ON IDEALS IN SUBTRACTION ALGEBRAS

Young Bae Jun and Hee Sik Kim

Received May 27, 2005

Abstract. The ideal generated by a set is established, and related results are discussed.

1. Introduction.

B. M. Schein [6] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [2]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [7] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. H. Kim and H. S. Kim [5] showed that a subtraction algebra is equivalent to an implicative BCK-algebra, and a subtraction semigroup is a special case of an **IS**-algebra, established by Y. B. Jun et al. [3], which is a generalization of a ring. The present authors with E. H. Roh [4] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In this paper, we establish an ideal generated by a subset of a subtraction algebra, and discuss related results.

2. Preliminaries

A *subtraction algebra* ([6]) is defined as an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [2], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [4]):

2000 *Mathematics Subject Classification.* 06F35.
Key words and phrases. Subtraction algebra, ideal (generated by a set).

(p1) $(x - y) - y = x - y$.

(p2) $x - 0 = x$ and $0 - x = 0$.

(p3) $(x - y) - x = 0$.

(p4) $x - (x - y) \leq y$.

(p5) $(x - y) - (y - x) = x - y$.

(p6) $x - (x - (x - y)) = x - y$.

(p7) $(x - y) - (z - y) \leq x - z$.

(p8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(p10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

3. Ideals generated by a subset

Definition 3.1 (Jun et al. [4]). A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in A$

(I2) $y \in A$ and $x - y \in A$ imply $x \in A$ for all $x, y \in X$.

Theorem 3.2. *Let A be a nonempty subset of a subtraction algebra X. Then the set*

$$K := \{x \in X \mid (\cdots((x - a_1) - a_2) - \cdots) - a_n = 0 \text{ for some } a_1, a_2, \cdots, a_n \in A\}$$

is a minimal ideal of X containing A.

Proof. Obviously $0 \in K$. Let $x, y \in X$ be such that $y \in K$ and $x - y \in K$. Then

$$(\cdots((y - a_1) - a_2) - \cdots) - a_n = 0$$

for some $a_1, a_2, \cdots, a_n \in A$, and

$$(\cdots(((x - y) - b_1) - b_2) - \cdots) - b_m = 0 \tag{3.1}$$

for some $b_1, b_2, \cdots, b_m \in A$. Applying (S3) to (3.1), we have

$$((\cdots((x - b_1) - b_2) - \cdots) - b_m) - y = 0,$$

that is, $(\cdots((x - b_1) - b_2) - \cdots) - b_m \leq y$. Using (p9) repeatedly, we get

$$\begin{aligned} & (\cdots(((\cdots((x - b_1) - b_2) - \cdots) - b_m) - a_1) - \cdots) - a_n \\ \leq \; & (\cdots((y - a_1) - a_2) - \cdots) - a_n = 0, \end{aligned}$$

and so $(\cdots(((\cdots((x - b_1) - b_2) - \cdots) - b_m) - a_1) - \cdots) - a_n = 0$. It follows that $x \in K$ so that K is an ideal of X. Let G be an ideal of X containing A and let $x \in K$. Then

$$(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0$$

for some $a_1, a_2, \cdots, a_n \in A$, which implies that $x \in G$ because G is an ideal of X and $a_1, a_2, \cdots, a_n \in G$. This completes the proof. □

The ideal K described in Theorem 3.2 is called the *ideal generated* by A, and denoted by $\langle A \rangle$.

Proposition 3.3. *Let a, x, and y be elements of a subtraction algebra X. If $a - x^m = 0$ and $a - y^n = 0$ for some $m, n \in \mathbb{N}$, then there exists $p \in \mathbb{N}$ such that $a - (x \wedge y)^p = 0$, where $a - x^k = (\cdots((a - x) - x) - \cdots) - x$ in which x occurs k-times.*

Proof. Let $m, n \in \mathbb{N}$ be such that

$$a - x^m = 0 \text{ and } a - y^n = 0. \tag{3.2}$$

Note that if $a - x^m = 0$, then $a - x^k = 0$ for $k \geq m$. Thus we can assume that $m = n$ in (3.2), and so it is sufficient to show that there exists $p \in \mathbb{N}$ such that

$$a - (x \wedge y)^p = 0 \text{ whenever } a - x^n = 0 = a - y^n. \tag{3.3}$$

The proof is by induction on n. For $n = 1$ we have $a \leq x$ and $a \leq y$, and so $a \leq x \wedge y$, that is, $a - (x \wedge y) = 0$. Suppose that (3.3) is true for n. Using (p2) and (S3), we have

$$\begin{aligned} 0 &= a - x^{n+1} = (a - x^{n+1}) - y^n \\ &= ((a - x^n) - x) - y^n \\ &= ((a - x^n) - y^n) - x, \end{aligned} \tag{3.4}$$

$$0 = a - y^{n+1} = (a - y^{n+1}) - x^n = ((a - x^n) - y^n) - y. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$((a - x^n) - y^n) - (x \wedge y) = 0.$$

It follows from (S3) that

$$\begin{aligned} 0 &= ((a - (x \wedge y)) - x^n) - y^n \\ &= (((a - (x \wedge y)) - y^n) - x^{n-1}) - x \end{aligned} \tag{3.6}$$

From $a - y^{n+1} = 0$, it follows by means of (p2) and (S3) that

$$(a - (x \wedge y)^k) - y^{n+1} = 0 \tag{3.7}$$

for any $k \in \mathbb{N}$. In particular, if $k = 1$ in (3.7) then

$$(((a - (x \wedge y)) - y^n) - x^{n-1}) - y = 0 \tag{3.8}$$

by (S3) and (p2). Combining (3.6) and (3.8), and using (S3), we obtain

$$((a - (x \wedge y)^2) - y^n) - x^{n-1} = 0.$$

In the same way, we can obtain

$$((a - (x \wedge y)^3) - y^n) - x^{n-2} = 0.$$

Continuing this process, we conclude that

$$(a - (x \wedge y)^{n+1}) - y^n = 0. \tag{3.9}$$

Similarly, we have

$$(a - (x \wedge y)^{n+1}) - x^n = 0. \tag{3.10}$$

Applying the induction hypothesis to (3.9) and (3.10), we have

$$0 = (a - (x \wedge y)^{n+1}) - (x \wedge y)^p = a - (x \wedge y)^{n+p+1}.$$

This completes the proof. □

Theorem 3.4. *Let A be an ideal of a subtraction algebra X and let $a, b \in X$. If $a \wedge b \in A$, then $\langle A \cup \{a\}\rangle \cap \langle A \cup \{b\}\rangle = A$.*

Proof. Let $a, b \in X$ be such that $a \wedge b \in A$. Obviously

$$A \subseteq \langle A \cup \{a\}\rangle \cap \langle A \cup \{b\}\rangle.$$

Let $x \in \langle A \cup \{a\}\rangle \cap \langle A \cup \{b\}\rangle$. Then $x \in \langle A \cup \{a\}\rangle$ and $x \in \langle A \cup \{b\}\rangle$. Hence there exist $a_1, a_2, \cdots, a_n \in \langle A \cup \{a\}\rangle$ and $b_1, b_2, \cdots, b_m \in \langle A \cup \{b\}\rangle$ such that

$$(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0$$

and

$$(\cdots((x - b_1) - b_2) - \cdots) - b_m = 0.$$

Using (S3) we can rewrite the above equalities in the following form

$$((\cdots((x - u_1) - u_2) - \cdots) - u_s) - a^k = 0,$$

$$((\cdots((x - v_1) - v_2) - \cdots) - v_t) - b^r = 0,$$

where

$$\{u_1, u_2, \cdots, u_s\} = \{a_1, a_2, \cdots, a_n\} \cap A$$

and

$$\{v_1, v_2, \cdots, v_t\} = \{b_1, b_2, \cdots, b_m\} \cap A.$$

It follows from (p2) and (S3) that

$$((\cdots(((\cdots((x - u_1) - u_2) - \cdots) - u_s) - v_1) - \cdots) - v_t) - a^k = 0,$$

$$((\cdots(((\cdots((x - u_1) - u_2) - \cdots) - u_s) - v_1) - \cdots) - v_t) - b^r = 0,$$

so from Proposition 3.3 that

$$((\cdots(((\cdots((x - u_1) - u_2) - \cdots) - u_s) - v_1) - \cdots) - v_t) - (a \wedge b)^p = 0$$

for some $p \in \mathbb{N}$. Since A is an ideal containing $a \wedge b$, we have $x \in A$, that is,

$$\langle A \cup \{a\}\rangle \cap \langle A \cup \{b\}\rangle \subseteq A.$$

This completes the proof. □

Lemma 3.5 (Jun et al. [4, Lemma 3.10])**.** *Every subtraction algebra satisfies the right self-distributive law, that is, the equality* $(x-y)-z=(x-z)-(y-z)$ *is valid.*

Theorem 3.6. *Let* X *be a subtraction algebra. For any* $a,b\in X$ *and* $n\in\mathbb{N}$*, the set*

$$[a;b^n]:=\{x\in X \mid (x-a)-b^n=0\}$$

is an ideal of X*.*

Proof. Obviously $0\in[a;b^n]$. Let $x,y\in X$ be such that $y\in[a;b^n]$ and $x-y\in[a;b^n]$. Using (S3), (p2) and Lemma 3.5, we have

$$\begin{aligned} 0 &= ((x-y)-a)-b^n \\ &= (((x-a)-(y-a))-b)-b^{n-1} \\ &= ((((x-a)-b)-((y-a)-b))-b)-b^{n-2} \\ &\cdots\cdots\cdots \\ &= ((x-a)-b^n)-((y-a)-b^n) \\ &= ((x-a)-b^n)-0 \\ &= (x-a)-b^n, \end{aligned}$$

and so $x\in[a;b^n]$. Therefore $[a;b^n]$ is an ideal of X. □

Using the set $[a;b^n]$ we establish a condition for a subset of a subtraction algebra X to be an ideal of X.

Theorem 3.7. *Let* A *be a nonempty subset of a subtraction algebra* X*. Then* A *is an ideal of* X *if and only if* $[a;b^n]\subseteq A$ *for every* $a,b\in A$ *and* $n\in\mathbb{N}$*.*

Proof. Assume that A is an ideal of X and let $a,b\in A$ and $n\in\mathbb{N}$. If $x\in[a;b^n]$, then $(x-a)-b^n=0$. Since $a,b\in A$, it follows that $x\in A$ by using (I2) repeatedly. Hence $[a;b^n]\subseteq A$. Conversely suppose that $[a;b^n]\subseteq A$ for every $a,b\in A$ and $n\in\mathbb{N}$. Obviously $0\in[a;b^n]\subseteq A$. Let $x,y\in X$ be such that $y\in A$ and $x-y\in A$. Then

$$\begin{aligned} (x-(x-y))-y^n &= ((x-(x-y))-y)-y^{n-1} \\ &= ((x-y)-(x-y))-y^{n-1} \\ &= 0-y^{n-1}=0, \end{aligned}$$

and thus $x\in[x-y;y^n]\subseteq A$. Hence A is an ideal of X. □

Corollary 3.8. *If* A *is an ideal of a subtraction algebra* X*, then* $A=\bigcup_{a,b\in A}[a;b^n]$ *for every* $n\in\mathbb{N}$*.*

Proof. Let A be an ideal of X. The inclusion $\bigcup_{a,b\in A}[a;b^n]\subseteq A$ is by Theorem 3.7. Let $x\in A$. Since $x\in[x;0^n]$, it follows that

$$A\subseteq\bigcup_{x\in A}[x;0^n]\subseteq\bigcup_{a,b\in A}[a;b^n].$$

This completes the proof. □

References

[1] J. C. Abbott, *Semi-Boolean Algebras,* Matemat. Vesnik **4** (1967), 177–198.

[2] J. C. Abbott, *Sets, Lattices and Boolean Algebras,* Allyn and Bacon, Boston 1969.

[3] Y. B. Jun, X. L. Xin and E. H. Roh, *A class of algebras related to BCI-algebras and semigroups,* Soochow J. Math. **24(4)** (1998), 309–321.

[4] Y. B. Jun, H. S. Kim and E. H. Roh, *Ideal theory of subtraction algebras,* Sci. Math. Jpn. Online **e-2004**, 397–402.

[5] Y. H. Kim and H. S. Kim, *Subtraction algebras and BCK-algebras*, Math. Bohemica **128(1)** (2003), 21–24.

[6] B. M. Schein, *Difference Semigroups,* Comm. in Algebra **20** (1992), 2153–2169.

[7] B. Zelinka, *Subtraction Semigroups,* Math. Bohemica, **120** (1995), 445–447.

Y. B. Jun: Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Korea
E-mail address: ybjun@gnu.ac.kr jamjana@korea.com

H. S. Kim: Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail address: heekim@hanyang.ac.kr