A NOTE ON $[\mathfrak{a},\mathfrak{b}]^r$-COMPACTNESS AND $[\mathfrak{a},\mathfrak{b}]^r$-REFINABILITY

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Abstract. In this paper, we shall show: (1) Properties $[\mathfrak{a},\mathfrak{b}]^r$-compactness, $[\mathfrak{a},\mathfrak{b}]^r$-refinability and weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinability are preserved under taking countable closed sums, F_σ-subsets and preimages of perfect mappings.

(2) (GCH) Let X be a space with $t(X) \le \mathfrak{n}$ and Y be a bounded $\mathfrak{n}$-compact space for some cardinal $\mathfrak{n}$. If X is $[\mathfrak{a},\mathfrak{b}]^r$-compact (resp. $[\mathfrak{a},\mathfrak{b}]^r$-refinable, weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable) and $L(Y) < \mathfrak{a}$, then $X \times Y$ is $[\mathfrak{a},\mathfrak{b}]^r$-compact (resp. $[\mathfrak{a},\mathfrak{b}]^r$-refinable, weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable).

(3) Suppose that $\mathfrak{a}$ is a regular cardinal with $\mathfrak{a} \ge \omega_1$. Let X be a separable metric space and Y be a $P(\omega)$-space. If Y is $[\mathfrak{a},\mathfrak{b}]^r$-compact (resp. $[\mathfrak{a},\mathfrak{b}]^r$-refinable, weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable), then $X \times Y$ is $[\mathfrak{a},\mathfrak{b}]^r$-compact (resp. $[\mathfrak{a},\mathfrak{b}]^r$-refinable, weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable).

1. Introduction

Throughout this paper, X and Y denote topological spaces and $\mathfrak{a},\mathfrak{b},\mathfrak{m}$ and $\mathfrak{n}$ denote infinite cardinal numbers. All spaces are assumed to be topological spaces with no separation axioms and all maps are assumed to be continuous.

In [1], Alexandroff and Urysohn introduced $[\mathfrak{a},\mathfrak{b}]^r$-compactness. After that, Hodel and Vaughan [5] investigated the relation between $[\mathfrak{a},\mathfrak{b}]^r$-compactness and $[\mathfrak{a},\mathfrak{b}]$-compatness and they introduced $[\mathfrak{a},\mathfrak{b}]^r$-refinability. Weak $[\mathfrak{a},\mathfrak{b}]^r$-refinability was introduced in Worrell and Wicke [9] where it was shown that weakly $[\omega_1,\infty)^r$-refinable, countable compact space is compact.

In this paper, we shall investigate $[\mathfrak{a},\mathfrak{b}]^r$-compactness, $[\mathfrak{a},\mathfrak{b}]^r$-refinability and weak $[\mathfrak{a},\mathfrak{b}]^r$-refinability.

The cardinality of a set A is denoted by $|A|$.

Definition 1. A space X is said to be $[\mathfrak{a},\mathfrak{b}]^r$-*compact* if every subset M of X such that $\mathfrak{a} \le |M| \le \mathfrak{b}$ and $|M| = \mathfrak{m}$ is a regular cardinal has a complete accumulation point, i.e., a point $p \in X$ such that for every neighborhood O of p, $|O \cap M| = |M|$.

A space X is $[\mathfrak{a},\infty)^r$-*compact* if it is $[\mathfrak{a},\mathfrak{b}]^r$-compact for every $\mathfrak{b} \ge \mathfrak{a}$.

Definition 2. A space X is said to be $[\mathfrak{a},\mathfrak{b}]^r$-*refinable* if for every open cover $\mathcal{U}$ of X such that $\mathfrak{a} \le |\mathcal{U}| \le \mathfrak{b}$ and $|\mathcal{U}| = \mathfrak{m}$ is a regular cardinal, there is a collection $\{\mathcal{V}_\alpha : \alpha \in A\}$ of open refinements of $\mathcal{U}$ with $|A| < \mathfrak{m}$ such that for each point $p \in X$, $\mathrm{ord}(p,\mathcal{V}_\alpha) < \mathfrak{m}$ for some $\alpha \in A$. Here $\mathrm{ord}(p,\mathcal{V}_\alpha) = |\{V : p \in V \in \mathcal{V}_\alpha\}|$.

A space X is $[\mathfrak{a},\infty)^r$-*refinable* if it is $[\mathfrak{a},\mathfrak{b}]^r$-refinable for every $\mathfrak{b} \ge \mathfrak{a}$.

Definition 3. A space X is said to be *weakly* $[\mathfrak{a},\mathfrak{b}]^r$-*refinable* if for every open cover $\mathcal{U}$ of X such that $\mathfrak{a} \le |\mathcal{U}| \le \mathfrak{b}$ and $|\mathcal{U}| = \mathfrak{m}$ is a regular cardinal, there is an open refinement

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$\mathcal{V} = \bigcup_{\alpha \in A} \mathcal{V}_\alpha$ of $\mathcal{U}$ with $|A| < \mathfrak{m}$ such that for each point $p \in X$, $0 <\mathrm{ord}(p, \mathcal{V}_\alpha) < \mathfrak{m}$ for some $\alpha \in A$.

A space X is *weakly* $[\mathfrak{a}, \infty)^r$*-refinable* if it is weakly $[\mathfrak{a}, \mathfrak{b}]^r$-refinable for every $\mathfrak{b} \geq \mathfrak{a}$.

It is clear that $\delta\theta$-refinable space is $[\omega_1, \infty)^r$-refinable space, and weakly $\delta\theta$-refinable space is weakly $[\omega_1, \infty)^r$-refinable space.

A cardinal is an initial ordinal and an ordinal is the set of its predecessors. Thus, for a subset M of a space X with $|M| = \mathfrak{m}$, we can denote $M = \{x_\alpha : \alpha < \mathfrak{m}\}$. Similarly, for a cover $\mathcal{U}$ of X with $|\mathcal{U}| = \mathfrak{m}$, we can denote $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{m}\}$.

The following theorem plays a fundamental role in the theory of $[\mathfrak{a}, \mathfrak{b}]^r$-compactness. We shall give this theorem with a proof for the convenience of readers.

Theorem 1. [1] *For any space X the following conditions are equivalent.*

(a) *X is $[\mathfrak{a}, \mathfrak{b}]^r$-compact.*

(b) *Every open cover $\mathcal{U}$ of X such that $\mathfrak{a} \leq |\mathcal{U}| \leq \mathfrak{b}$ and $|\mathcal{U}| = \mathfrak{m}$ is a regular cardinal has a subcover $\mathcal{U}'$ such that $|\mathcal{U}'| < \mathfrak{m}$.*

Proof. (a) $\Rightarrow$ (b). Assume that (a) holds and (b) does not hold. Then there is an open cover $\mathcal{U}$ of X such that $\mathfrak{a} \leq |\mathcal{U}| \leq \mathfrak{b}$ and $|\mathcal{U}| = \mathfrak{m}$ is a regular cardinal and $\mathcal{U}$ has no subcover whose cardinality $< \mathfrak{m}$. We can denote $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{m}\}$. For each $\alpha < \mathfrak{m}$, since $X \setminus \cup_{\beta<\alpha} U_\beta \neq \emptyset$, we can choose $x_\alpha \in X \setminus \cup_{\beta<\alpha} U_\beta$.

Put $M = \{x_\alpha : \alpha < \mathfrak{m}\}$. Then $|M| = \mathfrak{m}$. Since X is $[\mathfrak{a}, \mathfrak{b}]^r$-compact, there exists a complete accumulation point p of M.

Choose an $\alpha < \mathfrak{m}$ such that $p \in U_\alpha$. Since $x_\lambda \in X \setminus \cup_{\beta<\lambda} U_\beta$, $x_\lambda \notin U_\alpha$ for every $\lambda > \alpha$. Thus $|U_\alpha \cap M| < \mathfrak{m}$. This contradicts that p is a complete accumulation point of M.

(b) $\Rightarrow$ (a). Assume that (b) holds. Let M be a subset of X such that $\mathfrak{a} \leq |M| \leq \mathfrak{b}$ and $|M| = \mathfrak{m}$ is a regular cardinal. Assume that M has no complete accumulation point. Then, for each $x \in X$, there is a neighborhood O_x of x such that $|O_x \cap M| < \mathfrak{m}$.

We may assume that M is an well-orderd set and put $M = \{x_\alpha : \alpha < \mathfrak{m}\}$. For each $\alpha < \mathfrak{m}$, put $U_\alpha = \cup\{O : O \text{ is an open set of } X \text{ such that } \emptyset \neq O \cap M \subset \{x_\beta : \beta \leq \alpha\}\}$. Then $\overline{M} \subset \cup\{U_\alpha : \alpha < \mathfrak{m}\}$. To show this, let $x \in \overline{M}$. There is a neighborhood O_x of x such that $|O_x \cap M| < \mathfrak{m}$. Thus $O_x \cap M \subset \{x_\beta : \beta \leq \alpha\}$ for some $\alpha < \mathfrak{m}$. Since $O_x \cap M \neq \emptyset$, $O_x \subset U_\alpha$.

Put $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{m}\} \cup \{X \setminus \overline{M}\}$. Then $\mathcal{U}$ is an open cover of X and $|\mathcal{U}| = \mathfrak{m}$. By the condition (b), there is a subcover $\mathcal{U}'$ of $\mathcal{U}$ with $|\mathcal{U}'| < \mathfrak{m}$. Then there is a $\lambda < \mathfrak{m}$ such that $\mathcal{U}' = \{U_\alpha : \alpha < \lambda\} \cup \{X \setminus \overline{M}\}$. Therefore $M \subset \cup\{U_\alpha : \alpha < \lambda\}$. Since $\lambda < \mathfrak{m}$ and $|U_\alpha \cap M| < \mathfrak{m}$ for every $\alpha < \lambda$ and $\mathfrak{m}$ is a regular cardinal, $|M| = |\cup_{\alpha<\lambda} U_\alpha \cap M| = \sum_{\alpha<\lambda} |U_\alpha \cap M| < \mathfrak{m} = |M|$. This is a contradiction. □

We use Theorem 1 to prove our results in this note.

2. Countable closed sums and F_σ-subsets

A subset F is called an F_σ-set of X if F is presented by a countable union of closed subsets of X.

In this section we shall show that $[\mathfrak{a}, \mathfrak{b}]^r$-compactness, $[\mathfrak{a}, \mathfrak{b}]^r$-refinability and weakly $[\mathfrak{a}, \mathfrak{b}]^r$-refinability are preserved under taking F_σ-sets and countable closed sums.

Let $\mathfrak{a}$ be a cardinal with $\mathfrak{a} \geq \omega_1$.

First we shall prove the following.

Theorem 2. *Let Y be a closed subset of X.*

(1) *If X is $[\mathfrak{a}, \mathfrak{b}]^r$-compact, then Y is $[\mathfrak{a}, \mathfrak{b}]^r$-compact.*

(2) *If X is $[\mathfrak{a},\infty)^r$-compact, then Y is $[\mathfrak{a},\infty)^r$-compact.*
(3) *If X is $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then Y is $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(4) *If X is $[\mathfrak{a},\infty)^r$-refinable, then Y is $[\mathfrak{a},\infty)^r$-refinable.*
(5) *If X is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then Y is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(6) *If X is weakly $[\mathfrak{a},\infty)^r$-refinable, then Y is weakly $[\mathfrak{a},\infty)^r$-refinable.*

Proof. (2), (4) and (6) follow from (1), (3) and (5), respectively.

To prove (1), (3) and (5), let $\mathcal{U}$ be an open cover of Y such that $\mathfrak{a} \le |\mathcal{U}| \le \mathfrak{b}$ and $|\mathcal{U}| = \mathfrak{m}$ is a regular cardinal. We can write $\mathcal{U} = \{U_\lambda : \lambda < \mathfrak{m}\}$. For each $\lambda < \mathfrak{m}$, let G_λ be an open subset of X such that $U_\lambda = G_\lambda \cap Y$, and put $\mathcal{G} = \{G_\lambda : \lambda < \mathfrak{m}\} \cup \{X \setminus Y\}$. Then $\mathcal{G}$ is an open cover of X and $|\mathcal{G}| = \mathfrak{m}$.

(1). Since X is $[\mathfrak{a},\mathfrak{b}]^r$-conpact, there is a subcover $\mathcal{G}'$ of $\mathcal{G}$ with $|\mathcal{G}'| < \mathfrak{m}$.

Put $\mathcal{U}' = \{G \cap Y : G \in \mathcal{G}'\}$. Then $\mathcal{U}'$ is a subcover of $\mathcal{U}$ with $|\mathcal{U}'| < \mathfrak{m}$. Hence Y is $[\mathfrak{a},\mathfrak{b}]^r$-compact.

(3). Since X is $[\mathfrak{a},\mathfrak{b}]^r$-refinable, there is a collection $\{\mathcal{H}_\alpha : \alpha \in A\}$ of open refinements of $\mathcal{G}$ with $|A| < \mathfrak{m}$ such that for each $x \in X$, there is an $\alpha \in A$ such that $\mathrm{ord}(x, \mathcal{H}_\alpha) < \mathfrak{m}$.

Put $\mathcal{V}_\alpha = \{H \cap Y : H \in \mathcal{H}_\alpha\}$. Then $\{\mathcal{V}_\alpha : \alpha \in A\}$ is a collection of open refinements of $\mathcal{U}$, and for each $y \in Y$ there is an $\alpha \in A$ such that $\mathrm{ord}(y, \mathcal{V}_\alpha) < \mathfrak{m}$. Hence X is $[\mathfrak{a},\mathfrak{b}]^r$-refinable.

(5). Since X is weakly $[\mathfrak{a},\mathfrak{b}]^r$-rifinable, there is an open refinement $\mathcal{H} = \cup_{\alpha\in A}\mathcal{H}_\alpha$ of $\mathcal{G}$ with $|A| < \mathfrak{m}$ such that for each $x \in X$, there is an $\alpha \in A$ such that $0 < \mathrm{ord}(x, \mathcal{H}_\alpha) < \mathfrak{m}$.

Put $\mathcal{V}_\alpha = \{H \cap Y : H \in \mathcal{H}_\alpha\}$ and $\mathcal{V} = \cup_{\alpha\in A}\mathcal{V}_\alpha$. Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$, and for each $y \in Y$, there is an $\alpha \in A$ such that $0 <\mathrm{ord}(y, \mathcal{V}_\alpha) < \mathfrak{m}$. Hence Y is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable. □

Theorem 3. *Let X be a space and assume that X is the union of countably many closed subspaces $Y_n, n \in \omega$ of X.*

(1) *If each Y_n is $[\mathfrak{a},\mathfrak{b}]^r$-compact, then X is $[\mathfrak{a},\mathfrak{b}]^r$-compact.*
(2) *If each Y_n is $[\mathfrak{a},\infty)^r$-compact, then X is $[\mathfrak{a},\infty)^r$-compact.*
(3) *If each Y_n is $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then X is $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(4) *If each Y_n is $[\mathfrak{a},\infty)^r$-refinable, then X is $[\mathfrak{a},\infty)^r$-refinable.*
(5) *If each Y_n is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then X is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(6) *If each Y_n is weakly $[\mathfrak{a},\infty)^r$-refinable, then X is weakly $[\mathfrak{a},\infty)^r$-refinable.*

Proof. (2), (4) and (6) follow from (1), (3) and (5), respectively.

To prove (1), (3) and (5), let $\mathcal{U}$ be an open cover of X such that $\mathfrak{a} \le |\mathcal{U}| \le \mathfrak{b}$ and $|\mathcal{U}| = \mathfrak{m}$ is a regular cardinal. We can denote $\mathcal{U} = \{U_\lambda : \lambda < \mathfrak{m}\}$. For each $n \in \omega$, put $\mathcal{U}_n = \{U_\lambda \cap Y_n : \lambda < \mathfrak{m}\}$. Then $\mathcal{U}_n$ is an open cover of Y_n and $|\mathcal{U}_n| = \mathfrak{m}$.

(1). Since Y_n is $[\mathfrak{a},\mathfrak{b}]^r$-compact by the assumption, there is a subcover $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $|\mathcal{V}_n| < \mathfrak{m}$.

For each $V \in \mathcal{V}_n$, let us choose an element $U_V \in \mathcal{U}$ such that $V = U_V \cap Y_n$ and put $\mathcal{U}'_n = \{U_V : V \in \mathcal{V}_n\}$. Let $\mathcal{U}' = \cup_{n\in\omega}\mathcal{U}'_n$. Then $\mathcal{U}'$ is a subcover of $\mathcal{U}$ such that $|\mathcal{U}'| < \mathfrak{m}$. Hence X is $[\mathfrak{a},\mathfrak{b}]^r$-compact.

(3). Since Y_n is $[\mathfrak{a},\mathfrak{b}]^r$-refinable by the assumption, there is a collection $\{\mathcal{V}'_{n,\alpha} : \alpha \in A_n\}$ of open refinements of $\mathcal{U}_n$ with $|A_n| < \mathfrak{m}$ such that for each $y \in Y_n$, there is an $\alpha \in A_n$ such that $\mathrm{ord}(y, \mathcal{V}'_{n,\alpha}) < \mathfrak{m}$.

For each $V \in \mathcal{V}'_{n,\alpha}$, let us choose an open subset O_V of X such that $V = O_V \cap Y_n$ and an element $U_V \in \mathcal{U}$ such that $V \subset U_V$ and put $H_V = O_V \cap U_V$. Let $\mathcal{V}_{n,\alpha} = \{H_V : V \in \mathcal{V}'_{n,\alpha}\} \cup \{(X \setminus Y_n) \cap U : U \in \mathcal{U}\}$ and put $B = \cup_{n\in\omega}(\{n\} \times A_n)$. Then $|B| < \mathfrak{m}$ and $\{\mathcal{V}_{n,\alpha} : (n,\alpha) \in B\}$

is a collection of open refinements of $\mathcal{U}$. For each $x \in X$, there is an $n \in \omega$ such that $x \in Y_n$. Let us choose an $\alpha \in A_n$ such that $\mathrm{ord}(x, \mathcal{V}'_{n,\alpha}) < \mathfrak{m}$. Then $(n, \alpha) \in B$ and $\mathrm{ord}(x, \mathcal{V}_{n,\alpha}) < \mathfrak{m}$. Hence X is $[\mathfrak{a}, \mathfrak{b}]^r$-refinable.

(5). Since Y_n is weakly $[\mathfrak{a}, \mathfrak{b}]^r$-refinable by the assumption, there is an open refinement $\mathcal{V}'_n = \cup_{\alpha \in A_n} \mathcal{V}'_{n,\alpha}$ of $\mathcal{U}_n$ with $|A_n| < \mathfrak{m}$ such that for each $y \in Y_n$, there is an $\alpha \in A_n$ such that $0 < \mathrm{ord}(y, \mathcal{V}'_{n,\alpha}) < \mathfrak{m}$.

For each $V \in \mathcal{V}'_{n,\alpha}$, let us choose an open subset O_V of X such that $V = O_V \cap Y_n$ and an element $U_V \in \mathcal{U}$ such that $V \subset U_V$, and put $H_V = O_V \cap U_V$. Let $\mathcal{V}_{n,\alpha} = \{H_V : V \in \mathcal{V}'_{n,\alpha}\}$ and put $B = \cup_{n \in \omega}(\{n\} \times A_n)$. Then $|B| < \mathfrak{m}$ and $\mathcal{V} = \cup_{(n,\alpha) \in B} \mathcal{V}_{n,\alpha}$ is an open refinement of $\mathcal{U}$. For each $x \in X$, there is an $n \in \omega$ such that $x \in Y_n$. Let us choose an $\alpha \in A_n$ such that $0 < \mathrm{ord}(x, \mathcal{V}'_{n,\alpha}) < \mathfrak{m}$. Then $(n, \alpha) \in B$ and $0 < \mathrm{ord}(x, \mathcal{V}_{n,\alpha}) < \mathfrak{m}$. Hence X is weakly $[\mathfrak{a}, \mathfrak{b}]^r$-refinable. □

The following theorem follows from Theorems 2 and 3.

Theorem 4. *Let F be an F_σ-set of X.*

(1) *If X is $[\mathfrak{a}, \mathfrak{b}]^r$-compact, then F is $[\mathfrak{a}, \mathfrak{b}]^r$-compact.*
(2) *If X is $[\mathfrak{a}, \infty)^r$-compact, then F is $[\mathfrak{a}, \infty)^r$-compact.*
(3) *If X is $[\mathfrak{a}, \mathfrak{b}]^r$-refinable, then F is $[\mathfrak{a}, \mathfrak{b}]^r$-refinable.*
(4) *If X is $[\mathfrak{a}, \infty)^r$-refinable, then F is $[\mathfrak{a}, \infty)^r$-refinable.*
(5) *If X is weakly $[\mathfrak{a}, \mathfrak{b}]^r$-refinable, then F is weakly $[\mathfrak{a}, \mathfrak{b}]^r$-refinable.*
(6) *If X is weakly $[\mathfrak{a}, \infty)^r$-refinable, then F is weakly $[\mathfrak{a}, \infty)^r$-refinable.*

3. Mappings

A mapping $f : X \to Y$ is said to be *perfect* if f is a closed mapping with $f^{-1}(y)$ compact for each $y \in Y$.

Theorem 5. *Let f be a perfect map from X onto Y.*

(1) *If Y is $[\mathfrak{a}, \mathfrak{b}]^r$-compact, then X is $[\mathfrak{a}, \mathfrak{b}]^r$-compact.*
(2) *If Y is $[\mathfrak{a}, \infty)^r$-compact, then X is $[\mathfrak{a}, \infty)^r$-compact.*
(3) *If Y is $[\mathfrak{a}, \mathfrak{b}]^r$-refinable, then X is $[\mathfrak{a}, \mathfrak{b}]^r$-refinable.*
(4) *If Y is $[\mathfrak{a}, \infty)^r$-refinable, then X is $[\mathfrak{a}, \infty)^r$-refinable.*
(5) *If Y is weakly $[\mathfrak{a}, \mathfrak{b}]^r$-refinable, then X is weakly $[\mathfrak{a}, \mathfrak{b}]^r$-refinable.*
(6) *If Y is weakly $[\mathfrak{a}, \infty)^r$-refinable, then X is weakly $[\mathfrak{a}, \infty)^r$-refinable.*

Proof. (2), (4) and (6) follow from (1), (3) and (5), respectively.

To prove (1), (3) and (5), let $\mathcal{U}$ be an open cover of X such that $\mathfrak{a} \leq |\mathcal{U}| \leq \mathfrak{b}$ and $|\mathcal{U}| = \mathfrak{m}$ is a regular cardinal. Put $\mathcal{U}^{<\omega} = \{\mathcal{W} \subset \mathcal{U} : |\mathcal{W}| < \omega\}$ and $\mathcal{U}^F = \{\cup\mathcal{W} : \mathcal{W} \in \mathcal{U}^{<\omega}\}$. We represent $\mathcal{U}^F$ as $\{U_\alpha : \alpha \in A\}$. Then $|A| = \mathfrak{m}$.

For each $\alpha \in A$, put $G_\alpha = Y \setminus f(X \setminus U_\alpha)$. Then $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ is an open cover of Y.

(1). Since Y is $[\mathfrak{a}, \mathfrak{b}]^r$-compact, there is a subcover $\mathcal{G}' = \{G_\beta : \beta \in B\}$ of $\mathcal{G}$ such that $|B| < \mathfrak{m}$. Let $\mathcal{U}^{F'} = \{U_\beta \in \mathcal{U}^F : \beta \in B\}$. Since $f^{-1}(\mathcal{G}')$ is an open cover of X and $f^{-1}(G_\beta) \subset U_\beta$ for each $\beta \in B$, $\mathcal{U}^{F'}$ is an open cover of X.

For each $\beta \in B$, let us choose an element $\mathcal{W}_\beta \in \mathcal{U}^{<\omega}$ such that $U_\beta = \cup\mathcal{W}_\beta$. Put $\mathcal{U}' = \cup_{\beta \in B} \mathcal{W}_\beta$. Then $\mathcal{U}'$ is a subcover of $\mathcal{U}$ and $|\mathcal{U}'| < \mathfrak{m}$. Hence X is $[\mathfrak{a}, \mathfrak{b}]^r$-compact.

(3). Since Y is $[\mathfrak{a},\mathfrak{b}]^r$-refinable, there is a collection $\{\mathcal{H}_\beta : \beta \in B\}$ of open refinements of $\mathcal{G}$ with $|B| < \mathfrak{m}$ such that for each $y \in Y$, there is a $\beta \in B$ such that $\operatorname{ord}(y, \mathcal{H}_\beta) < \mathfrak{m}$.

For each $H \in \mathcal{H}_\beta$, let us choose an $\alpha(H) \in A$ such that $H \subset G_{\alpha(H)}$ and an element $\mathcal{W}_{\alpha(H)} \in \mathcal{U}^{<\omega}$ such that $U_{\alpha(H)} = \cup\mathcal{W}_{\alpha(H)}$. Let $\mathcal{V}_H = \{f^{-1}(H) \cap U : U \in \mathcal{W}_{\alpha(H)}\}$ and $\mathcal{V}_\beta = \cup_{H\in\mathcal{H}_\beta}\mathcal{V}_H$. Since $f^{-1}(\mathcal{H}_\beta)$ is an open cover of X and $f^{-1}(H) \subset f^{-1}(G_{\alpha(H)}) \subset U_{\alpha(H)}$ for each $H \in \mathcal{H}_\beta$, $\mathcal{V}_\beta$ is an open cover of X and a refinement of $\mathcal{U}$.

Then $\{\mathcal{V}_\beta : \beta \in B\}$ is a collection of open refinements of $\mathcal{U}$. Pick $x \in X$. If $y = f(x)$, there is a $\beta \in B$ such that $\operatorname{ord}(y, \mathcal{H}_\beta) < \mathfrak{m}$. Then $\operatorname{ord}(x, \mathcal{V}_\beta) < \mathfrak{m}$. Hence X is $[\mathfrak{a},\mathfrak{b}]^r$-refinable.

(5). Since Y is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable, there is an open refinement $\mathcal{H} = \cup_{\beta\in B}\mathcal{H}_\beta$ of $\mathcal{G}$ with $|B| < \mathfrak{m}$ such that for each $y \in Y$, there is a $\beta \in B$ such that $0 <\operatorname{ord}(y, \mathcal{H}_\beta) < \mathfrak{m}$.

For each $H \in \mathcal{H}_\beta$, let us choose an $\alpha(H) \in A$ such that $H \subset G_{\alpha(H)}$ and an element $\mathcal{W}_{\alpha(H)} \in \mathcal{U}^{<\omega}$ such that $U_{\alpha(H)} = \cup\mathcal{W}_{\alpha(H)}$. Let $\mathcal{V}_H = \{f^{-1}(H) \cap U : U \in \mathcal{W}_{\alpha(H)}\}$ and $\mathcal{V}_\beta = \cup_{H\in\mathcal{H}_\beta}\mathcal{V}_H$. Let $\mathcal{V} = \cup_{\beta\in B}\mathcal{V}_\beta$. Since $f^{-1}(\mathcal{H})$ is an open cover of X and $f^{-1}(H) \subset f^{-1}(G_{\alpha(H)}) \subset U_{\alpha(H)}$ for each $H \in \mathcal{H}$, $\mathcal{V}$ is an open cover of X.

Then $\mathcal{V} = \cup_{\beta\in B}\mathcal{V}_\beta$ is an open refinement of $\mathcal{U}$ with $|B| < \mathfrak{m}$. Pick $x \in X$. If $y = f(x)$, there is a $\beta \in B$ such that $0 <\operatorname{ord}(y, \mathcal{H}_\beta) < \mathfrak{m}$. Then $0 <\operatorname{ord}(x, \mathcal{V}_\beta) < \mathfrak{m}$. Hence X is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable. □

Lemma 1. *(GCH) Let $\mathfrak{m}$, $\mathfrak{n}$ be infinite cardinals. Let $\mathfrak{m}$ be a regular cardinal.*

(i) *If $\mathfrak{n} < \mathfrak{m}$, then $\mathfrak{m}^{\mathfrak{n}} = \mathfrak{m}$.*
(ii) *If $\mathfrak{a} < \mathfrak{m}$, then $\cup_{\mathfrak{n}<\mathfrak{a}}\mathfrak{m}^{\mathfrak{n}} = \mathfrak{m}$.*

Proof. (i) is from [6, p49, Corollary 2], and (ii) follows from (i). □

The smallest cardinal $\mathfrak{a}$ such that every open cover of a space X has an open refinement whose cardinality $\leq \mathfrak{a}$ is called *Lindelöf number* of the space X and is denoted by $L(X)$. ([4])

Theorem 6. *(GCH) Let f be a closed map from X onto Y and $L(f^{-1}(y)) < \mathfrak{a}$ for each $y \in Y$.*

(1) *If Y is $[\mathfrak{a},\mathfrak{b}]^r$-compact, then X is $[\mathfrak{a},\mathfrak{b}]^r$-compact.*
(2) *If Y is $[\mathfrak{a},\infty)^r$-compact, then X is $[\mathfrak{a},\infty)^r$-compact.*
(3) *If Y is $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then X is $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(4) *If Y is $[\mathfrak{a},\infty)^r$-refinable, then X is $[\mathfrak{a},\infty)^r$-refinable.*
(5) *If Y is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then X is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(6) *If Y is weakly $[\mathfrak{a},\infty)^r$-refinable, then X is weakly $[\mathfrak{a},\infty)^r$-refinable.*

Proof. We can prove similar to Theorem 5 replacing $\mathcal{U}^{<\omega}$ and $\mathcal{U}^F$ in proof of Theorem 5 by $\mathcal{U}^{<\mathfrak{a}} = \{\mathcal{W} \subset \mathcal{U} : |\mathcal{W}| < \mathfrak{a}\}$ and $\mathcal{U}^{(\mathfrak{a})} = \{\cup\mathcal{W} : \mathcal{W} \in \mathcal{U}^{<\mathfrak{a}}\}$ for an open cover $\mathcal{U}$ of X such that $\mathfrak{a} \leq |\mathcal{U}|$ $(\leq \mathfrak{b})$ and $|\mathcal{U}| = \mathfrak{m}$ is a regular cardinal. If we represent $\mathcal{U}^{(\mathfrak{a})}$ as $\{U_\alpha : \alpha \in A\}$, then $|A| = \mathfrak{m}$ by Lemma 1. □

4. Product spaces

In this section we assume that every space is a Hausdorff space.

Theorem 7. *Let X be a compact space.*

(1) *If Y is $[\mathfrak{a},\mathfrak{b}]^r$-compact, then $X \times Y$ is $[\mathfrak{a},\mathfrak{b}]^r$-compact.*
(2) *If Y is $[\mathfrak{a},\infty)^r$-compact, then $X \times Y$ is $[\mathfrak{a},\infty)^r$-compact.*
(3) *If Y is $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then $X \times Y$ is $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(4) *If Y is $[\mathfrak{a},\infty)^r$-refinable, then $X \times Y$ is $[\mathfrak{a},\infty)^r$-refinable.*
(5) *If Y is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then $X \times Y$ is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*

(6) *If* Y *is weakly* $[\mathfrak{a},\infty)^r$*-refinable, then* $X\times Y$ *is weakly* $[\mathfrak{a},\infty)^r$*-refinable.*

Proof. Let p_Y be the projection map from $X\times Y$ onto Y. Then p_Y is a perfect map. Hence $X\times Y$ is a space which has the same property as that of Y by Theorem 5. □

A cardinal number $\mathfrak{n}$ is the *tightness* of a space X if it is the smallest infinite cardinal such that, for every $x\in X$ and subset A of X, if $x\in\overline{A}$, then there exists a subset B of A such that $|B|\le\mathfrak{n}$ and $x\in\overline{B}$. This cardinal is denoted by $t(X)$. ([4])

A space X is an $\mathfrak{n}$-*bounded space* if for every subset M of X with $|M|\le\mathfrak{n}$, there exists a compact subset C of X such that $M\subset C$.

Lemma 2. [7, Lemma 5] *Let* X *be a space with* $t(X)\le\mathfrak{n}$*, and* Y *be an* $\mathfrak{n}$*-bounded space for some cardinal* $\mathfrak{n}$*. Let* p_X *be the projection map from* $X\times Y$ *onto* X*. Then* p_X *is a closed map.*

In [7, Lemma 5], Kombarov uses strongly $\mathfrak{n}$-compact space instead of $\mathfrak{n}$-bounded space.

Lemma 3. *Let* X *be a space with* $L(X)<\mathfrak{a}$*. Then* X *is* $[\mathfrak{a},\mathfrak{b}]^r$*-compact for every* $\mathfrak{b}>\mathfrak{a}$*.*

This proof is obvious.

Theorem 8. *(GCH) Let* X *be a space with* $t(X)\le\mathfrak{n}$ *and* Y *be an* $\mathfrak{n}$*-bounded space for some cardinal* $\mathfrak{n}$*.*

(1) *If* X *is* $[\mathfrak{a},\mathfrak{b}]^r$*-compact and* $L(Y)<\mathfrak{a}$*, then* $X\times Y$ *is* $[\mathfrak{a},\mathfrak{b}]^r$*-compact.*
(2) *If* X *is* $[\mathfrak{a},\infty)^r$*-compact and* $L(Y)<\mathfrak{a}$*, then* $X\times Y$ *is* $[\mathfrak{a},\infty)^r$*-compact.*
(3) *If* X *is* $[\mathfrak{a},\mathfrak{b}]^r$*-refinable and* $L(Y)<\mathfrak{a}$*, then* $X\times Y$ *is* $[\mathfrak{a},\mathfrak{b}]^r$*-refinable.*
(4) *If* X *is* $[\mathfrak{a},\infty)^r$*-refinable and* $L(Y)<\mathfrak{a}$*, then* $X\times Y$ *is* $[\mathfrak{a},\infty)^r$*-refinable.*
(5) *If* X *is weakly* $[\mathfrak{a},\mathfrak{b}]^r$*-refinable and* $L(Y)<\mathfrak{a}$*, then* $X\times Y$ *is weakly* $[\mathfrak{a},\mathfrak{b}]^r$*-refinable.*
(6) *If* X *is weakly* $[\mathfrak{a},\infty)^r$*-refinable and* $L(Y)<\mathfrak{a}$*, then* $X\times Y$ *is weakly* $[\mathfrak{a},\infty)^r$*-refinable.*

Proof. Let p_X be the projection map from $X\times Y$ onto X. Then p_X is a closed map by Lemma 2. Since for each $x\in X$, $L(p_X^{-1}(x))=L(\{x\}\times Y)=L(Y)<\mathfrak{a}$, $X\times Y$ is a space which has the same property as that of X by Theorem 6. □

Let Ω be a set. Denote $\Omega^n=\{(\alpha_0,\alpha_1,\cdots,\alpha_{n-1}):\alpha_i\in\Omega,\text{for each } i=0,1,\cdots,n-1\}$ for each $n\in\omega$, $\Omega^{<\omega}=\cup_{n\in\omega}\Omega^n$ and $\Omega^\omega=\{(\alpha_0,\alpha_1,\cdots,\alpha_n,\cdots):\alpha_n\in\Omega \text{ for each } n\in\omega\}$. For each $\sigma=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1})\in\Omega^n$ and $\alpha\in\Omega$, we denote $\sigma\vee\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1},\alpha)$. For each $\sigma=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1},\cdots)\in\Omega^\omega$, we denote $\sigma\restriction n=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1})$. It is ovbious that $\sigma\restriction n\in\Omega^n$.

A space X is said to be a P-*space* (resp. $P(\mathfrak{m})$-*space*) ([8]) if for any set Ω (resp. with $|\Omega|\le\mathfrak{m}$) and for any family $\{G(\sigma):\sigma\in\Omega^{<\omega}\}$ of open sets of X satisfying the following condition:

(P1) $G(\sigma)\subset G(\sigma\vee\alpha)$ for $\sigma\in\Omega^{<\omega}$ and $\alpha\in\Omega$,

there exists a family $\{F(\sigma):\sigma\in\Omega^{<\omega}\}$ of closed sets of X satisfying the following conditions:

(P2) $F(\sigma)\subset G(\sigma)$ for $\sigma\in\Omega^{<\omega}$,
(P3) for any $\sigma\in\Omega^\omega$, if $\cup_{n\in\omega}G(\sigma\restriction n)=X$, then $\cup_{n\in\omega}F(\sigma\restriction n)=X$.

The smallest cardinal of a base of a space X is said to be the *weight* of X and is denoted by $w(X)$.

Lemma 4. [3] *If* X *is a metrizable space, then for each* $n\in\omega$*, there are locally finite open covers* $\mathcal{H}_n$ *and* $\mathcal{B}_n$ *of* X *satisfying the following conditions:*

(i) $\mathcal{H}_n=\{H(\sigma):\sigma\in\Omega^n\}$, $\mathcal{B}_n=\{B(\sigma):\sigma\in\Omega^n\}$ *with* $|\mathcal{H}_n|=|\mathcal{B}_n|=w(X)$*,*
(ii) $\overline{B(\sigma)}\subset H(\sigma)$ *for each* $\sigma\in\Omega^n$*,*

(iii) *$H(\sigma)=\cup_{\alpha\in\Omega}H(\sigma\vee\alpha)$, $B(\sigma)=\cup_{\alpha\in\Omega}B(\sigma\vee\alpha)$ for each $\sigma\in\Omega^n$,*
(iv) *for each $x\in X$, there is a $\sigma\in\Omega^\omega$ such that $\{H(\sigma\restriction n):n\in\omega\}$ is a local base of x and $\{B(\sigma\restriction n):n\in\omega\}$ is a local base of x.*

The following theorem is due to the suggestion by Prof. K. Chiba.

Theorem 9. *Suppose that $\mathfrak{a}$ is a regular cardinal with $\mathfrak{a}\geq\omega_1$. Let X be a separable metric space and Y be a $P(\omega)$-space.*

(1) *If Y is $[\mathfrak{a},\mathfrak{b}]^r$-compact, then $X\times Y$ is $[\mathfrak{a},\mathfrak{b}]^r$-compact.*
(2) *If Y is $[\mathfrak{a},\infty)^r$-compact, then $X\times Y$ is $[\mathfrak{a},\infty)^r$-compact.*
(3) *If Y is $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then $X\times Y$ is $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(4) *If Y is $[\mathfrak{a},\infty)^r$-refinable, then $X\times Y$ is $[\mathfrak{a},\infty)^r$-refinable.*
(5) *If Y is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable, then $X\times Y$ is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable.*
(6) *If Y is weakly $[\mathfrak{a},\infty)^r$-refinable, then $X\times Y$ is weakly $[\mathfrak{a},\infty)^r$-refinable.*

Proof. (2), (4) and (6) follow from (1), (3) and (5), respectively.

To prove (1), (3) and (5), let $\mathcal{U}=\{U_\lambda:\lambda\in\Lambda\}$ be an open cover of $X\times Y$ with $\mathfrak{a}\leq|\Lambda|\leq\mathfrak{b}$ and $|\Lambda|=\mathfrak{m}$ is a regular cardinal. Since $w(X)=\omega$, there are sequences $\{\mathcal{H}_n:n\in\omega\}$ and $\{\mathcal{B}_n:n\in\omega\}$ of locally finite open covers with $|\Omega|=\omega$ satisfying the conditions in Lemma 4.

For each $\sigma\in\Omega^{<\omega}$ and $\lambda\in\Lambda$, let us define
$G_{\sigma,\lambda}=\cup\{G:G$ is an open subset of Y such that $H(\sigma)\times G\subset U_\lambda\}$. Then $G_{\sigma,\lambda}$ is an open subset of Y and $H(\sigma)\times G_{\sigma,\lambda}\subset U_\lambda$. For each $\sigma\in\Omega^{<\omega}$, put $G(\sigma)=\cup_{\lambda\in\Lambda}G_{\sigma,\lambda}$.

Let $\sigma\in\Omega^\omega$. If $\{H(\sigma\restriction n):n\in\omega\}$ is a local base of a point x of X, then $\cup_{n\in\omega}G(\sigma\restriction n)=Y$. For each $\sigma\in\Omega^{<\omega}$ and each $\alpha\in\Omega$, $G(\sigma)\subset G(\sigma\vee\alpha)$. Since Y is a $P(\omega)$-space, there is a closed cover $\{F(\sigma):\sigma\in\Omega^{<\omega}\}$ of Y satisfying the following conditions:

(P2) $F(\sigma)\subset G(\sigma)$ for each $\sigma\in\Omega^{<\omega}$,
(P3) for each $\sigma\in\Omega^\omega$, if $\cup_{n\in\omega}G(\sigma\restriction n)=Y$, then $\cup_{n\in\omega}F(\sigma\restriction n)=Y$.

Put $M_n=\{\overline{B(\sigma)}\times F(\sigma):\sigma\in\Omega^n\}$. Then M_n is a closed set of $X\times Y$ and we have $X\times Y=\cup_{n\in\omega}M_n$.

For each $\sigma\in\Omega^{<\omega}$, $\mathcal{G}_\sigma=\{G_{\sigma,\lambda}:\lambda\in\Lambda\}$ is a collection of open sets of Y, covers $F(\sigma)$ and $\mathcal{G}'_\sigma=\mathcal{G}_\sigma\cup\{Y\setminus F(\sigma)\}$ is an open cover of Y and $|\mathcal{G}'_\sigma|=\mathfrak{m}$.

(1). Since Y is $[\mathfrak{a},\mathfrak{b}]^r$-compact, there is a subcover $\mathcal{G}''_\sigma$ of $\mathcal{G}'_\sigma$ such that $|\mathcal{G}''_\sigma|<\mathfrak{m}$.

Put $\mathcal{O}_\sigma=\{G\in\mathcal{G}''_\sigma:G\cap F(\sigma)\neq\emptyset\}$, $\mathcal{V}(\sigma)=\{H(\sigma)\times O:O\in\mathcal{O}_\sigma\}$ and $\mathcal{V}=\cup_{\sigma\in\Omega^{<\omega}}\mathcal{V}(\sigma)$. Then since $|\Omega^{<\omega}|=\omega$ and $|\mathcal{V}(\sigma)|<\mathfrak{m}$ for each $\sigma\in\Omega^{<\omega}$ and $\mathfrak{m}$ is a regular caldinal, we have $|\mathcal{V}|<\mathfrak{m}$. Thus $\mathcal{V}$ is an open cover of $X\times Y$, and is a refinement of $\mathcal{U}$ with $|\mathcal{V}|<\mathfrak{m}$.

For each $V\in\mathcal{V}$, choose an element $U_V\in\mathcal{U}$ such that $V\subset U_V$. Then $\mathcal{U}'=\{U_V:V\in\mathcal{V}\}$ is a subcover of $\mathcal{U}$ with $|\mathcal{U}'|<\mathfrak{m}$. Hence $X\times Y$ is $[\mathfrak{a},\mathfrak{b}]^r$-compact.

(3). Since Y is $[\mathfrak{a},\mathfrak{b}]^r$-refinble, there is a collection $\{\mathcal{O}'_{\sigma,\alpha}:\alpha\in A_\sigma\}$ of open refinements of $\mathcal{G}'_\sigma$ with $|A_\sigma|<\mathfrak{m}$ such that for each $y\in Y$, there is an $\alpha\in A_\sigma$ such that $\mathrm{ord}(y,\mathcal{O}'_{\sigma,\alpha})<\mathfrak{m}$.

Put $\mathcal{O}_{\sigma,\alpha}=\{O\in\mathcal{O}'_{\sigma,\alpha}:O\cap F(\sigma)\neq\emptyset\}$. Then for each $\alpha\in A_\sigma$, $\mathcal{O}_{\sigma,\alpha}$ is a collection of open sets of Y, which covers $F(\sigma)$ and is a partial refinement of $\mathcal{G}'_\sigma$, such that for each $y\in Y$, there is an $\alpha\in A_\sigma$ with $\mathrm{ord}(y,\mathcal{O}_{\sigma,\alpha})<\mathfrak{m}$ and for each $y\in F(\sigma)$, there is an $\alpha\in A_\sigma$ with $0<\mathrm{ord}(y,\mathcal{O}_{\sigma,\alpha})<\mathfrak{m}$.

Put $A=\cup_{\sigma\in\Omega^{<\omega}}A_\sigma$. Then, since $|\Omega^{<\omega}|=\omega$ and $|A_\sigma|<\mathfrak{m}$ for each $\sigma\in\Omega^{<\omega}$ and $\mathfrak{m}$ is a regular cardinal, we have $|A|<\mathfrak{m}$. For each $\sigma\in\Omega^{<\omega}$, let us choose $\gamma_\alpha\in A_\sigma$ and define $\mathcal{O}'_{\sigma,\alpha}=\mathcal{O}'_{\sigma,\gamma_\alpha}$ for each $\alpha\in A\setminus A_\sigma$. Thus we may assume that $A_\sigma=A$ for each $\sigma\in\Omega^{<\omega}$. Define $\Gamma'(n,k)=\{n\}\times\{k\}\times(\Omega^n)^k\times A^k$, $\Gamma(n,k)=\{(n,k,(\sigma_i)_{i<k},(\gamma_i)_{i<k})\in$

$\Gamma'(n,k) : \gamma_i \in A_{\sigma_i}, i < k\}$ and $\Gamma = \cup_{n\in\omega} \cup_{k\in\omega} \Gamma(n,k)$. Then $|\Gamma| < \mathfrak{m}$. For each $\gamma = (n, k, (\sigma_i)_{i<k}, (\gamma_i)_{i<k}) \in \Gamma$, define $\mathcal{V}(\gamma) = \{H(\sigma_i) \times O_i : O_i \in \mathcal{O}_{\sigma_i,\alpha_i}, i < k\} \cup \{H(\sigma) \times G_{\sigma,\lambda} : \sigma \in \Omega^n \setminus \{\sigma_i : i < k\}\} \cup \{(X \times Y \setminus M_n) \cap U_\lambda : \lambda \in \Lambda\}$. Then $\mathcal{V}(\gamma)$ is an open cover of $X \times Y$ and $\mathcal{V}(\gamma)$ is a refinement of $\mathcal{U}$.

For each $(x,y) \in X \times Y$, there is a $\gamma \in \Gamma$ such that $\mathrm{ord}((x,y),\mathcal{V}(\gamma)) < \mathfrak{m}$. To show this, let $(x,y) \in X \times Y$. Then there is an $n \in \omega$ such that $(x,y) \in M_n$. Since $\mathcal{H}_n$ is locally finite, there is a finite set $\{\sigma_i : i < k\}$ of Ω^n such that $x \in H(\sigma_i)$ and $x \notin H(\sigma)$ if $\sigma \notin \{\sigma_i : i < k\}$. Since $(x,y) \in M_n$, $(x,y) \in H(\sigma) \times F(\sigma)$ for some $\sigma \in \Omega^n$. Without loss of generality, we may assume $\sigma = \sigma_0$. Since $y \in F(\sigma_0)$, there is a $\gamma_0 \in A_{\sigma_0}$ such that $0 < \mathrm{ord}(y, \mathcal{O}_{\sigma_0,\gamma_0}) < \mathfrak{m}$. For each $i = 1, \cdots, k-1$, there is a $\gamma_i \in A_{\sigma_i}$ such that $\mathrm{ord}(y, \mathcal{O}_{\sigma_i,\gamma_i}) < \mathfrak{m}$. Put $\gamma = (n, k, (\sigma_i)_{i<k}, (\gamma_i)_{i<k})$. Then $\mathrm{ord}((x,y),\mathcal{V}(\gamma)) \leq \sum_{i=0}^{k-1} \mathrm{ord}(y, \mathcal{O}_{\sigma_i,\gamma_i}) < \mathfrak{m}$.

Hence $X \times Y$ is $[\mathfrak{a},\mathfrak{b}]^r$-refinable.

(5). Since Y is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinble, there is an open refinement $\mathcal{O}'_\sigma = \cup_{\alpha\in A_\sigma} \mathcal{O}'_{\sigma,\alpha}$ of $\mathcal{G}'_\sigma$ with $|A_\sigma| < \mathfrak{m}$ such that for each $y \in Y$, there is an $\alpha \in A_\sigma$ such that $0 < \mathrm{ord}(y, \mathcal{O}'_{\sigma\alpha}) < \mathfrak{m}$.

Put $\mathcal{O}_{\sigma,\alpha} = \{O \in \mathcal{O}'_{\sigma,\alpha} : O \cap F(\sigma) \neq \emptyset\}$ and $\mathcal{O}_\sigma = \cup_{\alpha\in A_\sigma} \mathcal{O}_{\sigma,\alpha}$. Then $\mathcal{O}_\sigma$ is a collction of open sets of Y, which covers $F(\sigma)$ and is a partial refinement of $\mathcal{G}'_\sigma$, such that for each $y \in Y$, there is an $\alpha \in A_\sigma$ with $0 < \mathrm{ord}(y, \mathcal{O}_{\sigma,\alpha}) < \mathfrak{m}$ and for each $y \in F(\sigma)$, there is an $\alpha \in A_\sigma$ with $0 < \mathrm{ord}(y, \mathcal{O}_{\sigma,\alpha}) < \mathfrak{m}$.

Put $A = \cup_{\sigma\in\Omega^{<\omega}} (\{\sigma\} \times A_\sigma)$, $\mathcal{V}(\sigma,\alpha) = \{H(\sigma) \times O : O \in \mathcal{O}_{\sigma,\alpha}\}$ and $\mathcal{V} = \cup_{(\sigma,\alpha)\in A} \mathcal{V}(\sigma,\alpha)$. Then, since $|\Omega^{<\omega}| = \omega$ and $|A_\sigma| < \mathfrak{m}$ for each $\sigma \in \Omega^{<\omega}$ and $\mathfrak{m}$ is a regular cardinal, we have $|A| < \mathfrak{m}$. Thus $\mathcal{V}$ is an open cover of $X \times Y$ and is a refinement of $\mathcal{U}$ with $|A| < \mathfrak{m}$.

Let $(x,y) \in X \times Y$. Then $(x,y) \in H(\sigma) \times F(\sigma)$ for some $\sigma \in \Omega^{<\omega}$. Then, there is an $\alpha \in A_\sigma$ such that $0 < \mathrm{ord}(y, \mathcal{O}_{\sigma,\alpha}) < \mathfrak{m}$. It is easy to see that $0 < \mathrm{ord}((x,y), \mathcal{V}(\sigma,\alpha)) < \mathfrak{m}$. Hence $X \times Y$ is weakly $[\mathfrak{a},\mathfrak{b}]^r$-refinable. □

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