. In section 4, we calculate the ratio and difference of the continuous and discrete Fisher informations. Further we study the effect of discretization of observation.

2 Continuously observed case Let us consider the one dimensional Ornstein-Uhlenbeck process represented by stochastic differential equations :

$$dX_t = \theta X_t \, dt + \sigma \, dW_t, \quad X_0 = x, \tag{1}$$

2000 *Mathematics Subject Classification.* 62F12, 62M05.

Key words and phrases. continuous observations, discrete observations, Fisher information, maximum likelihood estimator, Ornstein-Uhlenbeck process.

where W_t is a standard Wiener process. We assume that the diffusion coefficient $\sigma > 0$ and the initial value x are given constants. By Ito's formula, the solution of the stochastic equation (1) is represented by the following Wiener integral :

$$X_t = xe^{\theta t} + \sigma e^{\theta t} \int_0^t e^{-\theta s}\, dW_s.$$

The property of the Wiener integral implies that, if t is fixed, X_t is normally distributed with mean $xe^{\theta t}$ and variance $v_t(\theta)$:

$$E(X_t) = xe^{\theta t}, \tag{2}$$

$$V(X_t) = \frac{\sigma^2}{2\theta}\left(e^{2\theta t} - 1\right) = v_t(\theta) \quad \text{(say)}. \tag{3}$$

That is,

$$X_t \sim N\left[\, xe^{\theta t},\ \frac{\sigma^2}{2\theta}\left(e^{2\theta t} - 1\right) \right]. \tag{4}$$

We note that $v_t(\theta)$ is the C^ω-class function of θ because

$$v_t(\theta) = \frac{\sigma^2}{2\theta}\left(e^{2\theta t} - 1\right) = \sigma^2 t \sum_{k=0}^{\infty} \frac{(2\theta t)^k}{(k+1)!},$$

and $v_t(0) = \lim_{\theta \to 0} v_t(\theta) = \sigma^2 t$ also gives the variance of the process $X_t = x + \sigma W_t$, which is the solution of the equation (1) for $\theta = 0$.

We obtain from Theorem 7.19 of Liptser and Shiryaev (2001) that the likelihood function of this process for continuous observations in time interval $[0, T]$ is given by

$$L_T(\theta) = \exp\left(\frac{\theta}{\sigma^2} \int_0^T X_t\, dX_t - \frac{\theta^2}{2\sigma^2} \int_0^T X_t^2\, dt \right).$$

Therefore, we have the log-likelihood function and its derivatives as follows :

$$\ell_T(\theta) = \frac{\theta}{\sigma^2} \int_0^T X_t\, dX_t - \frac{\theta^2}{2\sigma^2} \int_0^T X_t^2\, dt$$

$$\dot{\ell}_T(\theta) = \frac{1}{\sigma^2} \int_0^T X_t\, dX_t - \frac{\theta}{\sigma^2} \int_0^T X_t^2\, dt \tag{5}$$

$$\ddot{\ell}_T(\theta) = -\frac{1}{\sigma^2} \int_0^T X_t^2\, dt, \tag{6}$$

where the dot notation " $\cdot$ " denotes the differentiation with respect to the drift parameter θ .

The likelihood equation (5) leads to the maximum likelihood estimator :

$$\hat{\theta}_T = \frac{\int_0^T X_t\, dX_t}{\int_0^T X_t^2\, dt}.$$

¿From the equation (6), we calculate the Fisher information, immediately.

Lemma 1 *We have the Fisher information for the continuous observations of the drift parameter $\theta \neq 0$ of the Ornstein-Uhlenbeck process :*

$$\begin{aligned} I_T(\theta) &= \frac{x^2}{2\theta\sigma^2}\left(e^{2\theta T}-1\right)+\frac{1}{(2\theta)^2}\left(e^{2\theta T}-1\right)-\frac{T}{2\theta} \\ &= J_T(\theta)+K_T(\theta) \quad (say), \end{aligned} \tag{7}$$

where $J_T(\theta)$ is related to the initial value x and $K_T(\theta)$ is the remainder :

$$J_T(\theta) = \frac{x^2}{2\theta\sigma^2}\left(e^{2\theta T}-1\right), \tag{8}$$

$$K_T(\theta) = \frac{1}{(2\theta)^2}\left(e^{2\theta T}-1\right)-\frac{T}{2\theta}. \tag{9}$$

Proof
It is easy to see from (2) and (3) that

$$E(X_t^2)=\{E(X_t)\}^2+V(X_t)=x^2e^{2\theta t}+\frac{\sigma^2}{2\theta}\left(e^{2\theta t}-1\right),$$

and thus, we have from (6) the Fisher information :

$$\begin{aligned} I_T(\theta) &= E\left[-\ddot{\ell}_T(\theta)\right]=\frac{1}{\sigma^2}\int_0^T E(X_t^2)\,dt \\ &= \frac{1}{\sigma^2}\left\{x^2\int_0^T e^{2\theta t}dt+\frac{\sigma^2}{2\theta}\int_0^T\left(e^{2\theta t}-1\right)dt\right\} \\ &= \frac{x^2}{\sigma^2}\frac{1}{2\theta}\left(e^{2\theta T}-1\right)+\frac{1}{2\theta}\left\{\frac{1}{2\theta}\left(e^{2\theta T}-1\right)-T\right\} \\ &= \frac{x^2}{2\theta\sigma^2}\left(e^{2\theta T}-1\right)+\frac{1}{(2\theta)^2}\left(e^{2\theta T}-1-2\theta T\right). \end{aligned}$$

□

Remark Consider the case $\theta=0$. The same result

$$I_T(0)=\frac{x^2}{\sigma^2}T+\frac{T^2}{2}=J_T(0)+K_T(0)$$

follows by setting

$$J_T(0)=\lim_{\theta\to 0}J_T(\theta)=\frac{x^2}{\sigma^2}T \quad \text{and} \quad K_T(0)=\lim_{\theta\to 0}K_T(\theta)=\frac{T^2}{2}.$$

3 Discretely observed case Let us divide equally the total time interval $[0,T]$ by n and denote :

$$\Delta=\frac{T}{n}, \quad t_k=k\Delta, \quad \text{for } k=0,1,\ldots,n.$$

That is,

$$0=t_0<t_1<\cdots<t_{n-1}<t_n=T, \quad t_k-t_{k-1}=\Delta, \quad \text{for } k=1,\ldots,n.$$

We observe the Ornstein-Uhlenbeck process at discrete times $\{t_k : k = 0, 1, \ldots, n\}$ as observations :

$$X_k = X_{t_k}, \quad k = 0, 1, \ldots, n, \quad X_0 = x.$$

Setting $t = \Delta$ and $x = X_{k-1}$ in (4), we have from the Markov property that the transition density is the normal density with mean $e^{\theta\Delta} X_{k-1}$ and variance $v_\Delta(\theta)$:

$$p(X_k \mid X_{k-1}) = \frac{1}{\sqrt{2\pi v_\Delta(\theta)}} \exp\left\{-\frac{(X_k - e^{\theta\Delta}X_{k-1})^2}{2v_\Delta(\theta)}\right\}, \tag{10}$$

where

$$v_\Delta(\theta) = \frac{\sigma^2}{2\theta}\left(e^{2\theta\Delta} - 1\right) = v(\theta) \quad \text{(say)}. \tag{11}$$

Therefore, we obtain that the likelihood function for discrete observations is

$$\begin{aligned} L_n(\theta) &= \prod_{k=1}^n p(X_k \mid X_{k-1}) \\ &= \prod_{k=1}^n \frac{1}{\sqrt{2\pi v(\theta)}} \exp\left\{-\frac{(X_k - e^{\theta\Delta}X_{k-1})^2}{2v(\theta)}\right\}. \end{aligned}$$

Hence, we have the log-likelihood function and its derivatives with respect to θ as follows :

$$\begin{aligned} \ell_n(\theta) &= -\frac{n}{2}\log 2\pi v(\theta) - \frac{1}{2v(\theta)}\sum_{k=1}^n (X_k - e^{\theta\Delta}X_{k-1})^2, \\ \dot\ell_n(\theta) &= -\frac{n}{2}\frac{\dot v(\theta)}{v(\theta)} + \frac{\dot v(\theta)}{2v(\theta)^2}\sum_{k=1}^n (X_k - e^{\theta\Delta}X_{k-1})^2 \\ &\quad + \frac{\Delta e^{\theta\Delta}}{v(\theta)}\sum_{k=1}^n (X_k - e^{\theta\Delta}X_{k-1})X_{k-1}, \end{aligned} \tag{12}$$

$$\begin{aligned} \ddot\ell_n(\theta) &= -\frac{n}{2}\frac{\ddot v(\theta)v(\theta) - \dot v(\theta)^2}{v(\theta)^2} \\ &\quad + \frac{\ddot v(\theta)v(\theta) - 2\dot v(\theta)^2}{2v(\theta)^3}\sum_{k=1}^n (X_k - e^{\theta\Delta}X_{k-1})^2 \\ &\quad - 2\frac{\dot v(\theta)}{v(\theta)^2}\Delta e^{\theta\Delta}\sum_{k=1}^n (X_k - e^{\theta\Delta}X_{k-1})X_{k-1} \\ &\quad + \frac{\Delta^2 e^{\theta\Delta}}{v(\theta)}\sum_{k=1}^n \{(X_k - e^{\theta\Delta}X_{k-1})X_{k-1} - e^{\theta\Delta}X_{k-1}^2\}. \end{aligned} \tag{13}$$

In order to represent the discrete Fisher information clearly, we set the functions $\phi(y)$ and $\psi(y)$ as follows :

$$\phi(y) = \frac{y}{\sinh y} = \frac{2y}{e^y - e^{-y}}, \tag{14}$$

$$\begin{aligned} \psi(y) &= e^y\phi(y) - \frac{\{e^y\phi(y) - 1\}^2}{y} \\ &= \frac{2ye^y}{e^y - e^{-y}} - \frac{1}{y}\left\{\frac{2ye^y}{e^y - e^{-y}} - 1\right\}^2 \end{aligned} \tag{15}$$

for $y \neq 0$ and $\phi(0) = \psi(0) = 1$.

Lemma 2

(i) *The function* $\phi(y)$ *is continuous in* y*, and we have*

$$0 < \phi(y) < 1, \quad \textit{for} \ \ y \neq 0.$$

(ii) *The function* $\psi(y)$ *is also continuous in* y*, and further, we have*

$$0 < \psi(y) < 1, \quad \textit{for} \ \ y < 0, \quad \textit{and} \quad \psi(y) < 1, \quad \textit{for} \ \ y > 0.$$

Proof

(i) By the L'Hospital's theorem for the indeterminate form, we have

$$\lim_{y \to 0} \ \phi(y) = \lim_{y \to 0} \frac{2y}{e^y - e^{-y}} = \lim_{y \to 0} \frac{2}{e^y + e^{-y}} = 1.$$

Thus, ϕ is continuous in y. It is easy to see that

$$0 < \phi(y) < 1, \quad \text{for any} \ \ y \neq 0.$$

(ii) We see

$$1 - \psi(y) = \{1 - e^y \phi(y)\} \left\{ 1 + \frac{1 - e^y \phi(y)}{y} \right\},$$

and set

$$\begin{aligned} \psi_1(y) &= 1 - e^y \phi(y) = \frac{e^y - e^{-y} - 2ye^y}{e^y - e^{-y}}, \\ \psi_2(y) &= 1 + \frac{1 - e^y \phi(y)}{y} = 1 + \frac{e^y - e^{-y} - 2ye^y}{y(e^y - e^{-y})}. \end{aligned}$$

It follows from (i) of Lemma 2 that

$$\lim_{y \to 0} \ \psi_1(y) = 1 - \lim_{y \to 0} \ e^y \phi(y) = 1 - 1 = 0.$$

Similarly, by the L'Hospital's theorem, we have

$$\begin{aligned} \lim_{y \to 0} \ \psi_2(y) &= 1 + \lim_{y \to 0} \ \frac{1 - e^{-2y} - 2y}{y - ye^{-2y}} \\ &= 1 + \lim_{y \to 0} \ \frac{2e^{-2y} - 2}{1 - e^{-2y} + 2ye^{-2y}} \\ &= 1 + \lim_{y \to 0} \ \frac{-4e^{-2y}}{4e^{-2y} - 4ye^{-2y}} = 1 - 1 = 0. \end{aligned}$$

These lead to

$$\lim_{y \to 0} \ \psi(y) = 1 - \lim_{y \to 0} \ \psi_1(y)\psi_2(y) = 1.$$

Thus, ψ is continuous in y.

It is easy to see that

$$\begin{aligned} &\lim_{y \to \infty} \ \psi_1(y) = -\infty, \qquad \text{and} \qquad \lim_{y \to -\infty} \ \psi_1(y) = 1, \\ &\lim_{y \to \infty} \ \psi_2(y) = -1, \qquad \text{and} \qquad \lim_{y \to -\infty} \ \psi_2(y) = 1. \end{aligned}$$

Therefore, these lead to

$$\begin{aligned}\lim_{y\to\infty}\psi(y) &= 1-\lim_{y\to\infty}\psi_1(y)\psi_2(y)=-\infty,\\ \lim_{y\to-\infty}\psi(y) &= 1-\lim_{y\to-\infty}\psi_1(y)\psi_2(y)=0.\end{aligned}$$

Since

$$\begin{aligned}\psi_1'(y) &= -2\frac{(e^y+ye^y)(e^y-e^{-y})-ye^y(e^y+e^{-y})}{(e^y-e^{-y})^2}\\ &= -2\frac{e^{2y}-1-2y}{(e^y-e^{-y})^2}=-2\left\{\frac{e^{2y}-1-2y}{(2y)^2}\right\}\phi(y)^2<0,\end{aligned}$$

it follows from $\psi_1(0)=0$ that

$$\psi_1(y)>0 \quad \text{if } \ y<0, \quad \text{and} \quad \psi_1(y)<0 \quad \text{if } \ y>0.$$

Furthermore, since

$$\begin{aligned}\psi_2'(y) &= \frac{(2e^{-2y}-2)(y-ye^{-2y})-(1-e^{-2y}-2y)(1-e^{-2y}+2ye^{-2y})}{(y-ye^{-2y})^2}\\ &= \frac{1}{y^2}\{\phi(y)^2-1\}<0,\end{aligned}$$

it follows from $\psi_2(0)=0$ that

$$\psi_2(y)>0 \quad \text{if } \ y<0, \quad \text{and} \quad \psi_2(y)<0 \quad \text{if } \ y>0.$$

Therefore, we have

$$1-\psi(y)=\psi_1(y)\psi_2(y)>0, \quad \text{that is,} \quad \psi(y)<1, \quad \text{for } \ y\neq 0.$$

These facts and $\psi'(y)=-\psi_1'(y)\psi_2(y)-\psi_1(y)\psi_2'(y)$ imply that

$$\psi'(y)>0 \ \text{ if } \ y<0, \quad \text{and} \quad \psi'(y)<0 \ \text{ if } \ y>0,$$

and thus, that $\psi(y)$ is monotone increasing for $y<0$ and decreasing for $y>0$, and have the maximum value $\psi(0)=1$ at $y=0$. □

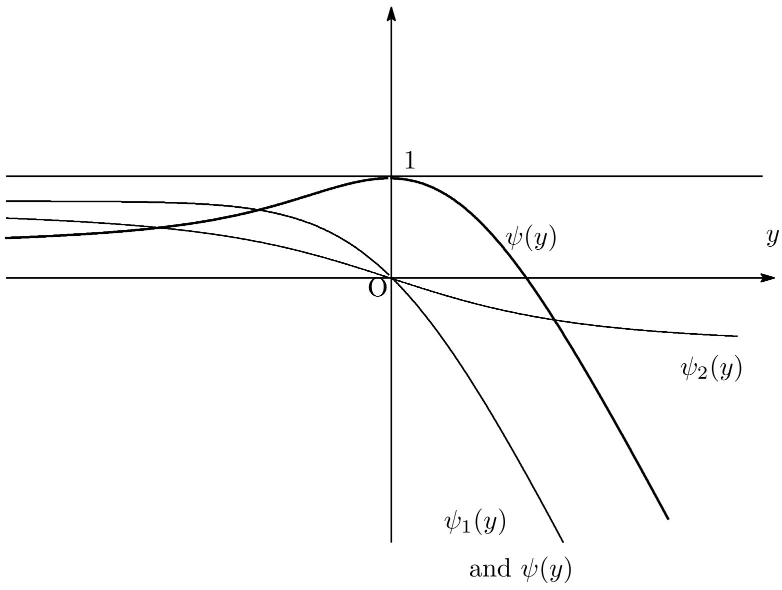

Figure 1 $\psi_1(y)$, $\psi_2(y)$

Lemma 3 *We have the Fisher information for the discrete observations of the drift parameter $\theta \neq 0$ of the Ornstein-Uhlenbeck process :*

$$\begin{aligned} I_n(\theta) &= \frac{x^2}{2\theta\sigma^2}\left(e^{2\theta T}-1\right)\phi(\theta\Delta)^2 \\ &\quad +\frac{1}{(2\theta)^2}\left(e^{2\theta T}-1\right)\phi(\theta\Delta)^2-\frac{T}{2\theta}\psi(\theta\Delta) \\ &= J_n(\theta)+K_n(\theta) \quad (say), \end{aligned} \tag{16}$$

where $J_n(\theta)$ is related to the initial value x and $K_n(\theta)$ is the remainder :

$$J_n(\theta) = \frac{x^2}{2\theta\sigma^2}\left(e^{2\theta T}-1\right)\phi(\theta\Delta)^2, \tag{17}$$

$$K_n(\theta) = \frac{1}{(2\theta)^2}\left(e^{2\theta T}-1\right)\phi(\theta\Delta)^2-\frac{T}{2\theta}\psi(\theta\Delta). \tag{18}$$

Proof
We see from (10), (11) and (13)

$$\begin{aligned} E\{\ddot{\ell}_n(\theta)\} &= -\frac{n}{2}\frac{\ddot{v}(\theta)v(\theta)-\dot{v}(\theta)^2}{v(\theta)^2}+\frac{\ddot{v}(\theta)v(\theta)-2\dot{v}(\theta)^2}{2v(\theta)^3}nv(\theta) \\ &\quad -\frac{\Delta^2 e^{2\theta\Delta}}{v(\theta)}\sum_{k=1}^{n}E(X_{k-1}^2) \\ &= -\frac{n}{2}\frac{\dot{v}(\theta)^2}{v(\theta)^2} \\ &\quad -\frac{\Delta^2 e^{2\theta\Delta}}{v(\theta)}\sum_{k=1}^{n}\left\{x^2e^{2\theta(k-1)\Delta}+\frac{\sigma^2}{2\theta}\left(e^{2\theta(k-1)\Delta}-1\right)\right\} \\ &= -\frac{x^2}{\sigma^2}\,\frac{2\theta\Delta^2e^{2\theta\Delta}}{e^{2\theta\Delta}-1}\,\frac{e^{2\theta\Delta n}-1}{e^{2\theta\Delta}-1} \\ &\quad -\frac{\Delta^2e^{2\theta\Delta}}{e^{2\theta\Delta}-1}\left(\frac{e^{2\theta\Delta n}-1}{e^{2\theta\Delta}-1}-n\right)-\frac{n}{2}\left(\frac{2\Delta e^{2\theta\Delta}}{e^{2\theta\Delta}-1}-\frac{1}{\theta}\right)^2. \end{aligned}$$

By using $T = n\Delta$ and the notations (14) and (15), we obtain the discrete Fisher information $I_n(\theta)$:

$$\begin{aligned} I_n(\theta) &= E\left\{-\ddot{\ell}_n(\theta)\right\} \\ &= \frac{x^2}{2\theta\sigma^2}\left(e^{2\theta T}-1\right)\left(\frac{\theta\Delta}{\sinh(\theta\Delta)}\right)^2 \\ &\quad +\frac{1}{(2\theta)^2}\left(e^{2\theta T}-1\right)\left(\frac{\theta\Delta}{\sinh(\theta\Delta)}\right)^2 \\ &\quad -\frac{T}{2\theta}\left\{\frac{e^{\theta\Delta}\theta\Delta}{\sinh(\theta\Delta)}-\frac{1}{\theta\Delta}\left(\frac{e^{\theta\Delta}\theta\Delta}{\sinh(\theta\Delta)}-1\right)^2\right\} \\ &= J_n(\theta)+K_n(\theta). \end{aligned}$$

□

Remark When $\theta = 0$, we can represent $I_n(0)$ equivalently. Let

$$J_n(0) = \lim_{\theta \to 0} J_n(\theta) = \frac{x^2}{\sigma^2} T \quad \text{and} \quad K_n(0) = \lim_{\theta \to 0} K_n(\theta) = \frac{T^2}{2},$$

then

$$I_n(0) = \frac{x^2}{\sigma^2} T + \frac{T^2}{2} = J_n(0) + K_n(0).$$

Hence, for all n and $T = n\Delta$, $I_T(0) = \frac{x^2}{\sigma^2} T + \frac{T^2}{2} = I_n(0)$.

4 Comparison between the continuous and discrete Fisher informations First, we consider the asymptotic relations between the continuous and discrete Fisher informations.

Theorem 4 *If the time T is fixed and the size of discrete observations n tends to infinity, in the situation where the discretizing time interval $\Delta = \frac{T}{n}$ becomes to tend to zero, then the discrete Fisher information converges to the continuous one :*

$$\lim_{n \to \infty} I_n(\theta) = I_T(\theta).$$

Proof
Since, by Lemma 2,

$$\lim_{y \to 0} \phi(y) = 1, \quad \text{and} \quad \lim_{y \to 0} \psi(y) = 1,$$

we have

$$\begin{aligned} J_n(\theta) &= J_T(\theta)\phi(\theta\Delta)^2 \to J_T(\theta), \\ K_n(\theta) &= \frac{1}{(2\theta)^2}\left(e^{2\theta T} - 1\right)\phi(\theta\Delta)^2 - \frac{T}{2\theta}\psi(\theta\Delta) \\ &\to \frac{1}{(2\theta)^2}\left(e^{2\theta T} - 1\right) - \frac{T}{2\theta} = K_T(\theta), \end{aligned}$$

as $\Delta \to 0$. These complete the proof of the theorem. □

Theorem 5 *If the discretizing time interval Δ is fixed and the size of discrete observations n be tended to infinity, where $T = n\Delta$ becomes to tend to infinity, then we have the following limit of the ratio of the discrete Fisher information to the continuous one :*

(i) *If $\theta > 0$,*

$$\lim_{n \to \infty} \frac{I_n(\theta)}{I_T(\theta)} = \phi(\theta\Delta)^2 < 1.$$

(ii) *If $\theta < 0$,*

$$\lim_{n \to \infty} \frac{I_n(\theta)}{I_T(\theta)} = \psi(\theta\Delta) < 1.$$

Proof
We rearrange the terms of two Fisher informations (7) and (16) as follows :

$$\begin{aligned} I_T(\theta) &= \left(e^{2\theta T}-1\right)\left\{\frac{x^2}{2\theta\sigma^2}+\frac{1}{(2\theta)^2}\right\}-\frac{T}{2\theta}, \\ I_n(\theta) &= \left(e^{2\theta T}-1\right)\left\{\frac{x^2}{2\theta\sigma^2}+\frac{1}{(2\theta)^2}\right\}\phi(\theta\Delta)^2-\frac{T}{2\theta}\psi(\theta\Delta). \end{aligned}$$

As $T \to \infty$, the leading term of the limitation is $e^{2\theta T}-1$ if $\theta > 0$, and $\frac{T}{2\theta}$ if $\theta < 0$. This implies the convergences of the theorem. Furthermore, it follows from Lemma 2 that $\phi(\theta\Delta)^2 < 1$ for any $\theta \neq 0$ and $\psi(\theta\Delta) < 1$ for $\theta < 0$. Hence, the proof of the theorem is completed. □

Now, we show that the continuous Fisher information $I_T(\theta)$ is exactly larger than the discrete Fisher information $I_n(\theta)$ under the fixed time-interval Δ and the total time-interval $[0, T]$.

Theorem 6 *Suppose that the time-interval Δ and the total time-interval $[0, T]$ of observation are fixed. Then, it holds that the difference of the continuous and discrete Fisher informations is exactly positive :*

$$I_T(\theta) - I_n(\theta) > 0, \quad \text{for } \theta \neq 0.$$

Proof
By Lemma 1 and 3, the difference is

$$I_T(\theta) - I_n(\theta) = \{J_T(\theta) - J_n(\theta)\} + \{K_T(\theta) - K_n(\theta)\}.$$

We denote three differences in the last equation by (8), (9), (17) and (18) as follows :

$$\begin{aligned} \mathcal{I}_n(\theta) &= I_T(\theta) - I_n(\theta), \\ \mathcal{J}_n(\theta) &= J_T(\theta) - J_n(\theta) \\ &= \frac{x^2}{2\theta\sigma^2}\left(e^{2\theta T}-1\right)\{1-\phi(\theta\Delta)^2\}, \\ \mathcal{K}_n(\theta) &= K_T(\theta) - K_n(\theta) \\ &= \frac{1}{(2\theta)^2}\left(e^{2\theta T}-1\right)\{1-\phi(\theta\Delta)^2\} - \frac{T}{2\theta}\{1-\psi(\theta\Delta)\}. \end{aligned}$$

By Lemma 2, we immediately obtain $\mathcal{J}_n(\theta)$, the part related to the initial value x, is nonnegative.

Now, we are going to prove that $\mathcal{K}_n(\theta)$, the remainder part, is positive. Recalling $y = \theta\Delta \neq 0$ and $T = n\Delta$, we rewrite it as follows :

$$\begin{aligned} \mathcal{K}_n(\theta) &= \frac{T}{2\theta}\left[\frac{e^{2\theta T}-1}{2\theta T}\left\{1-\phi(y)^2\right\}+\{\psi(y)-1\}\right] \\ &= \frac{n\Delta}{2\theta}\left[\frac{e^{2ny}-1}{2ny}\left\{1-\phi(y)^2\right\}+\{e^y\phi(y)-1\}-\frac{1}{y}\{e^y\phi(y)-1\}^2\right] \\ &= \frac{n}{2\theta^2}\left[\frac{e^{2ny}-1}{2n}\left\{1-\phi(y)^2\right\}+y\{e^y\phi(y)-1\}-\{e^y\phi(y)-1\}^2\right]. \end{aligned}$$

Set the part of [ ] in the last equation by $D_n(y)$:

$$D_n(y) = \frac{e^{2ny}-1}{2n}\left\{1-\phi(y)^2\right\} + y\left\{e^y\phi(y)-1\right\} - \left\{e^y\phi(y)-1\right\}^2.$$

Since $\frac{e^{2ny}-1}{2n}$ is increasing in n for any $y \neq 0$ and $1-\phi(y)^2 > 0$, we see $D_n(y) \geq D_1(y)$. Therefore, it is sufficient to show $D_1(y) > 0$ in order to prove $D_n(y) > 0$. In fact, we see

$$\begin{aligned} D_1(y) &= \frac{e^{2y}-1}{2}\left\{1-\phi(y)^2\right\} + y\left(e^y\phi(y)-1\right) - \left(e^y\phi(y)-1\right)^2 \\ &= e^y \sinh y\left(1-\frac{y^2}{\sinh^2 y}\right) + \left(\frac{y^2e^y}{\sinh y} - y\right) - \left(\frac{ye^y}{\sinh y} - 1\right)^2 \\ &= \frac{1}{\sinh^2 y}\left(e^y\sinh^3 y - y\sinh^2 y - y^2e^{2y} + 2ye^y\sinh y - \sinh^2 y\right). \end{aligned}$$

Here, we put the part of ( ) in the last equation by $\xi(y)$:

$$\begin{aligned} \xi(y) &= e^y\sinh^3 y - y\sinh^2 y - y^2e^{2y} + 2ye^y\sinh y - \sinh^2 y \\ &= \frac{1}{8}\left(e^{4y} - 5e^{2y} + 7 - 3e^{-2y}\right) + \frac{y}{4}\left(3e^{2y} - 2 - e^{-2y}\right) - y^2e^{2y}. \end{aligned}$$

Its derivatives with respect to y are

$$\begin{aligned} \xi'(y) &= \frac{1}{2}\left\{(e^{4y} - e^{2y} - 1 + e^{-2y}) - y(e^{2y} - e^{-2y}) - 4y^2e^{2y}\right\}, \\ \xi''(y) &= \frac{1}{2}\left\{(4e^{4y} - 3e^{2y} - e^{-2y}) - y(10e^{2y} + 2e^{-2y}) - 8y^2e^{2y}\right\} \\ &= \frac{1}{2}e^{2y}\left(4e^{2y} - 10y - 8y^2 - 3 - 2ye^{-4y} - e^{-4y}\right). \end{aligned}$$

Moreover, denoting the part of ( ) in the last equation by $\eta(y)$:

$$\eta(y) = 4e^{2y} - e^{-4y} - 2ye^{-4y} - 10y - 8y^2 - 3,$$

we have its derivatives :

$$\begin{aligned} \eta'(y) &= 8e^{2y} + 2e^{-4y} + 8ye^{-4y} - 10 - 16y, \\ \eta''(y) &= 16e^{2y} - 32ye^{-4y} - 16 \geq 16(e^{2y} - 1 - 2y) > 0. \end{aligned}$$

This means that $\eta'(y)$ is monotone increasing and $\eta'(0) = 0$ and thus, that $\eta'(y) < 0$, if $y < 0$ and $\eta'(y) > 0$, if $y > 0$. Consequently, it follows that $\eta(y)$ takes the minimum $\eta(0) = 0$ and thus, that $\xi''(y) > 0$ and $\xi'(y)$ is monotone-increasing. Both this and $\xi'(0) = 0$ mean that $\xi(x)$ takes the minimum $\xi(0) = 0$. Hence, we conclude that $D_1(y) > 0$ and at the same time, that $\mathcal{K}_n(\theta) > 0$. We therefore showed

$$\mathcal{I}_n(\theta) = \mathcal{J}_n(\theta) + \mathcal{K}_n(\theta) > 0.$$

The proof of the theorem is completed. □

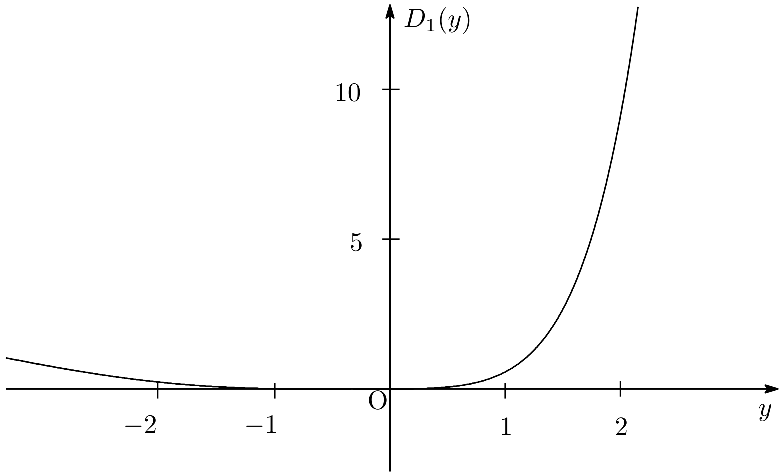

Figure 2 $D_1(y)$

5 Discussion Figure 2 shows that $D_1(y)$ is almost equal to zero around $y = 0$. But, in the case $y > 0$, $D_1(y)$ is increasing more rapidly than in the case $y < 0$. We conclude that the loss of information which arises from discrete observations depends on the product of θ and Δ rather than only on the discrete observation time-interval Δ, and then it becomes very small, if $y = \theta\Delta$ is sufficiently small.

References

[1] Dacunha-Castelle, D. and Florens-Zmirou, D. (1986) Estimation of the coefficients of a diffusion from discrete observations, *Stochastics*, 19, 263-284

[2] Kessler, M. (2000) Simple and explicit estimating functions for a discretely observed diffusion process, *Scandinavian Journal of Statistics*, 27, 65-82,

[3] Küchler, U. and Sørensen, M. (1997) *Exponential Families of Stochastic Processes*, Springer, New York.

[4] Liptser, R.S. and Shiryayev, A.N. (2001) *Statistics of Random Processes vol.1, 2nd. ed.*, Springer, New York.

Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama-cho, Toyonaka, Osaka 560-8531, Japan.

E-mail : fujii@sigmath.es.osaka-u.ac.jp