NOTE : A REPEATED ONE-PLAYER GAME OF DECEPTION WITH DISCOUNTING

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ABSTRACT. A repeated one-player game of deception with discounting is given and the solution is derived explicitly. It clearly shows a commonly conceivable fact that a player who deceives his (or her) opponent can get advantage.

1 No-Deception Case. Player I observes the r.v.s $X_t, t = 1, 2, \cdots$, sequentially one by one which are *i.i.d.* with distribution $U_{[0,1]}$. Facing the first r.v. $X_1 = x$, player I shows x to player II, and then II makes his choice whether to accept ($= A$) or reject ($= R$) it. If II chooses A, he receives the amount x from I. If II chooses R, the x is rejected and the next X_2 is observed by I and shown to II and the above choice process continues. The discount rate $\beta \in [0, 1]$ is introduced. The process ends as soon as II chooses A. Player II aims to maxmize his expected payoff.

Let v be the expected payoff for II obtained by employing his optimal strategy. Then

$$v = E[X \vee \beta v] \tag{1.1}$$

which gives the equation $v = \frac{1}{2}\left(1 + \beta^2 v^2\right)$, *i.e.*,

$$v = \frac{1}{\beta^2}\left(1 - \sqrt{1-\beta^2}\right), \tag{1.2}$$

since another root is evidently inappropriate. The optimal strategy for II is to choose $A(R)$, if $X = x > (<) \beta v$.

Some values of v for $\beta \in [0, 1]$ are given in Table 1.

Table 1. Game values for No-Deception case.

$\beta = 1$	0.8	3/4	1/2	1/4	0
$v = 1$	5/8(= 0.625)	0.6019	$2(2-\sqrt{3}) \approx 0.5359$	0.5081	1/2
$\beta v = 1$	1/2	0.4514	$2-\sqrt{3} \approx 0.2680$	0.1270	0

(The case $\beta = 1$ means that II always rejects every x seeking for the largest amount 1.
The case $\beta = 0$ means that II accepts the first r.v. ending the process.)

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2 The Case where Player I Deceives his Opponent. Let player I choose a number $a \in [\frac{1}{2}, 1]$. Facing the first r.v. $X_1 = x$, player I opens (cover) the x he privately observed, if $\overline{a} < x < a$ (if otherwise). He covers large x since he doesn't want this x to be accepted by player II. He covers small x since he wants this x to be accepted by II by II's mistake. When I covers the x, which II cannot know its value, II employs the mixed strategy $(A, R; p, \overline{p}), 0 \leq p \leq 1$. When I opens the x, then value becomes known to II, and II chooses $A(R)$, if $x > (<) \frac{1}{2}$ (see Remark 1). When I opens x and II rejects it, or when x is covered and II rejects it, then the process continues and the next r.v. X_2 is observed by I. This process ends as soon as II chooses A. Player I aims to minimize the expected payoff to II when the process ends. Figure 1 shows the choice-pairs for the players when $X_1 = x$.

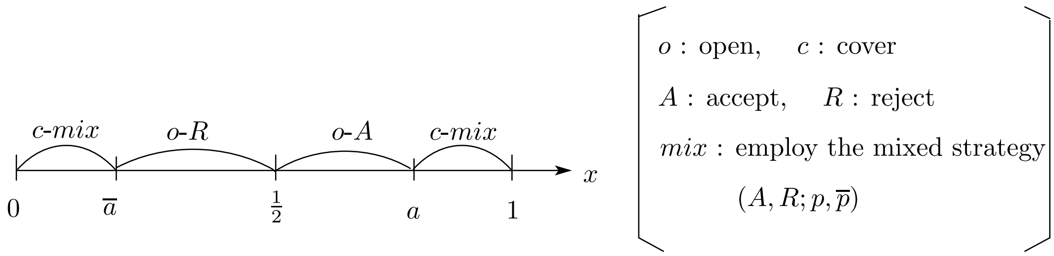

Figure 1. Players' behavior in the deception game

Let v be the value of the game where I is the minimizer. Also let $\beta \in [0, 1]$ be the discount rate. The case $\beta = 1$ means that the process continues until II accepts a r.v. without discounting. The case $\beta = 0$ means that II accepts the first r.v., thus ending the process, since if II chooses R he gets zero payoff. The expected payoff to II, when players behave as is shown in Figure 1, is

$$\text{(2.1)} \qquad M(a|p) = \left[\int_{\overline{a}}^{\frac{1}{2}} \beta v + \int_{1/2}^{a} x\right] dx + \left[\int_0^{\overline{a}} + \int_a^1\right] (px + \overline{p}\beta v)\, dx$$
$$= \frac{1}{2}a^2 - \{p - (2p-1)\beta v\}\, a + \left(\frac{3}{2} - 2p\right)\beta v + p - \frac{1}{8},$$

a convex function of a. Therefore the optimal choice for I is $a^* = p - (2p-1)\beta v = \overline{\beta v}p + \beta v\overline{p}$, which is in $\left[\frac{1}{2}, \overline{\beta v}\right]$, only if $\frac{1}{2} \leq p \leq 1$. Since

$$\text{(2.2)} \qquad M(a^*|p) = -\frac{1}{2}(2p-1)^2\beta^2 v^2 + \left(2p^2 - 3p + \frac{3}{2}\right)\beta v - \frac{1}{2}p^2 + p - \frac{1}{8},$$

we obtain, by equating $M(a^*|p)$ with v, the following quadratic equation

$$\text{(2.3)} \qquad -\frac{1}{2}(2p-1)^2\beta^2 v^2 + \left\{\left(2p^2 - 3p + \frac{3}{2}\right)\beta - 1\right\} v - \frac{1}{2}p^2 + p - \frac{1}{8} = 0.$$

Eq.(2.3) gives

$$4\,(1 - \beta/2)\, v = 1, \quad \beta^2 v^2 + (8 - 3\beta)v - \frac{11}{4} = 0, \quad \beta^2 v^2 + (2 - \beta)v - \frac{3}{4} = 0$$

for $p = 1/2, 3/4, 1$, respectively.

Moreover, (2.3), in the case $\beta = 1$, gives $v = \frac{1}{2}$ and hence $a^* = \overline{v}p + v\overline{p} = \frac{1}{2}$, for $\forall p \in (0, 1]$. Numerical values of a^* and v for some parameter-pairs of (β, p) are computed from (2.3), and are given in Table 2.

Table 2. Solution of the deception game.

	$p = 1/2$	$p = 0.6$	$p = 3/4$	$p = 1$	$(*)$
$\beta = 1$	$\alpha^* = 1/2$	1/2	1/2	1/2	
	$v = 1/2$	1/2	1/2	1/2	1
0.8	1/2	0.5295	0.5635	0.6044	
	$\frac{5}{12}(\approx 0.4167)$	0.4405	0.4662	0.4946	$5/8(= 0.625)$
1/2	1/2	0.56275	0.6459	0.7680	
	1/3	0.3725	0.4164	0.4641	0.5359
0	1/2	0.6	3/4	1	
	1/4	0.295	$\frac{11}{32}(= 0.34375)$	$3/8(= 0.375)$	1/2

The column $(*)$ in the table was moved from Table 1 for No-deception Case in Section 1, in order to make clear the advantage of deception for player I.

Remark 1 The decision threshold 1/2 for player II is conventionaly chosen by the reason that $EX = 1/2$. The other choice, for example, βv, will need more annoying computations.

Remark 2 The games discussed in this article are a *one-player game* for II in Section 1, and for I in Section 2. In Section 2, II has no decision variable, since $p \in \left[\frac{1}{2}, 1\right]$ is a predetermined parameter known for both of I and II.

Remark 3 Some *two-player* deception games with the similar nature as treated in the present note are discussed in Ref.[1~5]. Among these the most related one to the problem in this note is Ref.[3].

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