STABLE RANK OF C^*-TENSOR PRODUCTS WITH THE ALGEBRAS OF CONTINUOUS FUNCTIONS ON PRODUCT SPACES OF INTERVALS

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Abstract. In this paper we estimate the (topological) stable rank of C^*-tensor products of C^*-algebras with the algebras of continuous functions on product spaces of closed intervals. Moreover, we give an application of this result to C^*-tensor products with the algebras of continuous functions on locally compact Hausdorff spaces.

Introduction

This article answers a somewhat standing problem raised by Rieffel [Rf1, Question 1.8] and also conjectured but not solved in the paper by Nagisa, Osaka and Phillips [NOP, p. 990]. The problem is whether or not the stable rank of C^*-tensor products of a C^*-algebra $\mathfrak{A}$ (or more generally a complex Banach algebra) with the C^*-algebra $C([0,1]^2)$ of continuous functions on the product space of two closed intervals can be estimated by the stable rank of $\mathfrak{A}$ plus 1. Symbolically,

$$(F1): \quad \mathrm{sr}(C([0,1]^2) \otimes \mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1.$$

It has been known by Rieffel [Rf1, Corollary 7.2] that for any C^*-algebra $\mathfrak{A}$,

$$(F2): \quad \mathrm{sr}(C([0,1]) \otimes \mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1,$$

which is deduced from an interesting Rieffel's result: the same rank estimate as (F2) holds when $C([0,1]) \otimes \mathfrak{A}$ is replaced by a crossed product of $\mathfrak{A}$ by the integers. See [NOP] also for another proof of (F2) and the real rank version of the above estimate (F2), that is, $\mathrm{RR}(C([0,1]) \otimes \mathfrak{A}) \leq \mathrm{RR}(\mathfrak{A}) + 1$ for any C^*-algebra $\mathfrak{A}$ (cf. [BP] for the real rank). As a key idea of the proof of $(F1)$, we use the absolute connected stable rank for C^*-algebras, introduced by Nistor (a stronger version of the connected stable rank of Rieffel [Rf1]) and his result [Ns1, Lemma 2.4]. See [Eh], [Ns2], [Rf2], [Sd1-6] and [ST1-2] for some other related works on the stable rank.

We now review and set up some definitions and notations as follows:

Notation. For a C^*-algebra $\mathfrak{A}$ and a compact space X, denote by $C(X, \mathfrak{A})$ the C^*-algebra of continuous $\mathfrak{A}$-valued functions on X. Set $C(X) = C(X, \mathbb{C})$. It is well known that $C(X, \mathfrak{A})$ is isomorphic to the C^*-tensor product $C(X) \otimes \mathfrak{A}$ (cf. [Mp, Theorem 6.4.17]). For a C^*-algebra $\mathfrak{A}$ (or its unitization $\mathfrak{A}^+$), its (topological) stable rank, connected stable rank and absolute connected stable rank are denoted by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$ and $\mathrm{acsr}(\mathfrak{A})$ respectively ([Rf1],

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[Ns1]). By definition, these ranks take values in $\{1, 2, \cdots, \infty\}$, and $\mathrm{sr}(\mathfrak{A}) \leq n$ if and only if the open subspace $L_n(\mathfrak{A})$ of $\mathfrak{A}^n$ is dense in $\mathfrak{A}^n$, where $(a_j)_{j=1}^n \in L_n(\mathfrak{A})$ if and only if $\sum_{j=1}^n a_j^* a_j$ is invertible in $\mathfrak{A}$, and such elements are called left unimodular rows in algebraic K-theory. And $\mathrm{csr}(\mathfrak{A}) \leq n$ if and only if $L_m(\mathfrak{A})$ is path-connected for any $m \geq n$, and $\mathrm{acsr}(\mathfrak{A}) \leq n$ if and only if for any $m \geq n$, $V \cap L_m(\mathfrak{A})$ is nonempty and path-connected for any nonempty open path-connected subspace V of $L_m(\mathfrak{A})$ (Note that the original definition of $\mathrm{acsr}(\mathfrak{A})$ only assumes that $V \cap L_m(\mathfrak{A})$ is connected, but in fact the path-connected of V and $V \cap L_m(\mathfrak{A})$ is used in the proof of a fundamental property of $\mathrm{acsr}(\mathfrak{A})$, [Ns1, Lemma 2.4] used later. [cf. Remark of Proposition 1]). Refer to [Bl] for some elementary facts about the stable rank.

The main results

We first recall that a topological space X is locally path-connected at a point $p \in X$ if for any open subset U containing p, there exists an open subset V containing p such that any two points of V can be connected by a path in U. We say that a space X is locally path-connected if it is locally path-connected at any point $p \in X$.

Proposition 1. *Let X be a complete metric space and U a dense open, path-connected and locally path-connected subspace of X and V a nonempty open path-connected subspace of X. Then the intersection $U \cap V$ is also nonempty, path-connected and locally path-connected.*

Proof. It is clear that $U \cap V$ is nonempty since U is dense in X and V is nonempty and open.

Let $x, y \in U \cap V$. Since $x, y \in V$ and V is path-connected, there is a continuous path $f_t \in V$ for $t \in [0,1]$ such that $f_0 = x$ and $f_1 = y$. For any $t \in [0,1]$, there exists an open ball O_t of f_t with $O_t \subset V$. Since U is dense in X, there exists $g_t \in O_t \cap U$ for $t \in [0,1]$. When $|s - t|$ is small, $g_s \in O_t \cap U$. Since U is locally path-connected, g_s and g_t can be connected by a path in $O_t \cap U$. By induction and suitable adjustments by taking the open balls O_t small when t is near 0 or 1, and since U is path-connected we can find a continuous path $h_t \in U \cap V$ contained in a small neighborhood of the path space $\{f_t \,|\, t \in [0,1]\}$ such that $h_0 = x$ and $h_1 = y$.

The same argument as above can be used to show that $U \cap V$ is locally path-connected. □

Remark. Note by [Rf1, Corollary 8.5] that $L_n(\mathfrak{A})$ is connected if and only if $GL_n(\mathfrak{A})_0$ acts transitively on $L_n(\mathfrak{A})$. Moreover, this in fact implies that $L_n(\mathfrak{A})$ can be locally path-connected by considering the restriction of orbits under the multiplication action of $GL_n(\mathfrak{A})_0$ on $L_n(\mathfrak{A})$.

Theorem 2. *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\mathrm{sr}(C([0,1]^2) \otimes \mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1.$$

Proof. When $\mathfrak{A}$ is nonunital, we note that $C([0,1]^2) \otimes \mathfrak{A}$ is a closed ideal of $C([0,1]^2) \otimes (\mathfrak{A}^+)$. By [Rf, Theorem 4.4], we have $\mathrm{sr}(C([0,1]^2) \otimes \mathfrak{A}) \leq \mathrm{sr}(C([0,1]^2) \otimes (\mathfrak{A}^+))$. Thus, we may assume that $\mathfrak{A}$ is unital in the following.

Next note that $C([0,1]^2) \otimes \mathfrak{A} \cong C([0,1]) \otimes C([0,1]) \otimes \mathfrak{A}$. By [Ns1, Lemma 2.4],

$$\mathrm{sr}(C([0,1]^2) \otimes \mathfrak{A}) = \mathrm{acsr}(C([0,1]) \otimes \mathfrak{A}).$$

Suppose that $\mathrm{sr}(\mathfrak{A}) \leq n$. Note that $L_{n+1}(C[0,1] \otimes \mathfrak{A}) = C([0,1], L_{n+1}(\mathfrak{A}))$ as a space. Indeed, for any $(f_j)_{j=1}^{n+1} \in L_{n+1}(C[0,1] \otimes \mathfrak{A})$, the element $\sum_{j=1}^{n+1} f_j^* f_j$ is invertible in $C[0,1] \otimes$

$\mathfrak{A}$. Thus, $\sum_{j=1}^{n+1} f_j^* f_j(t)$ is invertible in $\mathfrak{A}$ for any $t \in [0,1]$. Therefore, the function $t \mapsto (f_j(t)) \in \mathfrak{A}^{n+1}$ belongs to $C([0,1], L_{n+1}(\mathfrak{A}))$. Conversely, for any $f \in C([0,1], L_{n+1}(\mathfrak{A}))$ with $f(t) = (f_j(t))_{j=1}^{n+1}$, the element $\sum_{j=1}^{n+1} f_j(t)^* f_j(t)$ is invertible in $\mathfrak{A}$ for any t. Thus the function $t \mapsto \sum_{j=1}^{n+1} f_j(t)^* f_j(t)$ is invertible in $C([0,1], \mathfrak{A})$. Hence, $(f_j)_{j=1}^{n+1} \in L_{n+1}(C[0,1] \otimes \mathfrak{A})$.

Let V be a nonempty open path-connected subset of $(C([0,1]) \otimes \mathfrak{A})^{n+1}$. Then we show that $W = V \cap C([0,1], L_{n+1}(\mathfrak{A}))$ is nonempty and path-connected. Since $\mathrm{sr}(\mathfrak{A}) \leq n$, then $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$ so that $L_{n+1}(\mathfrak{A})$ is dense in $\mathfrak{A}^{n+1}$. Note that $L_{n+1}(\mathfrak{A})$ is open in $\mathfrak{A}^{n+1}$ ([Rf1, Proposition 8.2]). Thus W is nonempty. Now denote by $\mathrm{ev}_t(V)$ ($t \in [0,1]$) the set of evaluations $(f_1(t), \cdots, f_{n+1}(t)) \in \mathfrak{A}^{n+1}$ for every $f \in V$. Since $\mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1 \leq n+1$ by [Rf1, Corollary 4.10], $L_{n+1}(\mathfrak{A})$ is path-connected and locally path-connected from the above remark. By Proposition 1, $\mathrm{ev}_t(V) \cap L_{n+1}(\mathfrak{A})$ is path-connected and locally path-connected for any $t \in [0,1]$ since $\mathrm{ev}_t(V)$ is open and path-connected. Therefore, any two elements $f, g \in W$ can be connected by a continuous path in W since $f(t)$ and $g(t)$ can be connected and locally connected in $\mathrm{ev}_t(W)$ for any $t \in [0,1]$. Indeed, note that $C([0,1], \mathfrak{A}^{n+1}) \cong C([0,1]) \otimes \mathfrak{A}^{n+1}$. Thus, f and g can be approximated by finite sums $\sum_{j=1}^n h_j \otimes a_j$ and $\sum_{j=1}^n k_j \otimes b_j$ for $a_j, b_j \in \mathfrak{A}^{n+1}$ and $h_j, k_j \in C([0,1])$ respectively (cf. [Mp, Theorem 6.4.17]). Now suppose that f and g are not connected by any path in W. Then there exists an open neighborhood Z in $[0,1]$ such that the restrictions of f, g to Z are not connected by any path in the restriction $W|_Z$ of W. We may assume that Z is a finite sum of small open intervals if necessary by considering reparameterization of the domains of f and g, or by induction (One can use the density of $L_{n+1}(\mathfrak{A})$ for replacing such a_j, b_j with elements of $L_{n+1}(\mathfrak{A})$ even in the inductive process). Moreover, we may assume that the supports of the functions $\{h_j\}_{j=1}^n$, $\{k_j\}_{j=1}^n$ are closed intervals $[s_{j-1}, s_j]$ with $s_0 = 0$ and $s_n = 1$ which are disjoint when $|j - j'| \geq 2$ for $j, j' \in \{1, \cdots, n\}$ (this is possible from that the covering dimension of $[0,1]$ is one (cf. [Ng] or [Pr])), and $\{h_j\}_{j=1}^n$, $\{k_j\}_{j=1}^n$ are constant on closed intervals $[s'_{j-1}, s'_j]$ contained in $[s_{j-1}, s_j]$ and not contained in other supports respectively. Furthermore, we may assume that Z is contained in a finite sum of some intervals $[s'_{j-1}, s'_j]$ for some j, which deduces the contradiction. Therefore, we obtain $\mathrm{acsr}(C([0,1]) \otimes \mathfrak{A}) \leq n+1$, as desired. □

Remark 2.1. It is clear that a unital C^*-algebra in the statement can be replaced by a unital complex Banach algebra. Note that the result of Nistor [Ns1, Lemma 2.4] holds even for unital complex Banach algebras. Also note in general that $\sum_{j=1}^n f_j^* f_j$ is invertible if and only if $\sum_{j=1}^n g_j f_j$ is invertible for some g_j $(1 \leq j \leq n)$.

Remark 2.2. Our theorem 2 is stronger than the usual product formula of the stable rank in the case of $C([0,1]^2) \otimes \mathfrak{A}$ for $\mathfrak{A}$ a C^*-algebra:

$$\mathrm{sr}(C([0,1]^2) \otimes \mathfrak{A}) \leq \mathrm{sr}(C([0,1]^2)) + \mathrm{sr}(\mathfrak{A}) = 2 + \mathrm{sr}(\mathfrak{A}),$$

which is obtained by using (F2) twice, or by Theorem 2. On the other hand, it is obtained by [NOP, Proposition 5.3] that $\mathrm{sr}(C([0,1]^2) \otimes \mathfrak{A}) \geq 2$ for any unital C^*-algebra $\mathfrak{A}$. Also note $\mathrm{sr}(C([0,1]^2) \otimes \mathfrak{A}) \geq \mathrm{sr}(C([0,1]) \otimes \mathfrak{A}) \geq \mathrm{sr}(\mathfrak{A})$ by [Rf1, Theorem 4.3] since $\mathfrak{A}$ is a quotient C^*-algebra of $C([0,1]) \otimes \mathfrak{A}$ which is also a quotient of $C([0,1]^2) \otimes \mathfrak{A}$.

Corollary 3. *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\mathrm{sr}(C([0,1]^n) \otimes \mathfrak{A}) \leq \begin{cases} \mathrm{sr}(\mathfrak{A}) + m & \text{if } n = 2m \text{ even}, \\ \mathrm{sr}(\mathfrak{A}) + m + 1 & \text{if } n = 2m+1 \text{ odd}. \end{cases}$$

Therefore, we obtain

$$\mathrm{sr}(C([0,1]^n) \otimes \mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + \{n/2\},$$

where $\{x\}$ *means the least integer greater than or equal to* x*.*

Proof. When $\mathfrak{A}$ is nonunital, we note that $C([0,1]^n)\otimes\mathfrak{A}$ is a closed ideal of $C([0,1]^n)\otimes(\mathfrak{A}^+)$. Thus we may assume that $\mathfrak{A}$ is unital as in the proof of Theorem 2. Note that $C([0,1]^n)\otimes\mathfrak{A}$ is isomorphic to $(\otimes^m C([0,1]^2))\otimes\mathfrak{A}$ when $n = 2m$, and to $(\otimes^m C([0,1]^2))\otimes C([0,1])\otimes\mathfrak{A}$ when $n = 2m+1$. Thus, use Theorem 2 m-times repeatedly when $n = 2m$. When $n = 2m+1$, use Theorem 2 m-times and (F2) once. $\square$

Remark. Note by [Rf1, Proposition 1.7] that $\mathrm{sr}(C([0,1]^n)) = m+1$ when $n = 2m$ or $n = 2m+1$.

More generally, it is obtained that

Theorem 4. *Let* X *be a locally compact Hausdorff space with* $n = \dim \beta X$ *for* βX *the Stone-Čech compactification of* X*. Then it follows that for any* C^**-algebra* $\mathfrak{A}$*,*

$$\mathrm{sr}(C_0(X)\otimes\mathfrak{A}) \leq \mathrm{sr}(C([0,1]^n)\otimes(\mathfrak{A}^+)) \leq \mathrm{sr}(\mathfrak{A}) + \{n/2\}.$$

Proof. Use [NOP, Theorem 1.13] for the left inequality, and use Corollary 3 for the right inequality. $\square$

Remark. This is the formula conjectured at [NOP, p.980] (precisely, when X is compact). When X is a finite CW-complex of dimension n and $\mathfrak{A}$ is unital, the first left inequality is in fact the equality by [NOP, Proposition 1.7]. When X is a normal (or σ-compact) locally compact Hausdorff space, we have $\dim X = \dim \beta X$ (cf. [Pr, Proposition 6.4.3 and Corollary 10.1.7]).

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