TOPOLOGICAL REPRESENTATION OF DISTRIBUTIVE SEMILATTICES

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Abstract. In this paper we give a topological representation of distributive meet-semilattice with last element 1. We study the notion of irreducible, weakly-irreducible filter and order-ideal in a meet-semilattice. We show a characterization of distributive semilatices by means of weakly-irreducible filters. These results are applied to give a topological representation by means of ordered topological spaces. Finally, we show a duality between meet-hemimorphism of distributive meet-semilattices and certain relations. In particular, we obtain a duality for homomorphism of distributive meet-semilattices by means of binary relations.

1 Introduction

In [3] George Grätzer introduces the class of distributive join-semilattice and gives a topological representation extending the known topological representation of Stone for distributive lattices and Boolean algebras. He mentioned that it is possible to give a topological representation for homomorphisms of join-semilattices, but such representation is not given. Our purpose is to develop a topological representation for distributive meet-semilattices with last element 1 using ordered topological spaces. This approach is useful especially to give a duality for homomorphism of distributive meet-semilattices.

In Section 2 we introduce the definitions and necessary notions to develop this paper. In Section 3 we study the irreducible filters of a meet-semilattice and we give a characterization of them. We introduce the notion of order-ideal and weakly irreducible filters and we prove that a meet-semilattice is distributive if and only if the set of irreducible filters agrees with the set of weakly irreducible filters. In Section 4 we present the topological representation of a distributive meet-semilattice by means of ordered topological spaces. The representation given in this work is a modification of the results of G. Grätzer on the topological representation for distributive join-semilattices. In Section 5 we introduce the notion of meet-hemimorphism. We prove that there exists a duality between this type of mapping and certain $(n+1)$-ary relations. In the case $n = 1$ we have the usual notion of homomorphism. Consequently, we have that the dual of a homomorphism between two distributive meet-semilattices is a binary relation defined between the dual spaces instead of a function defined between the dual spaces.

2 Preliminaries

Let us recall that a *meet-semilattice* with last element is an algebra $\langle A, \wedge, 1\rangle$ of type $(2, 1)$ such that the operation $\wedge$ is idempotent, commutative, associative, and $a \wedge 1 = a$, for

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all $a \in A$. As usual, the binary relation $\leq$ defined by $a \leq b$ if and only if $a \wedge b = a$ is an order. In what follows we will say *semilattices* instead of meet-semilattice.

A *filter* of a semilattice A is a subset $F \subseteq A$ such that $1 \in F$, if $a \leq b$ and $a \in F$, then $b \in F$, and if $a, b \in F$, then $a \wedge b \in F$. The filter generated by a subset $H \subseteq A$ we will denoted by $F(H)$. If $H = \{a\}$, then we will write $F(a)$. We will denote by $Fi(A)$ the set of all filters of A.

Let X be a set. The set of all subsets of X is denoted by $\mathcal{P}(X)$. A subset $K \subseteq \mathcal{P}(X)$ is called *dually directed* if for any $U, V \in K$ there exists $W \in K$ such that $W \subseteq U \cap V$.

Let us consider a poset $\langle X, \leq \rangle$. A subset $U \subseteq X$ is said to be *increasing* (*decreasing*) if for all $x, y \in X$ such that $x \in U$ ($y \in U$) and $x \leq y$, we have $y \in U$ ($x \in U$). The set of all increasing subsets of X is denoted by $\mathcal{P}_i(X)$. We note that if $\langle X, \leq \rangle$ is a poset then $\langle \mathcal{P}_i(X), \cap, X \rangle$ is a semilattice. For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X : \exists y \in Y : y \leq x\}$ ($(Y] = \{x \in X : \exists y \in Y : x \leq y\}$). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively.

Let $\langle X, \mathcal{T} \rangle$ be a topological space. We will denote by $\mathcal{KO}(X)$ the set of all *compact and open* subsets of X and by D the set $\{U : U^c = X - U \in \mathcal{KO}(X)\}$.

Let us recall that an ordered topological space is a triple $\langle X, \leq, \mathcal{T} \rangle$ where $\langle X, \mathcal{T} \rangle$ is a topological space and $\langle X, \leq \rangle$ is a poset.

3 Distributive semilattices

A semilattice A is *distributive* if for all $a, b, c \in A$ such that $a \wedge b \leq c$ there exist a_1, $b_1 \in A$ such that $a \leq a_1$, $b \leq b_1$ and $c = a_1 \wedge b_1$. Consequently, the set $Fi(A)$ is a lattice. We will denote by $\mathcal{DS}$ the class of distributive semilattices. The following result was proved by G. Grätzer in [3] for distributive join-semilattices.

Theorem 1 [3] *Let A be a semilattice. Then A is distributive if and only if the set $Fi(A)$, considered as a lattice, is distributive.*

Definition 2 Let A be a semilattice. A proper filter F of A is *irreducible* if for all $F_1, F_2 \in Fi(A)$ such that $F = F_1 \cap F_2$, then $F = F_1$ or $F = F_2$.

Definition 3 Let A be a semilattice. A subset I of A is called an *order-ideal* of A if:

ID1 If $b \in I$ and $a \leq b$, then $a \in I$.

ID2 If $a, b \in I$ then there exists an element $c \in I$ such that $a \leq c$ and $b \leq c$.

Definition 4 Let A be a semilattice. A proper filter F of A is *weakly irreducible* if $I = F^c = \{x \in A : x \notin F\}$ *is an order-ideal.*

We will denote by $X(A)$, $X_w(A)$ and $Id(A)$ the set of irreducible filters, weakly irreducible filters and proper order-ideals of A, respectively.

Lemma 5 *Let A be a semilattice. Then every weakly irreducible filter is an irreducible filter.*

Proof. Let $F \in X_w(A)$. Let $F_1, F_2 \in Fi(A)$ such that $F = F_1 \cap F_2$. If $F \neq F_1$ and $F \neq F_2$, then there exists $a \in F_1 - F$ and $b \in F_2 - F$. As $F \in X_w(A)$, there exists $c \in A$ such that $a \leq c$ and $b \leq c$. But this implies that $c \in F_1 \cap F_2 = F$, which is a contradiction. Thus, F is irreducible. ∎

Lemma 6 *Let A be a semilattice. Let F be a proper filter of A. Then F is irreducible if and only if for every $a, b \notin F$ there exists $c \notin F$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$.*

Proof. $\Rightarrow$) Let $a, b \notin F$. Let us consider the filters $F_a = F(F \cup \{a\})$ and $F_b = F(F \cup \{b\})$. If $F = F_a \cap F_b$, then $F = F_a$ or $F = F_b$, which is a contradiction. So, $F \subset F_a \cap F_b$. It follows that there exists $c \in F_a \cap F_b$ and $c \notin F$. Then there are $f_1, f_2 \in F$ such that $a \wedge f_1 \leq c$ and $b \wedge f_2 \leq c$. Since F is a filter, $f = f_1 \wedge f_2 \in F$ and consequently $a \wedge f \leq c$ and $b \wedge f \leq c$.

$\Leftarrow$) Let F, $F_1, F_2 \in Fi(A)$ such that $F = F_1 \cap F_2$, $F \neq F_1$ and $F \neq F_2$. Then there exist $a \in F_1 - F$ and $b \in F_2 - F$. By the assumption, there exists $c \notin F$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$. Since $f \in F_1 \cap F_2$, then $a \wedge f \in F_1$ and $b \wedge f \in F_2$. It follows that $c \in F_1 \cap F_2 = F$, which is a contradiction. ■

Lemma 7 *Let A be a semilattice. Let F be a proper filter of A. Then F is weakly irreducible if and only if for all $F_1, F_2 \in Fi(A)$ such that $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$.*

Proof. Assume that F is weakly irreducible. Let $F_1, F_2 \in Fi(A)$ such that $F_1 \cap F_2 \subseteq F$. If $F_1 \subsetneq F$ and $F_2 \subsetneq F$, then there exists $a \in F_1 - F$ and there exists $b \in F_2 - F$. So, there exists $c \notin F$ such that $a \leq c$ and $b \leq c$. But this implies that $c \in F_1 \cap F_2 \subseteq F$, which is a contradiction. Thus, $F_1 \subseteq F$ or $F_2 \subseteq F$.

Let $a, b \notin F$. Then $F(a) \subsetneq F$ and $F(b) \subsetneq F$. By assumption, $F(a) \cap F(b) \subsetneq F$, i.e., there exists $c \in A$ such that $a \leq c$, $b \leq c$ and $c \notin F$. ■

The Prime Filter theorem is one of the most important results in the theory of distributive lattices. In [3] a similar result was established for distributive join-semilattices. Now, we shall give an analogous theorem for semilattices.

Theorem 8 *Let A be a semilattice. Let $F \in Fi(A)$ and $I \in Id(A)$ such that $F \cap I = \emptyset$. Then there exists $P \in X(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let us consider the set $\mathcal{F} = \{H \in Fi(A) : F \subseteq H \text{ and } H \cap I = \emptyset\}$. Since $F \in \mathcal{F}$, then $\mathcal{F} \neq \emptyset$. It is clear that the union of a chain of elements of $\mathcal{F}$ is also in $\mathcal{F}$. So, by Zorn's lemma, there exists a filter P maximal in $\mathcal{F}$. We prove that $P \in X(A)$. Let $a, b \notin P$ and let us consider the filters $P_a = F(P \cup \{a\})$ and $P_b = F(P \cup \{b\})$. Clearly, $P \subset P_a$ and $P \subset P_b$. Then, $P_a, P_b \notin \mathcal{F}$. Thus, $P_a \cap I \neq \emptyset$ and $P_a \cap I \neq \emptyset$. It follows that there exist $p_1, p_2 \in P$ and there exist $x, y \in I$ such that $p_1 \wedge p_2 \wedge a \leq x$ and $p_1 \wedge p_2 \wedge b \leq y$. Since I is an order-ideal, there exists $c \in I$ such that $x \leq c$ and $y \leq c$. So, $p_1 \wedge p_2 \wedge a \leq c$ and $p_1 \wedge p_2 \wedge b \leq c$. Therefore, by Lemma 6, we conclude that $P \in X(A)$. ■

Corollary 9 *In a semilattice A every proper filter is the intersection of irreducible filters.*

Proof. Let F be a proper filter of A. For each $a \notin F$, we have $F \cap (a] = \emptyset$. Since $(a] \in Id(A)$, then there exists $P_a \in X(A)$ such that $F \subseteq P_a$ and $a \notin P_a$. Thus, $F = \bigcap \{P_a : P_a \in X(A), a \notin F\}$. ■

In a distributive lattice, a filter is prime if and only if is an irreducible one (see [1]). Also, it is known that a lattice is distributive if and only if every irreducible filter is prime. In the case of distributive semilattices we can give a similar result, using irreducible filters and weakly irreducible filters.

Theorem 10 *Let A be a semilattice. Then the following conditions are equivalent:*

1. $X(A) = X_w(A)$.

2. *A is distributive.*

Proof. $1 \Rightarrow 2$. Using Theorem 1 we prove that the set $Fi(A)$, considered as a lattice, is distributive. Let $F_1, F_2, F_3 \in Fi(A)$. It is enough to prove that that :

$$F_1 \cap (F_2 \vee F_3) \subseteq (F_1 \cap F_2) \vee (F_1 \cap F_3).$$

Suppose the contrary. Then there exists $a \in F_1 \cap (F_2 \vee F_3) - (F_1 \cap F_2) \vee (F_1 \cap F_3)$. So, by Theorem 8, there exists $P \in X_w(A)$ such that $(F_1 \cap F_2) \vee (F_1 \cap F_3) \subseteq P$ and $a \notin P$. So, $F_1 \cap F_2 \subseteq P$ and $F_1 \cap F_2 \subseteq P$. By Lemma 7, $F_1 \subseteq P$ or $F_2 \subseteq P$ and $F_1 \subseteq P$ or $F_3 \subseteq P$. Since $a \in F_1$ and $a \notin P$, then $F_2 \subseteq P$ and $F_3 \subseteq P$. So, $F_2 \vee F_3 \subseteq P$ and consequently $a \in F_1 \cap (F_2 \vee F_3) \subseteq P$, which is a contradiction.

$2 \Rightarrow 1$. It is clear that $X_w(A) \subseteq X(A)$. Let $P \in X(A)$ and $F_1, F_2 \in Fi(A)$ such that $F_1 \cap F_2 \subseteq P$. Then, since $Fi(A)$ is distributive, $P = (F_1 \cap F_2) \vee P = (F_1 \vee P) \cap (F_2 \vee P)$. As $P \in X(A)$, $P = F_1 \vee P$ or $P = F_2 \vee P$, which implies that $F_1 \subseteq P$ or $F_2 \subseteq P$. Thus, by Lemma 7, $P \in X_w(A)$. ■

4 Topological Representation

We shall modify the topological representation for distributive join-semilattice given by G. Grätzer [3] introducing an order in the topology. First, we shall give a representation theorem for semilattices. Let us recall that if $\langle X, \leq \rangle$ is a poset then $\langle \mathcal{P}_i(X), \cap, X \rangle$ is a semilattice

Let $A \in \mathcal{DS}$. Let us consider the poset $\langle X(A), \subseteq \rangle$ and let us consider the mapping

$$\beta_A : A \to \mathcal{P}_i(X(A))$$

defined by $\beta_A(a) = \{P \in X(A) : a \in P\}$. Let $\beta_A(A)^c = \{\beta_A(a)^c : a \in A\}$.

Theorem 11 (Representation theorem) *Let $A \in \mathcal{DS}$. Then, A is isomorphic to the subalgebra $\beta_A(A) = \{\beta_A(a) : a \in A\}$ of $\mathcal{P}_i(X(A))$.*

Proof. It is clear that $\beta_A(a) \in \mathcal{P}_i(X(A))$ for all $a \in A$, $\beta_A(a \wedge b) = \beta_A(a) \cap \beta_A(b)$ and $\beta_A(1) = X(A)$. The injectivity of β_A follows by Theorem 8. Thus, $A \cong \beta_A(A)$. ■

Proposition 12 *Let $A \in \mathcal{DS}$. Then:*

1. $X(A) = \bigcup \{\beta_A(a)^c : a \in A\}$.
2. *For any $a, b \in A$ and for any $P \in X(A)$ such that $P \in \beta_A(a)^c \cap \beta_A(b)^c$ there exists $c \in A$ such that $P \in \beta_A(c)^c \subseteq \beta_A(a)^c \cap \beta_A(b)^c$.*
3. *For every $a \in A$ and for every $B \subseteq A$ if $\beta_A(a) = \bigcap \{\beta_A(b) : b \in B\}$, then there exists a finite subset B_0 of B such that $\beta(a) = \bigcap \{\beta_A(b) : b \in B_0\}$.*

Proof. 1. It is immediate, because any irreducible filter is proper. The assertion 2. follows by the definition of weakly irreducible filter.

3. Let $a \in A$ and let $B \subseteq A$ such that $\beta_A(a) = \bigcap \{\beta_A(b) : b \in B\}$. Let $F(B)$ be the filter generated by B. Then, $a \in F(B)$, because in the opposite case, by Theorem 8, there exists $P \in X(A)$ such that $F(B) \subseteq P$ and $a \notin P$. But this implies that $P \in \bigcap \{\beta_A(b) : b \in B\}$ and $P \notin \beta_A(a)$, which is impossible. Thus, there exists $B_0 = \{b_1, ..., b_n\} \subseteq B$ such that $b_1 \wedge ... \wedge b_n \leq a$. So, $\beta_A(b_1 \wedge ... \wedge b_n) = \beta_A(b_1) \cap ... \cap \beta_A(b_n) \subseteq \beta_A(a)$. It is clear that

$\beta_A(a) \subseteq \beta_A(b_1) \cap ... \cap \beta_A(b_n)$. Thus, $\beta_A(b_1) \cap ... \cap \beta_A(b_n) = \beta_A(a)$. ■

Let $A \in \mathcal{DS}$. By 1. of the previous theorem we have that the family

$$\beta_A(A)^c = \{\beta_A(a)^c = X(A) - \beta_A(a) : a \in A\}$$

is a subbasis for a topology defined on $X(A)$. By 2. of the previous theorem we have that $\beta_A(A)^c$ is a base for a topology on $X(A)$. So, the structure $\mathcal{F}(A) = \langle X(A), \subseteq, \beta_A(A)\rangle$ can be considered as an ordered topological space where $\beta_A(A)^c$ is a base for the topology. This space will be called the *dual space of* A.

Proposition 13 *Let $A \in \mathcal{DS}$. Let $\mathcal{F}(A)$ be the dual space of A. Then:*

1. *A subset $U \subseteq X(A)$ is open in $\mathcal{F}(A)$ if and only if there exists a filter F of A such that $U = \beta_A(F)^c$, where $\beta_A(F) = \{P \in X(A) : F \subseteq P\}$.*
2. *A subset $U \subseteq X(A)$ is a compact-open in $\mathcal{F}(A)$ if and only if there exists $a \in A$ such that $U = \beta_A(a)^c$.*
3. *Let F be a closed subset in $\mathcal{F}(A)$ and let $K = \{U_i : i \in I\}$ be a dually directed subfamily of compact-open subsets such that $F \cap U_i \neq \emptyset$ for any $i \in I$. Then, $F \cap \bigcap\{U_i : i \in I\} \neq \emptyset$.*

Proof. 1. Let U be an open subset in $\mathcal{F}(A)$. Since $\beta_A(A)^c$ is a base of the space $\mathcal{F}(A)$, $U = \bigcup\{\beta_A(a)^c : a \in B \subseteq A\}$. Let us consider the filter $F = F(B)$. It is easy to see that $U = \beta_A(F)^c$.

On the other hand it is immediate to check that if F is a filter of A, then $\beta_A(F) = \bigcap\{\beta_A(a) : a \in F\}$. Thus, $\beta_A(F)^c$ is open in $\mathcal{F}(A)$.

2. Let U be a compact-open in $\mathcal{F}(A)$. By 1 above ,

$$U = \beta_A(F)^c = \bigcup\{\beta_A(a)^c : a \in F\}$$

for some filter F of A. Since U is compact, there exists $\{a_1, ..., a_n\} \subseteq F$ such that

$$U = \beta_A(a_1)^c \cup ... \cup \beta_A(a_n)^c = (\beta_A(a_1) \cap ... \cap \beta_A(a_n))^c = \beta_A(a_1 \wedge ... \wedge a_n)^c.$$

The other direction follows by 3 of Proposition 12.

3. Let F be a closed subset in $\mathcal{F}(A)$ and let $K = \{U_i : i \in I\}$ be a family of dually directed compact-open subsets of $\mathcal{F}(A)$ such that $F \cap U_i \neq \emptyset$ for any $i \in I$. By 2. above, for each $i \in I$ there exists $a_i \in A$ such that $U_i = \beta_A(a_i)^c$. Let us consider the set $H = \{a_i \in A : U_i = \beta_A(a_i)^c\}$ and let us consider the decreasing subset generated by H:

$$(H] = \{x \in A : x \leq c \text{ for some } c \in H\}.$$

We prove that $(H]$ is an order-ideal of A. Let $a, b \in (H]$. Then there exist $c_1, c_2 \in H$ such that $a \leq c_1$ and $b \leq c_2$. Since $\beta_A(c_1)^c, \beta_A(c_2)^c \in K$, then there exists $c \in H$ such that $\beta_A(c)^c \subseteq \beta_A(c_1)^c \cap \beta_A(c_2)^c$. So, $a \leq c$ and $b \leq c$ and thus $(H] \in Id(A)$. Since F is a closedsubset, using the assertion 1. above we have $F = \beta_A(D)$ for some $D \in Fi(A)$. We prove that

$$D \cap (H] = \emptyset.$$

Suppose the contrary. Then there exists $a \in D$ and $c \in H$ such that $a \leq c$. By hypothesis, $\beta_A(D) \cap \beta_A(c)^c \neq \emptyset$. So, there exists $P \in X(A)$ such that $D \subseteq P$ and $c \notin P$. But as $a \in D$,

$a \in P$ and consequently $c \in P$, which is a contradiction. Thus, $D \cap (H] = \emptyset$. By Theorem 8, there exists $P \in X(D)$ such that $D \subseteq P$ and $H \cap P = \emptyset$. Therefore, $P \in \beta_A(D) = F$ and $P \in \bigcap \{U_i : i \in I\}$. ■

Guided by the previous result we introduce the following definition.

Definition 14 An *DS-space* is an ordered topological space $\mathcal{F} = \langle X, \leq, \mathcal{T} \rangle$ such that:

1. The set of all compact and open subsets $\mathcal{KO}(X)$ forms a basis for the topology.
2. All closed subsets are increasing.
3. For every $x, y \in X$, if $x \nleq y$, then there exists $U \in D$ such that $x \in U$ and $y \notin U$.
4. If F is a closed subset and $K = \{U_i : i \in I\}$ is a dually directed subfamily of $\mathcal{KO}(X)$ such that $F \cap U_i \neq \emptyset$ for all $i \in I$, then $F \cap \bigcap U_i \neq \emptyset$.

We note that by the condition 2 of the above definition we have that $D = \{U : U^c \in \mathcal{KO}(X)\} \subseteq \mathcal{P}_i(X)$. It is clear that if $\mathcal{F} = \langle X, \leq, \mathcal{T} \rangle$ is a DS-space then $\langle D, \cap, X \rangle$ is a distributive semilattice.

An ordered topological space $\langle X, \leq, \mathcal{T} \rangle$ where $\mathcal{KO}(X)$ is a basis for the topology $\mathcal{T}$ will be denoted by $\langle X, \leq, D \rangle$.

Lemma 15 *Let $\mathcal{F} = \langle X, \leq, D \rangle$ be a DS-space. Then for each $x \in X$, the set $H_X(x) = \{U \in D : x \in U\}$ belongs to $X(D)$. Thus, the mapping $H_X : X \to X(D)$ is well-defined.*

Proof. *Let* $x \in X$. It is clear that $H_X(x)$ is a filter of D. We prove that is it irreducible. Let $U, V \in D$ such that $x \in U^c \cap V^c$. As $U^c \cap V^c$ is open and $\mathcal{KO}(X)$ is a basis, then there exists $O^c \in \mathcal{KO}(X)$ such that $x \in O^c \subseteq U^c \cap V^c$. So, $H_X(x)$ is an irreducible filter. ■

Proposition 16 *Let $\mathcal{F} = \langle X, \leq, D \rangle$ be a DS-space. Then the following conditions are equivalents:*

1. *H_X is onto*
2. *For every closed subset F and for every dually directed subfamily $K = \{U_i : i \in I\}$ of $\mathcal{KO}(X)$ such that $F \cap U_i \neq \emptyset$ for all $i \in I$, then*

$$F \cap \bigcap \{U_i : i \in I\} \neq \emptyset.$$

Proof. $1 \Rightarrow 2$. Let F be a closed and let $K = \{U_i : i \in I\}$ be a dually directed subfamily of $\mathcal{KO}(X)$ such that $F \cap U_i \neq \emptyset$ for all $i \in I$. Since F^c is open and $\mathcal{KO}(X)$ is a basis, then $F = \bigcap \{U : U \in B \subseteq D\}$. Let us consider the set $H = \{U_i^c : U_i \in K\} \subseteq D$. Since K is dually directed, then the decreasing subset $(H]$ is an order-ideal of D. We prove that

$$F(B) \cap (H] = \emptyset.$$

Suppose the contrary. Then, there exists $U_k^c \in H$ and there exist $U_1, ..., U_n \in B$ such that $U_1 \cap ... \cap U_n \subseteq U_k^c$. Since $F \cap U_k \neq \emptyset$, there exists $x \in X$ such that $x \in F$ and $x \in U_k$. So, $x \in U_1 \cap ... \cap U_n \subseteq U_k^c$, which is a contradiction. Thus, there exists $P \in X(D)$ such that $F(B) \subseteq P$ and $P \cap (H] = \emptyset$. As H_X is onto, there exists $x \in X$ such that $P = H_X(x)$. Thus, $x \in F \cap \bigcap \{U_i : i \in I\}$.

$2 \Rightarrow 1$. Let $P \in X(D)$. Let us consider the set $K = \left\{U_j^c : U_j \notin P\right\} \subseteq \mathcal{KO}(X)$. The set $F = \bigcap \{U_i : U_i \in P\}$ is closed and $F \cap U_i^c \neq \emptyset$ for $U_i^c \in K$, because in the opposite case

$$U_j^c \subseteq \bigcup \{U_i^c : U_i \in P\},$$

and since U_j^c is compact, then $U_j^c \subseteq U_0^c \cup U_1^c \cup ... \cup U_n^c$, i.e., $U_1 \cap ... \cap U_n \subseteq U_j$. It follows that $U_j \in P$, which is a contradiction. By assumption, $F \cap \bigcap \left\{U_j^c : U_j \notin P\right\} \neq \emptyset$. So, there exists $x \in \bigcap \{U_i : U_i \in P\} \cap \bigcap \left\{U_j^c : U_j \notin P\right\}$, which implies that $P = H_X(x)$. ■

Theorem 17 *Let $A \in \mathcal{DS}$. Then $\mathcal{F}(A) = \langle X(A), \subseteq, \beta_A(A)\rangle$ is a DS-space and the mapping $\beta_A : A \to \beta_A(A)$ is a DS-isomorphism.*

Theorem 18 *Let $\mathcal{F} = \langle X, \leq, D\rangle$ be a DS-space. Then the mapping $H_X : X \to X(D)$ is an order-isomorphism and a homeomorphism.*

Proof. By the condition 4 of Definition 14 it follows that H_X is an order-isomorphism. By Proposition 16, H_X is onto. Thus, it is enough to check that if U is an open subset in $X(D)$, then $H_X^{-1}(U)$ is open in X. By Proposition 13, given an open U there exists a filter F of D such that $U^c = \beta_D(F)$. Let $V = \bigcap \{O : O \in F\}$. Then, V is closed in X. It is easy to see that $V^c = H_X^{-1}(U)$. Thus, H_X is a homeomorphism. ■

By the two previous theorems we can assert that there exists a duality between distributive semilattices and DS-spaces.

5 Duality for Homomorphisms

Let $X_1, ..., X_n$ be sets. The Cartesian product of the family $\{X_i : 1 \leq i \leq n\}$ is the set $\prod_{i=1}^n X_i = \{\vec{x} = (x_1, ..., x_n) : x_i \in X_i\}$. If $X_i = X$, we will write $\prod_{i=1}^n X_i = X^n$.

The sum of the sets $X_1, ..., X_n$ is the set

$$X_1 + \cdots + X_n = \{\vec{x} = (x_1, ..., x_n) : (\exists 1 \leq i \leq n) \quad x_i \in X_i\}.$$

Definition 19 Let $A, A_1, ..., A_n \in \mathcal{DS}$. A mapping $h : \prod_{i=1}^n A_i \to A$ is a *meet-hemimorphism* if

H1. $h(x_1, ..., a, ..., x_n) \wedge h(x_1, ..., b, ..., x_n) = h(x_1, ..., a \wedge b, ..., x_n)$,

H2. $h(x_1, ..., 1, ..., x_n) = 1$.

We note that a meet-hemimorphism h is monotonic in each variable, i.e., if $a \leq b$, then $h(x_1, ..., a, ..., x_n) \leq h(x_1, ..., b, ..., x_n)$. We will denote by $\mathcal{MH}(\prod_{i=1}^n A_i, A)$ the set of all meet-hemimorphism of $\prod_{i=1}^n A_i$ into A. A *homomorphism* between two distributive semilattices A and B is an element of $\mathcal{MH}(A, B)$

Our next aim is to prove a duality for meet-hemimorphism of distributive semilattices. In the context of lattices theory this type of mapping plays an important role in the theory of distributive lattices with operators (see [4] or [5]). Following the lines of the work [5], we shall prove a duality between meet-hemimorphism and certain $(n+1)$-ary relations. The following example is crucial in the next results.

Example 20 Let $\mathcal{F}$, $\mathcal{F}_1, ..., \mathcal{F}_n$ be DS-spaces. Let $R \subseteq X \times X_1 \times ... \times X_n$ be an $(n+1)$-ary relation. For each $x \in X$, let $R(x) = \{\vec{x} \in \prod_{i=1}^n X_i : (x, \vec{x}) \in R\}$. Define the mapping $h_R : \prod_{i=1}^n D_i \to D$ by

$$h_R(U_1, ..., U_n) = \{x \in X : R(x) \subseteq U_1 + ... + U_n\}.$$

It is easy to check that $h_R \in \mathcal{MH}(\prod_{i=1}^n D_i, D)$.

Proposition 21 *Let $\mathcal{F}$, $\mathcal{F}_1, ..., \mathcal{F}_n$ be DS-spaces. Let $R \subseteq X \times X_1 \times ... \times X_n$ be an $(n+1)$-ary relation. Suppose that for every $(U_1, ..., U_n) \in \prod_{i=1}^{n} D_i$, $h_R(U_1, ..., U_n) \in D$. Then the following conditions are equivalent:*

1. $R(x) = \bigcap \{U_1 + ... + U_n : R(x) \subseteq U_1 + ... + U_n\}$, *for all* $x \in X$.
2. *For all* $x \in X$ *and for all* $\vec{x} \in \prod_{i=1}^{n} X_i$,
   $(x, \vec{x}) \in R$ *if and only if* $h_R^{-1}(H_X(x)) \subseteq H_{X_1}(x_1) + ... + H_{X_n}(x_n)$,
   where $h_R^{-1}(H_X(x)) = \left\{ \overrightarrow{U} \in \prod_{i=1}^{n} D_i : h_R\left(\overrightarrow{U}\right) \in H_X(x) \right\}$.

If 1. or 2. is valid, then for all $x, y \in X$, $R(y) \subseteq R(x)$ *when* $x \leq y$.

Proof. $1 \Rightarrow 2$. Suppose that there exist $x \in X$ and $\vec{x} \in \prod_{i=1}^{n} X_i$ such that $(x, \vec{x}) \notin R$. Then there exist $U_i \in D_i$ for $1 \leq i \leq n$, such that $R(x) \subseteq U_1 + ... + U_n$ and $x_i \notin U_i$ for all $1 \leq i \leq n$. Thus, $x \in h_R(U_1, ..., U_n)$ and $\vec{x} \notin U_1 + ... + U_n$, i.e., $h_R^{-1}(H(x)) \subsetneq H_1(x_1) + ... + H_n(x_n)$.

$2 \Rightarrow 1$. It is easy and left to the reader. ■

Definition 22 Let $\mathcal{F}$, $\mathcal{F}_1, ..., \mathcal{F}_n$ be DS-spaces. We shall say that a subset $R \subseteq X \times X_1 \times ... \times X_n$ is a *meet-relation,* if:

1. For every $(U_1, ..., U_n) \in \prod_{i=1}^{n} D_i$, $h_R(U_1, ..., U_n) \in D$,
2. $R(x) = \bigcap \{U_1 + ... + U_n : R(x) \subseteq U_1 + ... + U_n\}$, *for all* $x \in X$.

Let $A, A_1, ..., A_n \in \mathcal{DS}$. Let $h \in \mathcal{MH}(\prod_{i=1}^{n} A_i, A)$. Define an $(n+1)$-ary relation $R_h \subseteq X(A) \times X(A_1) \times ... \times X(A_n)$ by:

$$(P, P_1, ..., P_n) \in R_h \Leftrightarrow h^{-1}(P) \subseteq P_1 + ... + P_n,$$

where $h^{-1}(P) = \{\vec{x} \in \prod_{i=1}^{n} A_i : h(\vec{x}) \in P\}$.

Proposition 23 *Let* $h \in \mathcal{MH}(\prod_{i=1}^{n} A_i, A)$.

1. *For all* $P \in X(A)$ *and for all* $\vec{a} \in \prod_{i=1}^{n} A_i$, $h(\vec{a}) \notin P$ *if and only if there exists* $P_i \in X(A_i)$ *for* $1 \leq i \leq n$, *such that* $(P, P_1, ..., P_n) \in R_h$ *and* $a_i \notin P_i$ *for all* $1 \leq i \leq n$.
2. *The relation* R_h *is a meet-relation.*
3. *The mapping* $h_{R_h} : \beta_{A_1}(A_1) \times ... \times \beta_{A_n}(A_n) \to \beta_A(A)$ *defined as in the example 20 satisfies* $h_{R_h}(\beta_{A_1}(a_1), ..., \beta_{A_n}(a_n)) = \beta_A(h(a_1, ..., a_n))$, *for all* $(a_1, ..., a_n) \in \prod_{i=1}^{n} A_i$.

Proof. 1. We prove first the case $n = 1$. Let $P \in X(A)$ and $h(a) \notin P$. Since h is a homomorphism, $h^{-1}(P)$ is a filter of A_1. So, there exists $P_1 \in X(A_1)$ such that $h^{-1}(P) \subseteq P_1$ and $a \notin P_1$.

We assume that $n > 1$. Let $P \in X(A)$ and let $\vec{a} \in \prod_{i=1}^{n} A_i$ such that $h(\vec{a}) \notin P$. We shall determine filters $F_1, ..., F_n$ and irreducible filters $P_1, ..., P_n$ in $A_1, ..., A_n$, respectively, by recursion as follows. Let

$$F_1 = \{x \in A_1 : h(x, a_2, ..., a_n) \in P\}.$$

Then, $F_1 \in Fi(A)$. Indeed, since $h(1, a_2, ..., a_n) = 1 \in P$, $1 \in F_1$. Let $x \leq y$ and $x \in F_1$. Since h is increasing in each coordinate, $y \in F_1$. If $x, y \in F_1$, by H1 of Definition 19, it follows that $x \wedge y \in F_1$. Thus, F_1 is a filter of A_1. Moreover, as $h(a_1, a_2, ..., a_n) \notin P$, $a_1 \notin F_1$. Consequently there exists $P_1 \in X(A_1)$ such that $F_1 \subseteq P_1$ and $a_1 \notin P_1$.

Suppose that we have determinated filters $F_1, ..., F_k$ and irreducible filters $P_1, ..., P_k$ in $A_1, ..., A_k$, respectively, such that

1. $F_i \subseteq P_i$ and $a_i \notin P_i$ for $1 \leq i \leq k$, and
2. $F_k = \{x \in A_k : h(x_1, ..., x_{k-1}, x, a_{k+1}, ..., a_n) \in P \text{ for some } x_i \notin P_i, 1 \leq i \leq k-1\}$.

Let us define the set

$$F_{k+1} = \{x \in A_{k+1} : h(x_1, ..., x_k, x, a_{k+2}, ..., a_n) \in P \text{ for some } x_i \notin P_i, 1 \leq i \leq k\}.$$

We prove that $F_{k+1} \in Fi(A_{k+1})$. Let $x, y \in F_{k+1}$. Then there exists $x_i, y_i \notin P_i$ for each $1 \leq i \leq k$ such that:

$h(x_1, ..., x_k, x, a_{k+2}, ..., a_n) \in P$ and $h(x_1, ..., x_k, y, a_{k+2}, ..., a_n) \in P$.

Since each P_i is an irreducible filter, then there exists $c_i \notin P_i$ such that $x_i, y_i \leq c_i$ for each $1 \leq i \leq k$. As h is increasing in each coordinate, $h(c_1, ..., c_k, x, a_{k+2}, ..., a_n) \in P$ and $h(c_1, ..., c_k, x, a_{k+2}, ..., a_n) \in P$. Then,

$$h(c_1, ..., c_k, x, a_{k+2}, ..., a_n) \wedge h(c_1, ..., c_k, y, a_{k+2}, ..., a_n) = h(c_1, ..., c_k, x \wedge y, a_{k+2}, ..., a_n) \in P$$

So, $x \wedge y \in F_{k+1}$, and thus F_{k+1} is a filter of A_{k+1}. Since $a_{k+1} \notin F_{k+1}$, there exists $P_{k+1} \in X(A_{k+1})$ such that $F_{k+1} \subseteq P_{k+1}$ and $a_{k+1} \notin P_{k+1}$. Therefore, we have filters $F_1, ..., F_n$ and irreducible filters P_1 ,...,P_n in $A_1, ..., A_n$, respectively, such that $a_i \notin P_i$ for $1 \leq i \leq n$, and

(1) $\qquad F_n = \{x \in A_n : h(x_1, ..., x_{n-1}, x) \in P \text{ for some } x_1 \notin P_1, ..., x_{n-1} \notin P_{n-1}\}$

It is clear that $\vec{a} \notin P_1 + \cdots + P_n$. Moreover, if $h(b_1, ..., b_n) \in P$ and $b_i \notin P_i$ for $1 \leq i \leq n-1$, then by (1), we get $b_n \in P_n$ and this implies that $(b_1, ..., b_n) \in P_1 + \cdots + P_n$, i.e., $h^{-1}(P) \subseteq P_1 + \cdots + P_n$.

The other direction is immediate.

2. By 1. it follows that $R_h(P) = \bigcap \{\beta_{A_1}(a_1) + \cdots + \beta_{A_n}(a_n) : h(a_1, ..., a_n) \in P\}$. Thus, we have proved 2. The assertion 3. also follows by 1. ■

Let $A, A_1, ..., A_n \in \mathcal{DS}$. By Proposition 21, for each $h \in \mathcal{H}_n\left(\prod_{i=1}^{n} A_i, A\right)$ there exists a meet- relation $R_h \subseteq X(A) \times X(A_1) \times \cdots \times X(A_n)$ such that $\beta_A(h(\vec{a})) = h_{R_h}(\beta_{A_1}(a_1), ..., \beta_{A_n}(a_n))$. And if $\mathcal{F}, \mathcal{F}_1, ..., \mathcal{F}_n$ are DS-spaces then for each meet relation $R \subseteq X \times X_1 \times \cdots \times X_n$ there exists $h_R \in \mathcal{H}_n\left(\prod_{i=1}^{n} D_i, D\right)$ such that $(x, \vec{x}) \in R$ if and only if $h_R^{-1}(H_X(x)) \subseteq H_{X_1}(x_1) + ... + H_{X_n}(x_n)$. In particular, we deduce that there exists a duality between the category of distributive semilattices with homomorphisms and the category of DS-spaces with meet binary relations.

We finish this section characterizing the injective and surjective homomorphisms of distributive semilattices. Let A and $B \in \mathcal{DS}$. Let $h : A \to B$ be a homomorphism. It is easy to see that, for each $P \in X(A)$, the subset of B defined by

$$(h(P^c)] = \{y \in B : y \leq h(p) \text{ for some } p \notin P\}$$

is an order-ideal of B. This fact will be used in the following results.

Theorem 24 *Let A and B be $\in \mathcal{DS}$ and let $h : A \to B$ be a homomorphism. Then*

1. *h is injective if and only if $\forall P \in X(A)\ \exists Q \in X(B)$ such that $R_h(Q) = [P)$, i.e., $h^{-1}(Q) = P$.*

2. *h is surjective if and only if $\forall Q \in X(B)\, \exists P \in X(A)$ such that $R_h(Q) = [P)$ and $\forall P, Q \in X(B)$ if $R_h(Q) \subseteq R_h(P)$, then $P \subseteq Q$.*

Proof. 1. $\Rightarrow$) Let $P \in X(A)$. Let us consider the filter $F(h(P))$. Since h is injective it is easy to see that $F(h(P)) \cap (h(P^c)] = \emptyset$. Thus, by Theorem 8, there exists $Q \in X(B)$ such that $h(P) \subseteq Q$ and $h(P^c) \cap Q = \emptyset$, i.e., $h^{-1}(Q) = P$.

$\Leftarrow$) Let $a, b \in A$. Suppose that $a \nleq b$. Then there exists $P \in X(A)$ such that $a \in P$ and $b \notin P$. By assumption, there exists $Q \in X(B)$ such that $P = h^{-1}(Q)$. Then, $h(a) \in Q$ and $h(b) \notin Q$. It follows that $h(a) \nleq h(b)$, which implies that h is injective.

2. It is easy to check that the condition $\forall Q \in X(B)\, \exists P \in X(A)$ such that $R_h(Q) = [P)$ is equivalent to the condition $\forall Q \in X(B)$, $h^{-1}(Q) \in X(A)$.

$\Rightarrow$) Let $Q \in X(B)$. Since h is a homomorphism, $h^{-1}(Q)$ is a filter. Let $a, b \in A$ such that $h(a), h(b) \notin Q$. Then there exists $c \notin Q$ such that $h(a) \leq c$ and $h(b) \leq c$. Since h is surjective, there exists $d \in A$ such that $c = h(d)$. So, $h(a) \leq h(d)$ and $h(b) \leq h(d)$ with $d \notin h^{-1}(Q)$. Thus, $h^{-1}(Q) \in X(A)$.

Let $P, Q \in X(B)$. We note that $R_h(Q) \subseteq R_h(P)$ if and only if $h^{-1}(P) \subseteq h^{-1}(Q)$. Suppose then that $h^{-1}(P) \subseteq h^{-1}(Q)$. So it is easy to check that $P \subseteq Q$.

$\Leftarrow$) Let $b \in B$ and assume that $b \notin h(A) = \{h(a) : a \in A\}$. Let us consider the filter $F\left(h\left(h^{-1}(F(b))\right)\right)$. Since $b \notin h(A)$, then it is easy to check that

$$F\left(h\left(h^{-1}(F(b))\right)\right) \cap (b] = \emptyset.$$

So, by Theorem 8, there exists $P \in X(B)$ such that $h\left(h^{-1}(F(b))\right) \subseteq P$ and $b \notin P$. We prove now that

$$(P^c \cap h(A)] = \{y \in B : y \leq x \text{ for some } x \in P^c \cap h(A)\}$$

is an order-ideal of B. It is enough to prove that if $x, y \in P^c \cap h(A)$, then there exists $c \in P^c \cap h(A)$ such that $x \leq c$ and $y \leq c$. Let $x, y \in P^c \cap h(A)$. Then $x = h(a)$ and $y = h(b)$ for some $a, b \in A$. Since $a, b \notin h^{-1}(P)$ and by assumption, $h^{-1}(P) \in X(B)$, there exists $c \notin h^{-1}(P)$ such that $a \leq c$ and $b \leq c$. So, $x \leq h(c)$ and $y \leq h(c)$. Thus, $(P^c \cap h(A)]$ is an order-ideal of B.

We prove that

$$F(b) \cap (P^c \cap h(A)] = \emptyset.$$

Suppose the contrary. Then there exists $q \in P^c$ and $z \in A$ such that $b \leq q = h(z)$. Thus, $h(z) \in F(b)$. It follows that $z \in h^{-1}(F(b)) \subseteq h^{-1}(P)$. So $h(z) \in P$, which is a contradiction. Thus, by Theorem 8, there exists $Q \in X(B)$ such that $Q \cap P^c \cap h(A) = \emptyset$ and $b \in Q$. So, $h(A) \cap Q \subseteq h(A) \cap P$, and this implies that $h^{-1}(Q) \subseteq h^{-1}(P)$ and by hyphotesis we conclude that $Q \subseteq P$. Thus, $b \in P$, which is a contradiction. So $b \in h(A)$ and therefore h is surjective. ■

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