INTUITIONISTIC FUZZY SETS IN SEMIGROUPS

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Received November 15, 2006

ABSTRACT. Further properties of intuitionistic fuzzy setting of several ideals of a semigroup are discussed.

1. INTRODUCTION.

After the introduction of fuzzy sets by Zadeh [10], several researchers were conducted on the generalizations of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1, 2] as a generalization of the notion of fuzzy set. In [7], Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals in semigroups. In [6, 5], Kim and Jun considered the intuitionistic fuzzification of the concept of several ideals in a semigroup, and investigated some properties of such ideals. In this paper, we discuss further properties of intuitionistic fuzzy setting of several ideals in a semigroup.

2. PRELIMINARIES

Let G be a semigroup. By a *subsemigroup* of G we mean a non-empty subset U of G such that $U^2 \subseteq U$, and by a *left* (*right*) *ideal* of G we mean a non-empty subset U of G such that $GU \subseteq U$ ($UG \subseteq U$). By *two-sided ideal* or simply *ideal*, we mean a non-empty subset of G which is both a left and a right ideal of G. A subsemigroup U of a semigroup G is called a *bi-ideal* of G if $UGU \subseteq U$. A semigroup G is said to be *right* (resp. *left*) *zero* if $xy = y$ (resp. $xy = x$) for all $x, y \in G$. A semigroup G is said to be *regular* if, for each element $a \in G$, there exists an element x in G such that $a = axa$. A semigroup G is said to be *left* (resp. *right*) *simple* if G itself is the only left (resp. right) ideal of G.

After the introduction of fuzzy sets by Zadeh [10], several researces were conducted on the generalizations of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1, 2], as a generalization of the notion of fuzzy set. An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$. An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ in X can be identified to an ordered pair (μ_A, γ_A) in $I^X \times I^X$, where $I = [0, 1]$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$.

2000 *Mathematics Subject Classification.* 20M12, 03E72, 03F55.
Key words and phrases. intuitionistic fuzzy (interior, bi-) ideal, intuitionistic fuzzy left simple semigroup.
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3. Intuitionistic fuzzy ideals

In what follows, let G be a semigroup unless otherwise specified.

Definition 3.1. [6] For an IFS $A = (\mu_A, \gamma_A)$ in G, consider the following axioms:

(S1) $(\forall x, y \in G)$ $(\mu_A(xy) \geq \min\{\mu_A(x),\, \mu_A(y)\})$,
(S2) $(\forall x, y \in G)$ $(\gamma_A(xy) \leq \max\{\gamma_A(x),\, \gamma_A(y)\})$.

Then $A = (\mu_A, \gamma_A)$ is called a *first* (resp. *second*) *intuitionistic fuzzy subsemigroup* ($IFSS_1$ (resp. $IFSS_2$)) of G if it satisfies (S1) (resp. (S2)). A first and second intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ is called an *intuitionistic fuzzy subsemigroup* ($IFSS$) of G.

Theorem 3.2. *If U is a subsemigroup of G, then the IFS $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFSS$ of G.*

Proof. Let $x, y \in G$. If $x, y \in U$, then $xy \in U$ since U is a subsemigroup of G. Hence

$$\chi_U(xy) = 1 \geq \min\{\chi_U(x),\, \chi_U(y)\}$$

and

$$\begin{aligned}\bar{\chi}_U(xy) &= 1 - \chi_U(xy) \leq 1 - \min\{\chi_U(x),\, \chi_U(y)\} \\ &= \max\{1 - \chi_U(x),\, 1 - \chi_U(y)\} = \max\{\bar{\chi}_U(x),\, \bar{\chi}_U(y)\}.\end{aligned}$$

If $x \notin U$ or $y \notin U$, then $\chi_U(x) = 0$ or $\chi_U(y) = 0$. Thus

$$\chi_U(xy) \geq 0 = \min\{\chi_U(x),\, \chi_U(y)\}$$

and

$$\begin{aligned}\max\{\bar{\chi}_U(x),\, \bar{\chi}_U(y)\} &= \max\{1 - \chi_U(x),\, 1 - \chi_U(y)\} \\ &= 1 - \min\{\chi_U(x),\, \chi_U(y)\} = 1 \geq \bar{\chi}_U(xy).\end{aligned}$$

Therefore $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFSS$ of G. □

Theorem 3.3. *Let U be a nonempty subset of G. If $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFSS_1$ or $IFSS_2$ of G, then U is a subsemigroup of G.*

Proof. Assume that $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFSS_1$ of G. Let $x \in U^2$. Then $x = ab$ for some $a, b \in U$. It follows from (S1) that

$$\chi_U(x) = \chi_U(ab) \geq \min\{\chi_U(a),\, \chi_U(b)\} = 1$$

so that $\chi_U(x) = 1$, i.e., $x \in U$. Hence U is a subsemigroup of G. Now suppose that $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFSS_2$ of G. Let $x \in U^2$. Then $x = ab$ for some $a, b \in U$. Using (S2), we have

$$\begin{aligned}\bar{\chi}_U(x) &= \bar{\chi}_U(ab) \leq \max\{\bar{\chi}_U(a),\, \bar{\chi}_U(b)\} \\ &= \max\{1 - \chi_U(a),\, 1 - \chi_U(b)\} = 0,\end{aligned}$$

and thus $1 - \chi_U(x) = \bar{\chi}_U(x) = 0$, i.e., $\chi_U(x) = 1$. This shows that $x \in U$, completing the proof. □

Definition 3.4. [6] For an IFS $A = (\mu_A, \gamma_A)$ in G, consider the following axioms:

(I1) $(\forall x, y \in G)$ $(\mu_A(xy) \geq \mu_A(y))$,
(I2) $(\forall x, y \in G)$ $(\gamma_A(xy) \leq \gamma_A(y))$.

Then $A = (\mu_A, \gamma_A)$ is called a *first* (resp. *second*) *intuitionistic fuzzy left ideal* ($IFLI_1$ (resp. $IFLI_2$)) of G if it satisfies (I1) (resp. (I2)). A first and second intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ is called an *intuitionistic fuzzy left ideal* ($IFLI$) of G. An intuitionistic fuzzy right ideal ($IFRI$) of G is defined in an analogous way. Both an $IFLI$ and an $IFRI$ is called an *intuitionistic fuzzy ideal* (IFI).

Proposition 3.5. *Let U be a left zero subsemigroup of G. If $A = (\mu_A, \gamma_A)$ is an $IFLI$ of G, then $A(x) = A(y)$ for all $x, y \in U$, that is, the restriction function of A into U is constant.*

Proof. Let $x, y \in U$. Then $xy = x$ and $yx = y$. Thus

$$\mu_A(x) = \mu_A(xy) \geq \mu_A(y) = \mu_A(yx) \geq \mu_A(x)$$

and

$$\gamma_A(x) = \gamma_A(xy) \leq \gamma_A(y) = \gamma_A(yx) \leq \gamma_A(x).$$

Hence $A(x) = A(y)$. □

Lemma 3.6. *If U is a left ideal of G, then $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFLI$ of G.*

Proof. Let $x, y \in G$. If $y \in U$, then $xy \in U$. Hence $\chi_U(xy) = 1 = \chi_U(y)$ and $\bar{\chi}_U(xy) = 1 - \chi_U(xy) = 0 = 1 - \chi_U(y) = \bar{\chi}_U(y)$. If $y \notin U$, then $\chi_U(xy) \geq 0 = \chi_U(y)$ and $\bar{\chi}_U(y) = 1 - \chi_U(y) = 1 \geq \bar{\chi}_U(xy)$. Therefore $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFLI$ of G. □

Theorem 3.7. *Let $A = (\mu_A, \gamma_A)$ be an $IFLI$ of G. If the set of all idempotent elements of G forms a left zero subsemigroup of G, then $A(u) = A(v)$ for all idempotent elements u and v of G.*

Proof. Let E_G be the set of all idempotent elements of G and assume that E_G is a left zero subsemigroup of G. For any $u, v \in E_G$, we have $uv = u$ and $vu = v$, and so

$$\mu_A(u) = \mu_A(uv) \geq \mu_A(v) = \mu_A(vu) \geq \mu_A(u)$$

and

$$\gamma_A(u) = \gamma_A(uv) \leq \gamma_A(v) = \gamma_A(vu) \leq \gamma_A(u).$$

This means that $A(u) = A(v)$ for all $u, v \in E_G$. □

Theorem 3.8. *Let G be a regular semigroup. If, for every subset U of G, $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFLI_1$* (or, *$IFLI_2$*) *of G, then E_G, the set of all idempotent elements of G, is a left zero subsemigroup of G.*

Proof. Note that $E_G \neq \emptyset$ since G is regular, and obviously E_G is a subsemigroup of G. Let $u, v \in E_G$. Since G is regular, $L[u] = Gu$ which is a left ideal of G (see [3, Lemma 2.13]). Hence $\widetilde{L[u]} = (\chi_{L[u]}, \bar{\chi}_{L[u]})$ is an $IFLI_1$ (or, $IFLI_2$) of G by Lemma 3.6. Thus $\chi_{L[u]}(v) = \chi_{L[v]}(u) = 1$ (or, $\bar{\chi}_{L[u]}(v) = \bar{\chi}_{L[v]}(u) = 0$). It follows that $v \in L[u] = Gu$ so that $v = xu = xuu = vu$ for some $x \in G$. This means that E_G is a left zero subsemigroup of G. □

Definition 3.9. [5] For an IFS $A = (\mu_A, \gamma_A)$ in G, consider the following axioms:

(I3) $(\forall a, x, y \in G)$ $(\mu_A(xay) \geq \mu_A(a))$,

(I4) $(\forall a, x, y \in G)$ $(\gamma_A(xay) \leq \gamma_A(a))$.

Then $A = (\mu_A, \gamma_A)$ is called a *first* (resp. *second*) *fuzzy intuitionistic fuzzy interior ideal* ($IFII_1$ (resp. $IFII_2$)) of G if it is an $IFSS_1$ (resp. $IFSS_2$) satisfying the condition (I3) (resp. (I4)). If $A = (\mu_A, \gamma_A)$ is both an $IFII_1$ and an $IFII_2$, we say that it is an *intuitionistic fuzzy interior ideal* ($IFII$) of G.

It is clear that every IFI is an $IFII$.

Theorem 3.10. *If G is regular, then every $IFII$ is an IFI.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an $IFII$ of G. Let a and b be any elements of G. Then there exists x and y in G such that $a = axa$ and $b = byb$. Thus

$$\mu_A(ab) = \mu_A((axa)b) = \mu_A((ax)(ab)) \geq \mu_A(a)$$

and

$$\gamma_A(ab) = \gamma_A((axa)b) = \gamma_A((ax)(ab)) \leq \mu_A(a).$$

This shows that $A = (\mu_A, \gamma_A)$ is an $IFLI$ of G. Similarly, we can see that $A = (\mu_A, \gamma_A)$ is an $IFRI$ of G, completing the proof. □

Theorem 3.11. *If U is an interior ideal of G, then the IFS $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFII$ of G.*

Proof. Let a, x and y be any elements of G. If $a \in U$, then $xay \in GUG \subseteq U$ and so $\chi_U(xay) = 1 = \chi_U(a)$ and

$$\bar{\chi}_U(xay) = 1 - \chi_U(xay) = 1 - 1 = 0 = 1 - \chi_U(a) = \bar{\chi}_U(a).$$

Since $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFSS$ of G by Theorem 3.2, we conclude that $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFII$ of G. □

Theorem 3.12. *Suppose that G is regular and let U be a nonempty subset of G. If $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFII_1$ or $IFII_2$ of G, then U is an interior ideal of G.*

Proof. Let $z \in GUG$. Then $z = xay$ for some $x, y \in G$ and $a \in U$. If $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFII_1$ of G, then $\chi_U(z) = \chi_U(xay) \geq \chi_U(a) = 1$, and so $\chi_U(z) = 1$, that is, $z \in U$. If $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFII_2$ of G, then

$$\bar{\chi}_U(z) = \bar{\chi}_U(xay) \leq \bar{\chi}_U(a) = 1 - \chi_U(a) = 0,$$

which implies that $1 - \chi_U(z) = \bar{\chi}_U(z) = 0$. Hence $\chi_U(z) = 1$, that is, $z \in U$. This completes the proof. □

Definition 3.13. A semigroup G is said to be *first* (resp. *second*) *intuitionistic fuzzy left simple* if every $IFLI_1$ (resp. $IFLI_2$) of G is a constant function. A semigroup G is said to be *intuitionistic fuzzy left simple* if it is first and second intuitionistic fuzzy left simple, i.e., every $IFLI$ of G is constant.

Theorem 3.14. *If G is left simple, then G is intuitionistic fuzzy left simple.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an $IFLI$ of G and let $a, b \in G$. Since G is left simple, it follows from [3, p. 6] that there exist elements x and y in G such that $b = xa$ and $a = yb$. Using (I1) and (I2), we have

$$\mu_A(a) = \mu_A(yb) \geq \mu_A(b) = \mu_A(xa) = \mu_A(a)$$

and

$$\gamma_A(a) = \gamma_A(yb) \leq \gamma_A(b) = \gamma_A(xa) = \gamma_A(a),$$

and so $\mu_A(a) = \mu_A(b)$ and $\gamma_A(a) = \gamma_A(b)$. Thus $A(a) = A(b)$, which means A is a constant function because a and b are any elements of G. Therefore G is intuitionistic fuzzy left simple. □

Theorem 3.15. *If G is first* (or *second*) *intuitionistic fuzzy left simple, then G is left simple.*

Proof. Let U be a left ideal of G. Assume that G is first (or, second) intuitionistic fuzzy left simple. Since $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFLI$ of G by Lemma 3.6, χ_U (or, $\bar{\chi}_U$) is a constant function. Since U is nonempty, it follows that $\chi_U = \mathbf{1}$ (or, $\bar{\chi}_U = \mathbf{0}$), where $\mathbf{1}$ and $\mathbf{0}$ are fuzzy sets in G given by $\mathbf{1}(x) = 1$ and $\mathbf{0}(x) = 0$ for all $x \in G$, respectively. Thus every element of G is in U, and so G is left simple. □

Theorem 3.16. *If G is simple, then every $IFII$ of G is constant.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an $IFII$ of G, and a and b any elements of G. Since G is simple, it follows from [9, I.3.9] that there exist elements x and y in G such that $a = xby$. Since $A = (\mu_A, \gamma_A)$ is an $IFII$ of G, we have $\mu_A(a) = \mu_A(xby) \geq \mu_A(b)$ and $\gamma_A(a) = \gamma_A(xby) \leq \gamma_A(b)$. It can be seen in a similar way that $\mu_A(b) \geq \mu_A(a)$ and $\gamma_A(b) \leq \gamma_A(a)$. Since a and b are arbitrary elements, this means that $A = (\mu_A, \gamma_A)$ is a constant function. □

Definition 3.17. [6] An $IFSS$ $A = (\mu_A, \gamma_A)$ of G is called an *intuitionistic fuzzy bi-ideal* ($IFBI$) of G if

(I5) $(\forall w, x, y \in G)$ $(\mu_A(xwy) \geq \min\{\mu_A(x), \mu_A(y)\})$,
(I6) $(\forall w, x, y \in G)$ $(\gamma_A(xwy) \leq \max\{\gamma_A(x), \gamma_A(y)\})$.

Note that every $IFLI$ (resp. $IFRI$) of G is an $IFBI$ of G (see [6]).

Theorem 3.18. *If G is left simple, then every $IFBI$ of G is an $IFRI$ of G.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an $IFBI$ of G, and a and b any elements of G. Since G is left simple, there exists $x \in G$ such that $b = xa$. Since $A = (\mu_A, \gamma_A)$ is an $IFBI$ of G, it follows that

$$\mu_A(ab) = \mu_A(axa) \geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a)$$

and

$$\gamma_A(ab) = \gamma_A(axa) \leq \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a)$$

so that $A = (\mu_A, \gamma_A)$ is an $IFRI$ of G. □

Lemma 3.19. *If U is a bi-ideal of G, then the IFS $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFBI$ of G.*

Proof. Let $w, x, y \in G$. If $x, y \in U$, then $xwy \in UGU \subseteq U$, and so

$$\chi_U(xwy) = 1 = \min\{\chi_U(x),\ \chi_U(y)\}$$

and

$$\bar{\chi}_U(xwy) = 1 - \chi_U(xwy) = 0 = \max\{\bar{\chi}_U(x),\ \bar{\chi}_U(y)\}.$$

If $x \notin U$ or $y \notin U$, then $\chi_U(x) = 0$ or $\chi_U(y) = 0$. Hence

$$\chi_U(xwy) \geq 0 = \min\{\chi_U(x),\ \chi_U(y)\}$$

and

$$\bar{\chi}_U(xwy) \leq 1 = \max\{\bar{\chi}_U(x),\ \bar{\chi}_U(y)\}.$$

Therefore $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an $IFBI$ of G. □

Theorem 3.20. *Let G be a regular semigroup. For every $IFBI$ $A = (\mu_A, \gamma_A)$ of G, if $\mu_A(u) = \mu_A(v)$* (or, *$\gamma_A(u) = \gamma_A(v)$*) *for all idempotents u and v of G, then G is a group.*

Proof. Assume that $\mu_A(u) = \mu_A(v)$ (or, $\gamma_A(u) = \gamma_A(v)$) for all idempotents u and v of G. Denote by $B[x]$ the principle bi-ideal of G generated by x in G, that is, $B[x] = \{x\} \cup \{x^2\} \cup xGx$. Since G is regular, it follows that $B[x] = xGx$. Using Lemma 3.19, $\widetilde{B[v]} = (\chi_{B[v]}, \bar{\chi}_{B[v]})$ is an $IFBI$ of G. Since $v \in B[v]$, we have $\chi_{B[v]}(u) = \chi_{B[v]}(v) = 1$ (or, $\bar{\chi}_{B[v]}(u) = 1 - \chi_{B[v]}(u) = 0$), and so $u \in B[v] = vGv$. Hence $u = vxv$ for some $x \in G$. It can be obtained in a similar way that $v = uyu$ for some $y \in G$. Therefore

$$u = vxv = vxvv = uv = uuyu = uyu = v,$$

which means that, since G is regular, E_G is nonempty and G contains exactly one idempotent. It follows from [3, p.33] that G is a group. □

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