MULTI-ROUND CARD GAME WITH ARBITRATION AND RANDOM AMOUNT OF BET

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Abstract. We consider zero-sum game which is usually called poker. Each of two players (I and II) independently draws a number (x and y, respectively) according to uniform distribution in $[0,1]$. Also they jointly draw a number z according to the distribution with cdf $F(z), 0 < z < \infty$. After observing his number and the "center card" z each player chooses either to accept (A) or to reject (R) his number. If both players choose A, showdown is made and I wins (loses) if $x > (<)\ y$, getting reward z ($-z$). If both players choose R, then the numbers are rejected and the game proceeds to the next round with the newly drawn numbers x', y', and z'. If players choose different choices, then arbitration comes in, and forces to take the same choices as I's (II's) with probability p ($\bar{p}$). Arbitration is fair (unfair) if $p = (\neq)\ \frac{1}{2}$. The game is played in n rounds . Player I (II) aims to maximize (minimize) the expected reward to I. Explicit solution is derived to this sequential game, and examples are shown for the cases where z obeys discrete and continuous uniform distributions.

1 Introduction. Let $(x_i), (y_i), (z_i), i = 1, 2, \cdots, n$, be *i.i.d.* with $U_{[0,1]}, U_{[0,1]}$ and *c.d.f.* $F(x)$ on $(0, \infty)$, respectively, and assume that these three sequences are independent. Player I [II] privately observes $(x_i)[(y_i)]$ and the "center card" z_i is jointly observed by both players. All sequences x_i, y_i, z_i are observed sequentially one-by-one. In the i-th round of the game, after observing x_i and z_i (y_i and z_i), player I (II) chooses one of the two choices Accept or Reject. If choice-pair is R-R, then the game proceeds to the next round and both players are dealt new hands x_{i+1} and y_{i+1} and jointly observe a new center card z_{i+1}. If the choice-pair is A-A, then showdown occurs and the game ends with I's reward z_i sgn $(x_i - y_i)$. [*c.f.* sgn $t = 1\ (0, 1)$, if $t > (=, <)\ 0$]. If players choose different choices, then arbitrater comes in and forces them to take the same choices as I's (II's) with probability p ($\bar{p}$). Arbitration is fair (unfair), if $p = (\neq)\ \frac{1}{2}$. This zero-sum game is played in n rounds, and player I (II) aims to maximize (minimize) the expected reward to I.

In the area of various card games, poker is the subject of numerous mathematical studies. A detailed description and discussion on mathematical model of two-person poker is given in Ref.[1,3,5].

In the game-theoretic viewpoint, poker is one of the simplest examples of games with incomplete information, in the sense that both players have to select his (or her) choice by his own private information *i.e.,* hand dealt by the dealer. A distinguished feature of the model analysed in the present paper is : (1) Each player has his own "weight", which is reflected in the arbitrators decision, when players want to choose different choices. (2) Players jointly observe "center card", that is, they play for a random amount of bet. So, players have information which has two parts – private and common. In papers [4,7] the idea along (1) is further developed. In the idea (2) the full-information secretary problem is closely related with. Players must consider the possibility of facing the larger amount of

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bet appearing in the future round. See Ref. [2; Section 5].

2 Problem and its Solution. Let $\phi_n(x \mid z)(\psi_n(y \mid z))$ be the probability that player I (II) chooses A when his hand is x (y) and the center card is $z > 0$. Let $v_n(z)$ be the value for I of the n round game when the center card is z, and also let $V_n = E_z v_n(z)$. Then we have the Optimality Equation

$$(1)\quad v_n(z) = \underset{(\phi_n(\cdot|z),\psi_n(\cdot|z))}{\text{val}} E_{(x,y)}[(\bar{\phi}_n(x \mid z), \phi_n(x \mid z))M_n(x, y \mid z)(\bar{\psi}_n(y \mid z), \psi_n(y \mid z))^T]$$

where

$$(2)\qquad M_n(x,y \mid z) = \begin{array}{c} \\ \mathrm{R} \\ \mathrm{A} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{R} & \mathrm{A} \end{array} \\ \begin{bmatrix} V_{n-1} & \bar{p}z \ \mathrm{sgn}(x-y) + pV_{n-1} \\ pz \ \mathrm{sgn}(x-y) + \bar{p}V_{n-1} & z \ \mathrm{sgn}(x-y) \end{bmatrix} \end{array}$$

$$(n \geq 1,\ V_0 = 0)$$

and we want to solve this equation.

Since

$$M_n(x,y \mid z) = V_{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (z \ \mathrm{sgn}(x-y) - V_{n-1})\begin{bmatrix} 0 & \bar{p} \\ p & 1 \end{bmatrix},$$

it is evident that

$$(3)\qquad \begin{aligned} A_n(\phi, \psi \mid z) &\equiv E_{(x,y)}[(\bar{\phi}, \phi)M_n(x, y \mid z)(\bar{\psi}, \psi)^T] \\ &= V_{n-1} + E_{(x,y)}[(z \ \mathrm{sgn}(x-y) - V_{n-1})(p\phi + \bar{p}\psi)] \end{aligned}$$

and $v_n(z) = \mathrm{val}_{(\phi,\psi)} A_n(\phi, \psi \mid z)$, where $\phi_n(x \mid z)$ and $\psi_n(y \mid z)$ are abreviated by ϕ and ψ. We repeatedly use this simplification throughout this paper.

Remark 1. If $p = \frac{1}{2}$ (*i.e.,* the game is symmetric), then $V_n \equiv 0$, and the common optimal strategy is to choose A (R) independently of z, if his hand is $>$ $(<)$ $\frac{1}{2}$.

Theorem. *Let* $\frac{1}{2} < p \leq 1$. *The optimal strategy-pair is*

$$(4)\qquad \phi_n^*(x \mid z) = I\ (z > V_{n-1} \ \text{ and } \ \frac{1}{2}(1 + z^{-1}V_{n-1}) \leq x \leq 1).$$

$$(5)\qquad \psi_n^*(y \mid z) = I\ (z < V_{n-1} \ \text{ or } \ \frac{1}{2}(1 - z^{-1}V_{n-1}) \leq y \leq 1).$$

and the value V_n *of the game is determined by the recursion*

$$(6)\quad V_n = \left\{pF(V_{n-1}) + \frac{1}{2}\bar{F}(V_{n-1})\right\}V_{n-1} + \left(\frac{p-\bar{p}}{4}\right)\int_{V_{n-1}}^{\infty}(z + z^{-1}V_{n-1}^2)dF(z),$$

$$(n \geq 1,\ V_0 = 0)$$

Moreover

$$(7)\qquad 0 < V_{n-1} \leq V_n.$$

Proof. We have from (3)

$$
\begin{aligned}
(8)\qquad A_n(\phi,\psi \mid z) - V_{n-1} &= pE_x[\{z(2x-1) - V_{n-1}\}\phi(x \mid z)] + \text{terms indep. of } \phi \\
&= \bar{p}E_y[\{z(1-2y) - V_{n-1}\}\psi(y \mid z)] + \text{terms indep. of } \psi.
\end{aligned}
$$

Therefore the optimal strategy-pair has the form of

$$\phi_n^*(x \mid z) = I\ ((2x-1)z - V_{n-1} > 0) \quad \text{and} \quad \psi_n^*(y \mid z) = I\ ((2y-1)z + V_{n-1} > 0),$$

and these are rewritten as (4)-(5). See Figure 1, where $a_n(z) \equiv \frac{1}{2}(1 + z^{-1}V_{n-1})$.

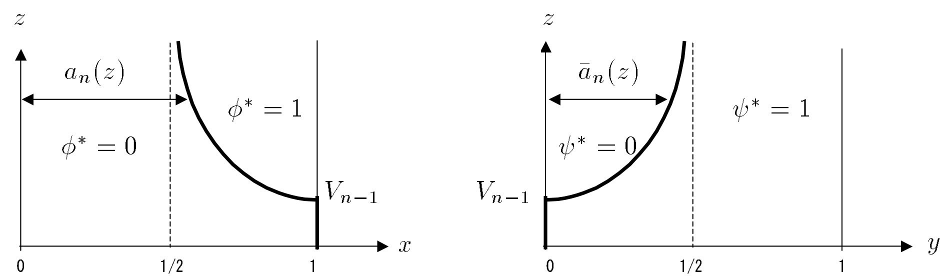

Figure 1. Optimal strategy-pair

Now, (3) becomes

$$A_n(\phi^*,\psi^* \mid z) = V_{n-1} + \bar{p}E_{(x,y)}[z\ \mathrm{sgn}(x-y) - V_{n-1}] = pV_{n-1}, \quad \text{for} \ \ 0 < z < V_{n-1}.$$

For $V_{n-1} < z$, it becomes

$$
\begin{aligned}
(9)\qquad & A_n(\phi^*,\psi^* \mid z) - V_{n-1} \\
&= E_{(x,y)}[(z\ \mathrm{sgn}(x-y) - V_{n-1})(p\phi^* + \bar{p}\psi^*)] \\
&= \left[\int_{a_n}^1 dx \int_{\bar{a}_n}^1 + p\int_{a_n}^1 dx \int_0^{\bar{a}_n} + \bar{p}\int_0^{a_n} dx \int_{\bar{a}_n}^1\right](z\ \mathrm{sgn}(x-y) - V_{n-1})dy \\
&= \left[\quad \cdots\cdots \quad\right] z\ \mathrm{sgn}(x-y)dy - (p\bar{a}_n + \bar{p}a_n)V_{n-1},
\end{aligned}
$$

where $a_n(z)$ is abreviated by a_n. The first term in the above expression is equal to z times

$$
\begin{aligned}
(10)\qquad \left[\quad \cdots\cdots \quad\right]\mathrm{sgn}(x-y)dy &= \bar{a}_n(a_n - \bar{a}_n) + p\bar{a}_n^2 - \bar{p}\{a_n^2 - (a_n - \bar{a}_n)^2\} \\
&= (p - \bar{p})a_n\bar{a}_n.
\end{aligned}
$$

Also, since $a_n(z) = \frac{1}{2}(1 + z^{-1}V_{n-1})$, we have

$$(11)\qquad pa_n + \bar{p}\bar{a}_n = \frac{1}{2}\left\{1 + (p-\bar{p})z^{-1}V_{n-1}\right\} \quad \text{and} \quad a_n\bar{a}_n = \frac{1}{4}(1 - z^{-2}V_{n-1}^2).$$

Substituting (10) and (11) into (9) we finally obtain, for $z > V_{n-1}$,

$$A_n(\phi^*,\psi^* \mid z) = \frac{1}{4}[(p-\bar{p})z + 2V_{n-1} + (p-\bar{p})z^{-1}V_{n-1}^2].$$

It then follows that

$$\begin{aligned} V_n &= EA_n(\phi^*, \psi^* \mid z) \\ &= pV_{n-1}F(V_{n-1}) + \frac{1}{4}\int_{V_{n-1}}^{\infty}[(p-\bar{p})z + 2V_{n-1} + (p-\bar{p})z^{-1}V_{n-1}^2]dF(z), \end{aligned}$$

which becomes (6).

Next we have to prove (7). Let the l.h.s. of (6) be rewritten by

$$D(v) \equiv pv + \frac{1}{4}(p-\bar{p})\int_v^{\infty}(\sqrt{z} - v/\sqrt{z})^2 dF(z),$$

with $v = V_{n-1}$. Then

$$D'(v) = pF(v) + \frac{1}{2}\bar{F}(v) + \frac{1}{2}(p-\bar{p})\int_v^{\infty}(v/z)dF(z) > 0,$$

that is, $D(v)$ is increasing in $v > 0$.

Therefore if we assume that $V_n \geq V_{n-1}$, then from (6)

$$V_{n+1} = D(V_n) \geq D(V_{n-1}) = V_n$$

and $V_1 = \left(\frac{p-\bar{p}}{4}\right)E(z) > 0 = V_0$, We conclude that (7) is vallid. This completes the proof of the theorem. □

3 Corollaries. We derive some immediate corollaries.

Colollary 1. *If $z = B(> 0)$, with probability 1, then $B^{-1}V_n(= \alpha_n$, say) satisfies the recursion*

$$(12)\quad \alpha_n = \begin{cases} \frac{1}{2}\alpha_{n-1} + \frac{1}{4}(p-\bar{p})(1+\alpha_{n-1}^2), & \text{if } \alpha_{n-1} < 1. \\ p\alpha_{n-1}, & \text{if } \alpha_{n-1} > 1, \quad (n \geq 1,\ \alpha_0 = 0) \end{cases}$$

and, as $n \to \infty$,

$$(13)\qquad \alpha_n \uparrow \alpha = \frac{1 - 2\sqrt{p\bar{p}}}{p-\bar{p}} = \frac{\sqrt{p}-\sqrt{\bar{p}}}{\sqrt{p}+\sqrt{\bar{p}}}.$$

The optimal strategy-pair is:

Choose A (R), *if* $x > (<)\ \frac{1}{2}(1+\alpha_{n-1})$, *for* I,

Choose A (R), *if* $y > (<)\ \frac{1}{2}(1-\alpha_{n-1})$, *for* II.

Proof. Equation (6) gives

$$V_n = \begin{cases} \frac{1}{2}V_{n-1} + \frac{1}{4}(p-\bar{p})(B - B^{-1}V_{n-1}^2), & \text{if } V_{n-1} < B \\ pV_{n-1}, & \text{if } V_{n-1} > B \end{cases}$$

from which we find (12). The limit α of α_n is a unique root of the equation

$$(p-\bar{p})(1+\alpha^2)=2\alpha,$$

i.e., (13). $\square$

The function α in (13) is convex and increasing with values 0 at $p=1/2$ and 1 at $p=1$. Corollary 1 is identical to Theorem 1 in the author's previous paper [7] which contains a table of α_n and α for $p=0.6, 0.8$ and 1.0.

Corollary 2. *If $z=B_j (j=1,2,\cdots,k; 0<B_1<\cdots<B_k)$ with probability k^{-1} each, then $\{V_n\}$ satisfies the recursion*

$$(14)\qquad V_n=\begin{cases}\left\{\left(p-\dfrac{1}{2}\right)(j-1)/k+\dfrac{1}{2}\right\}V_{n-1}+\left(\dfrac{p-\bar{p}}{4}\right)\displaystyle\sum_{m=j}^{k}(B_m+B_m^{-1}V_{n-1}^2), & \\ \qquad\qquad \text{if } B_{j-1}<V_{n-1}<B_j,\ j=1,\cdots,k\ (B_0\equiv 0) & \\ pV_{n-1}, \qquad \text{if } V_{n-1}>B_k & \end{cases}$$

So, for example, when $k=2, B_1=1$ and $B_2=2$, we have

$$(15)\qquad V_n=\begin{cases}\dfrac{1}{2}V_{n-1}+\dfrac{3}{8}(p-\bar{p})\left(1+\dfrac{1}{2}V_{n-1}^2\right), & \text{if } \ 0<V_{n-1}<1,\\ \left(\dfrac{1}{4}(p-\bar{p})+\dfrac{1}{2}\right)V_{n-1}+\dfrac{1}{8}(p-\bar{p})\left(2+\dfrac{1}{2}V_{n-1}^2\right), & \text{if } \ 1\le V_{n-1}\le 2,\\ pV_{n-1}, & \text{if } \ V_{n-1}>2.\end{cases}$$

The optimal strategy-pair is:

Choose A (R), *if* $x>(<)\ a_n(z)$, *for* I,

Choose A (R), *if* $y>(<)\ \bar{a}_n(z)$, *for* II,

where $a_n(z)=\frac{1}{2}(1+z^{-1}V_{n-1})\wedge 1,\ z=1,2.$

Proof is immediate from the Theorem.

The values of V_n and $a_n(z)$ are given for some n and p in Table 1. If $n=10$, $p=0.6$, and the first r.v. happens to be $z=\left\{\begin{array}{c}1\\2\end{array}\right\}$, for example, then

I chooses A, if and only if $x>\left\{\begin{array}{c}0.5757\\0.5379\end{array}\right\}$

and II chooses A, if and only if $y>\left\{\begin{array}{c}0.4243\\0.4621\end{array}\right\}$.

The value of the game is $V_{10}=0.1516$.

Table 1. Solution for the simplest discrete uniform distribution.

	$p=0.6$			$P=1.0$		
	V_n	$a_n(1)$	$a_n(2)$	V_n	$a_n(1)$	$a_n(2)$
n=1	0.0750	0.5000	0.5000	0.3750	0.5000	0.5000
2	0.1127	0.5375	0.5188	0.5889	0.6875	0.5938
3	0.1318	0.5564	0.5282	0.7345	0.7945	0.6472
4	0.1416	0.5659	0.5330	0.8435	0.8673	0.6836
5	0.1466	0.5708	0.5354	0.9302	0.9218	0.7109
6	0.1491	0.5733	0.5637	1.0023	0.9651	0.7326
7	0.1504	0.5746	0.5373	1.0645	1.0000	0.7506
8	0.1510	0.5752	0.5376	1.1192	1.0000	0.7661
9	0.1514	0.5755	0.5378	1.1677	1.0000	0.7798
10	0.1516	0.5757	0.5379	1.2110	1.0000	0.7919
Limit	0.1517(∗)			2.0000		

(∗) is the root of the equation $3\alpha^2 - 40\alpha + 6 = 0$.

Corollary 3. *If z is distributed as $U_{[0,1]}$, then $\{V_n\}$ satisfies the recursion*

$$V_n = \frac{1}{2}V_{n-1} + \frac{1}{4}(p-\bar{p})\left[\frac{1}{2} + \left(\frac{3}{2} - \log V_{n-1}\right)V_{n-1}^2\right], \quad (n \geq 1,\ V_0 = 0) \tag{16}$$

As $n \to \infty$, $V_n \uparrow \alpha$, where $\alpha \in (0,1)$ is a unique root of the equation

$$-2\alpha \log \alpha = \frac{4}{p-\bar{p}} - 3\alpha - \alpha^{-1}. \tag{17}$$

The optimal strategy-pair is given by (4)-(5).

Proof. Substituting $F(z) = z$ into (6), we get (16). $\{V_n\}$ is bounded by 1 from above and is increasing by (7). Therefore it has the limit α which satisfies the equation

$$\alpha = \left(\frac{p-\bar{p}}{2}\right)\left\{\frac{1}{2} + \left(\frac{3}{2} - \log \alpha\right)\alpha^2\right\},$$

i.e., (17). It is easy to find that (17) has a unique root in $(0,1)$, since the r.h.s. is a concave function with values $-\infty$ at $\alpha = 0+0$, and $8\bar{p}/(p-\bar{p})$ at $\alpha = 1-0$ and maximum value $\frac{4}{p-\bar{p}} - 2\sqrt{3}$ (> 4) at $\alpha = \frac{1}{\sqrt{3}}$. □

Table 2. Solution for the case $z \sim U_{[0,1]}$.

	$p=0.6$	$p=0.8$	$p=1.0$
n=1	V_n=0.025(=1/40)	V_n=0.075(=3/40)	V_n=0.125(=1/8)
2	0.0377	0.1160	0.2015
3	0.0442	0.1404	0.2573
4	0.0476	0.1554	0.3009
5	0.0493	0.1649	0.3366
6	0.0502	0.1709	0.3666
7	0.0507	0.1748	0.3924
8	0.0509	0.1773	0.4150
9	0.0510	0.1789	0.4349
10	0.0511	0.1799	0.4528
Limit	0.051	0.182	1

The values of V_n and $\alpha = \lim V_n$ are given for various n and p in Table 2. For example, if $n = 10, p = 0.6$, and the first r.v. happens to be z, then the optimal strategy- pair is given by (4)-(5), and the value of the game is $V_{10} = 0.0511$.

Remark 2. It is instructive to compare the game values for the cases in Corollaries 1 $\sim$ 3, See Table 3.

Table 3. Game values for various distributions of z.

	$p = 0.6$	$p = 0.8$	$p = 1.0$
	0.0250	0.0750	0.1250
n=1	0.0500	0.1500	0.2500
	0.0750		0.3750
	0.0377	0.1160	0.2015
2	0.0752	0.2284	0.3906
	0.1127		0.5889
	0.0493	0.1649	0.3366
5	0.0975	0.3119	0.6008
	0.1466		0.9302
	0.0511	0.1799	0.4528
10	0.1009	0.3317	0.7415
	0.1516		1.2110
	0.051	0.182	1
Limit	0.101	1/3	1
	0.1517	0.5081	2

Top:$z \sim U_{[0,1]}$. Middle:$z = 1$, w.p.1 (Reprodused from Table 1 in [7])
Bottom: $z = 1, 2$, w.p.1/2 each.

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