ON CLASSES OF OPERATORS GENERALIZING CLASS A AND PARANORMALITY

MASATOSHI ITO

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ABSTRACT. Recently, we introduced class A defined by an operator inequality, and also the definition of class A is similar to that of paranormality defined by a norm inequality. As generalizations of class A and paranormality, Fujii-Nakamoto introduced class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality respectively. These classes are related to p-hyponormality and log-hyponormality. The author showed more precise inclusion relations among the families of class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality than the results by Fujii-Nakamoto, and he also showed the results on powers of class $\mathrm{F}(p,r,q)$ operators. But some of the results on class $\mathrm{F}(p,r,q)$ require the assumption of invertibility.

In this paper, we shall remove the assumption of invertibility from the results on invertible class $\mathrm{F}(p,r,q)$ operators. Moreover we shall show that the families of class $\mathrm{F}(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality are proper on p.

1. INTRODUCTION

In this paper, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx,x) \geq 0$ for all $x \in H$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

As extensions of hyponormal operators, i.e., $T^*T \geq TT^*$, it is well known that p-hyponormal operators for $p > 0$ are defined by $(T^*T)^p \geq (TT^*)^p$ and invertible log-hyponormal operators are defined by $\log T^*T \geq \log TT^*$ for an invertible operator T, and also an operator T is said to be p-quasihyponormal for $p > 0$ if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$. We remark that we treat only invertible log-hyponormal operators in this paper (see also [19]). It is easily obtained that every p-hyponormal operator is q-hyponormal for $p > q > 0$ by Löwner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$,"* and every invertible p-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We remark that log-hyponormality is sometimes regarded as 0-hyponormality since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for $X > 0$. An operator T is paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. Ando [2] showed that every p-hyponormal operator for $p > 0$ and (invertible) log-hyponormal operator is paranormal.

Recently, in [10], we introduced class A defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every invertible log-hyponormal operator belongs to class A and every class A operator is paranormal. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms. And also Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [6] introduced class $\mathrm{A}(p,r)$

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and Yamazaki-Yanagida [21] introduced absolute-(p,r)-paranormality as follows: An operator T belongs to class $\mathrm{A}(p,r)$ for $p>0$ and $r>0$ if $(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$, and also an operator T is absolute-(p,r)-paranormal if $\left\| |T|^p|T^*|^r x \right\|^r \geq \left\| |T^*|^r x \right\|^{p+r}$ for every unit vector $x \in H$. We remark that class $\mathrm{A}(1,1)$ equals class A and also absolute-$(1,1)$-paranormality equals paranormality. These classes are generalizations of class $\mathrm{A}(k)$ and absolute-k-paranormality introduced as two families of classes based on class A and paranormality in [10], and also absolute-(p,r)-paranormality is a generalization of p-paranormality in [5]. We should remark that the families of class $\mathrm{A}(p,r)$ determined by operator inequalities and absolute-(p,r)-paranormality determined by norm inequalities constitute two increasing lines on $p>0$ and $r>0$ whose origin is (invertible) log-hyponormality.

Moreover, as a continuation of the discussion in [6], Fujii-Nakamoto [7] introduced the following classes of operators.

Definition ([7]). *For each $p>0$, $r \geq 0$ and $q>0$,*

(i) *An operator T belongs to class $F(p,r,q)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \tag{1.1}$$

(ii) *An operator T is (p,r,q)-paranormal if*

$$\| |T|^pU|T|^r x\|^{\frac{1}{q}} \geq \| |T|^{\frac{p+r}{q}} x\| \tag{1.2}$$

for every unit vector $x \in H$, where $T=U|T|$ is the polar decomposition of T. In particular, if $r>0$ and $q \geq 1$, then (1.2) is equivalent to

$$\left\| |T|^p|T^*|^r x \right\|^{\frac{1}{q}} \geq \left\| |T^*|^{\frac{p+r}{q}} x \right\| \tag{1.3}$$

for every unit vector $x \in H$ ([13]).

We remark that class $\mathrm{F}(p,r,\frac{p+r}{r})$ equals class $\mathrm{A}(p,r)$ and also $(p,r,\frac{p+r}{r})$-paranormality equals absolute-(p,r)-paranormality. In [13], we obtained the parallel result to that of class $\mathrm{A}(p,r)$ and absolute-(p,r)-paranormality that invertible class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality constitute two increasing lines on $p \geq \delta > 0$ and $r \geq r_0 > 0$ whose origin is δ-quasihyponormality. And also we showed the result on powers of invertible class $\mathrm{F}(p,r,q)$ operators. Thus many reseachers have been discussed parallel families of classes of operators which are generalizations of class A and paranormality.

In this paper, we shall remove the assumption of invertibility from the results on invertible class $\mathrm{F}(p,r,q)$ operators in [13]. Moreover we shall show that the families of class $\mathrm{F}(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality are proper on p.

2. Preliminaries

Fujii-Nakamoto [7] observed that class $\mathrm{F}(p,r,q)$ derives from the following Theorem 2.A shown in [8] and (p,r,q)-paranormality corresponds to class $\mathrm{F}(p,r,q)$.

We remark that alternative proofs of Theorem 2.A were given in [4] and [16] and also an elementary one page proof in [9]. Tanahashi [17] showed that the domain drawn for p,q and r in the Figure 1 is the best possible one for Theorem 2.A.

Theorem 2.A (Furuta inequality [8]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

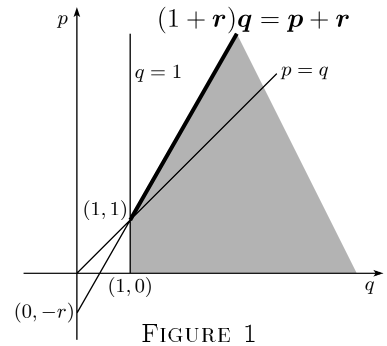

Figure 1

Fujii-Nakamoto [7] and the author [13] obtained the results on inclusion relations among the families of class $\mathrm{F}(p, r, q)$ and (p, r, q)-paranormality.

Theorem 2.B ([7]).

(i) *For a fixed $k > 0$, T is k-hyponormal if and only if T belongs to class $F(2kp, 2kr, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., T belongs to class $F(p, r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(k+r)q \geq p+r$.*

(ii) *If T belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then T belongs to class $F(p_0, r_0, q)$ for any $q \geq q_0$.*

(iii) *If T is (p_0, r_0, q_0)-paranormal for $p_0 > 0$, $r_0 \geq 0$ and $q_0 > 0$, then T is (p_0, r_0, q)-paranormal for any $q \geq q_0$.*

(iv) *If T belongs to class $F(p, r, q)$ for $p > 0$, $r \geq 0$ and $q \geq 1$, then T is (p, r, q)-paranormal.*

Theorem 2.C ([13]).

(i) *For each $p > 0$ and $r > 0$,*

  (i-1) *T is p-quasihyponormal if and only if T belongs to class $F(p, r, 1)$ if and only if T is $(p, r, 1)$-paranormal.*

  (i-2) *T is p-quasihyponormal if and only if T is $(p, 0, 1)$-paranormal.*

(ii) *Let T be a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$.*

  (ii-1) *If T is invertible and $0 \leq \delta \leq p_0$, then T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.*

  (ii-2) *If $-r_0 < \delta \leq p_0$, then T belongs to class $F(p_0, r, \frac{p_0+r}{\delta+r})$ for any $r \geq r_0$.*

(iii) *Let T be a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$.*

  (iii-1) *If $0 \leq \delta \leq p_0$, then T is $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p \geq p_0$ and $r \geq r_0$.*

  (iii-2) *If $-r_0 < \delta \leq p_0$, then T is $(p_0, r, \frac{p_0+r}{\delta+r})$-paranormal for any $r \geq r_0$.*

  (iii-3) *If $0 \leq \delta$, then T is $(p, r_0, \frac{p+r_0}{\delta+r_0})$-paranormal for any $p \geq p_0$.*

We remark that only (ii-1) of Theorem 2.C requres invertibility of T, and also we obtained in [14] that every class $\mathrm{A}(p_0, r_0)$ operator for $p_0 > 0$ and $r_0 > 0$ belongs to class $\mathrm{A}(p, r)$ for any $p \geq p_0$ and $r \geq r_0$ (without assumption of invertibility).

The following Figure 2 represents the inclusion relations among the families of class $\mathrm{F}(p, r, q)$ and (p, r, q)-paranormality.

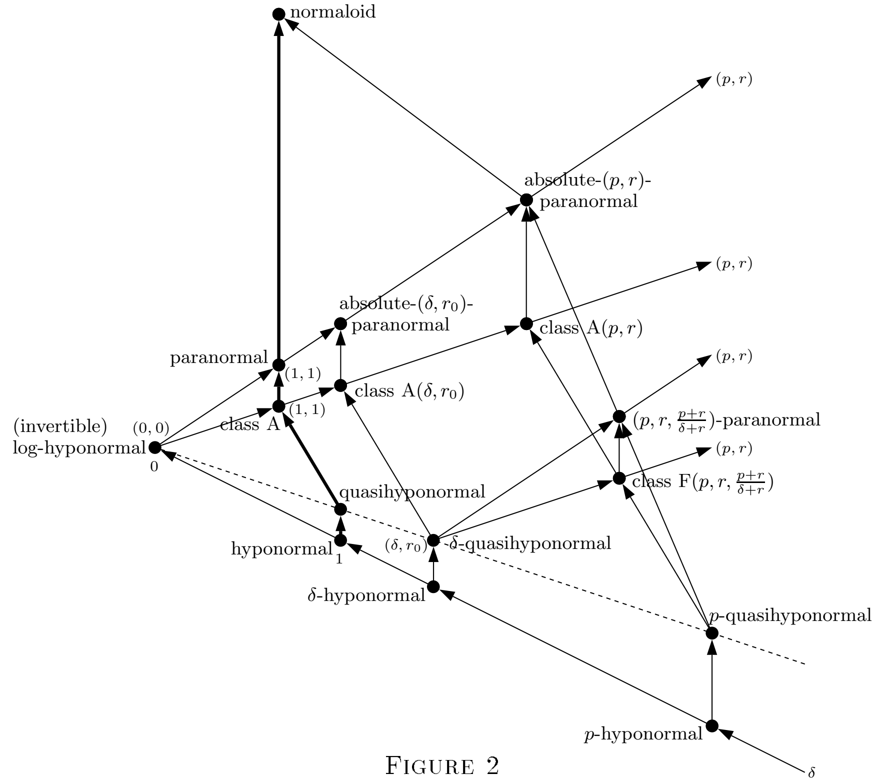

FIGURE 2

On the other hand, we obtained the results on powers of p-hyponormal, class A(p, r) and invertible class F(p, r, q) operators.

Theorem 2.D.

(i) *Let T be a p-hyponormal operator for $0 < p \leq 1$. Then T^n is $\frac{p}{n}$-hyponormal for all positive integer n* ([1]).

(ii) *Let T be a class $A(p, r)$ operator for $0 < p \leq 1$ and $0 < r \leq 1$. Then T^n belongs to class $A(\frac{p}{n}, \frac{r}{n})$ for all positive integer n* ([14]).

(iii) *Let T be an invertible class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then T^n belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n* ([13]).

We remark that (iii) interpolates (i) and (ii) if T is invertible in Theorem 2.D. In fact, (iii) yields (i) by putting $q = 1$ and $r = 0$, and also (iii) yields (ii) by putting $q = \frac{p+r}{r}$.

Moreover we have another result on powers of class A operators by combining [22, Theorem 1] and [14, Theorem 3].

Theorem 1. *If T is a class A operator, then*

$$|T|^2 \leq |T^2| \leq \cdots \leq |T^n|^{\frac{2}{n}} \quad \text{and} \quad |T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n^*}|^{\frac{2}{n}}$$

hold for all positive integer n.

We remark that (ii) of Theorem 2.D and Theorem 1 in case of invertible operators were shown in [20] and [12], respectively.

3. Main results

In this section, we shall show the results which remove the assumption of invertibility from (ii-1) of Theorem 2.C and (iii) of Theorem 2.D.

Theorem 2. *Let T be a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $0 \leq \delta \leq p_0$. Then T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.*

Theorem 3. *Let T be a class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then T^n belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n.*

We need the following two results in order to prove Theorem 2.

Theorem 3.A ([14]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$,*

(i) *If $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$.*

(ii) *If $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subset N(B)$, then $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

Theorem 3.B ([22]). *If $A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ holds for positive operators A and B and fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then*

$$A^{\alpha} \geq (A^{\frac{\alpha}{2}} B^{\beta_0} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta_0}}$$

holds for any $\alpha \geq \alpha_0$. Moreover, for each fixed $\gamma \geq -\beta_0$,

$$g_{\beta_0,\delta}(\alpha) = (B^{\frac{\beta_0}{2}} A^{\alpha} B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha+\beta_0}}$$

is an increasing function for $\alpha \geq \max\{\alpha_0, \delta\}$. Hence $(B^{\frac{\beta_0}{2}} A^{\alpha_2} B^{\frac{\beta_0}{2}})^{\frac{\alpha_1+\beta_0}{\alpha_2+\beta_0}} \geq B^{\frac{\beta_0}{2}} A^{\alpha_1} B^{\frac{\beta_0}{2}}$ holds for any α_1 and α_2 such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Proof of Theorem 2. In case $r_0 = 0$, it is already shown in (i) of Theorem 2.B since class $\mathrm{F}(p_0, 0, \frac{p_0}{\delta})$ for $0 < \delta \leq p_0$ equals δ-hyponormality. So we may assume $r_0 > 0$. Suppose that T belongs to class $\mathrm{F}(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 > 0$ and $0 \leq \delta \leq p_0$, i.e.,

$$(|T^*|^{r_0} |T|^{2p_0} |T^*|^{r_0})^{\frac{\delta+r_0}{p_0+r_0}} \geq |T^*|^{2(\delta+r_0)}. \tag{3.1}$$

Applying Löwner-Heinz theorem to (3.1), we have

$$(|T^*|^{r_0} |T|^{2p_0} |T^*|^{r_0})^{\frac{r_0}{p_0+r_0}} \geq |T^*|^{2r_0},$$

and also we have

$$|T|^{2p_0} \geq (|T|^{p_0} |T^*|^{2r_0} |T|^{p_0})^{\frac{p_0}{p_0+r_0}} \tag{3.2}$$

by (i) of Theorem 3.A. By applying Theorem 3.B to (3.2), we obtain that

$$g_{r_0,\delta}(p) = (|T^*|^{r_0} |T|^{2p} |T^*|^{r_0})^{\frac{\delta+r_0}{p+r_0}} \tag{3.3}$$
is an increasing function for $p \geq \max\{p_0, \delta\} = p_0$.

Therefore we have

$$\begin{aligned}(|T^*|^{r_0}|T|^{2p}|T^*|^{r_0})^{\frac{\delta+r_0}{p+r_0}} &= g_{r_0,\delta}(p) \\ &\geq g_{r_0,\delta}(p_0) && \text{by (3.3)} \\ &= (|T^*|^{r_0}|T|^{2p_0}|T^*|^{r_0})^{\frac{\delta+r_0}{p_0+r_0}} \\ &\geq |T^*|^{2(\delta+r_0)} && \text{by (3.1)}\end{aligned}$$

for any $p \geq p_0$, i.e., T belongs to class $\mathrm{F}(p, r_0, \frac{p+r_0}{\delta+r_0})$ for any $p \geq p_0$. Hence T belongs to class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$ by (ii-2) of Theorem 2.C. □

To prove Theorem 3, we prepare the following result which is a slight modification of [22, Lemma 5].

Lemma 4. *Let A, B and C be positive operators, $p > 0$, $0 < r \leq 1$ and $q \geq 1$ with $rq \leq p + r \leq (1+r)q$. If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$ and $B \geq C$, then $(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{1}{q}} \geq C^{\frac{p+r}{q}}$.*

Proof. The hypothesis $B \geq C$ ensures $B^r \geq C^r$ for $r \in (0, 1]$ by Löwner-Heinz theorem. By Douglas' theorem [3], there exists an operator X such that

$$B^{\frac{r}{2}}X = X^*B^{\frac{r}{2}} = C^{\frac{r}{2}} \tag{3.4}$$

and $\|X\| \leq 1$. Then we have

$$\begin{aligned}(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{1}{q}} &= (X^*B^{\frac{r}{2}}A^pB^{\frac{r}{2}}X)^{\frac{1}{q}} \\ &\geq X^*(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}}X && \text{by Hansen's inequality [11]} \\ &\geq X^*B^{\frac{p+r}{q}}X && \text{by the hypothesis} \\ &= C^{\frac{r}{2}}B^{\frac{p+r}{q}-r}C^{\frac{r}{2}} && \text{by (3.4) since } \tfrac{p+r}{q}-r \in [0,1] \\ &\geq C^{\frac{p+r}{q}} && \text{by Löwner-Heinz theorem.}\end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 3. Let T be a class $\mathrm{F}(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$, i.e.,

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \tag{1.1}$$

Class $\mathrm{F}(p, r, q)$ operator T for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$ belongs to class $\mathrm{F}(1, 1, 2)$, i.e., class A by (ii) of Theorem 2.B and Theorem 2, and also

$$|T^n|^{\frac{2}{n}} \geq |T|^2 \tag{3.5}$$

and

$$|T^*|^2 \geq |T^{n^*}|^{\frac{2}{n}} \tag{3.6}$$

hold for all positive integer n by Theorem 1. By applying Lemma 4 to (1.1) and (3.6), we have

$$(|T^{n^*}|^{\frac{r}{n}}|T|^{2p}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}} \geq |T^{n^*}|^{\frac{2}{n}\frac{p+r}{q}} \tag{3.7}$$

for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$ since $p + r \leq (1 + r)q$ always holds. Hence we obtain

$$\begin{aligned}(|T^{n^*}|^{\frac{r}{n}}|T^n|^{\frac{2p}{n}}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}} &\geq (|T^{n^*}|^{\frac{r}{n}}|T|^{2p}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}} \quad \text{by (3.5) and Löwner-Heinz theorem} \\ &\geq |T^{n^*}|^{\frac{2}{q}(\frac{p}{n}+\frac{r}{n})} \quad \text{by (3.7)}\end{aligned}$$

for all positive integer n, that is, T^n belongs to class $\mathrm{F}(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n. □

4. Properness of class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality

In this section, we shall show the results on inclusion relation among the families of p-quasihyponormality, class $\mathrm{F}(p, r, q)$ and (p, r, q)-paranormality.

Theorem 5. *For each $p_0 > 0$, there exists a p_0-quasihyponormal operator T such that T is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$.*

Theorem 6. *For each $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$,*

(i) *There exists a p_0-quasihyponormal operator T such that T is not p-quasihyponormal for any $p > 0$ such that $0 < p < p_0$.*
(ii) *There exists a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator T such that T does not belong to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$.*
(iii) *There exists a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal operator T such that T is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$.*

In Theorem 6, (i) has been obtained in [18], and also (ii) and (iii) asserts that the families of class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality are proper on p. Moreover we remark that these properness on p has no connection with r, and also we have the following corollary by putting $r = r_0$ in Theorem 6.

Corollary 7. *For each $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$,*

(i) *There exists a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator T such that T does not belong to class $F(p, r_0, \frac{p+r_0}{\delta+r_0})$ for any $p > 0$ such that $\delta \leq p < p_0$.*
(ii) *There exists a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal operator T such that T is not $(p, r_0, \frac{p+r_0}{\delta+r_0})$-paranormal for any $p > 0$ such that $\delta \leq p < p_0$.*

Here we shall show two propositions as a preparation of the proof of Theorem 5. We remark that these propositions are similar arguments to [2], [10], [15] and so on.

Firstly we shall give a characterization of (p, r, q)-paranormal operators.

Proposition 8. *For each $p > 0$, $r > 0$ and $-r < \delta \leq p$, an operator T is $(p, r, \frac{p+r}{\delta+r})$-paranormal if and only if*

$$(\delta + r)|T^*|^r|T|^{2p}|T^*|^r - (p + r)\lambda^{p-\delta}|T^*|^{2(\delta+r)} + (p - \delta)\lambda^{p+r} \geq 0 \quad \textit{for all } \lambda > 0.$$

Proof. Suppose that T is $(p,r,\frac{p+r}{\delta+r})$-paranormal for $p>0$, $r>0$ and $-r<\delta\leq p$, i.e.,

$$\left\||T|^p|T^*|^r x\right\|^{\frac{\delta+r}{p+r}} \geq \left\||T^*|^{\delta+r}x\right\| \quad \text{for every unit vector } x\in H. \tag{1.3}$$

(1.3) holds iff

$$\left\||T|^p|T^*|^r x\right\|^{\frac{\delta+r}{p+r}}\left\|x\right\|^{\frac{p-\delta}{p+r}} \geq \left\||T^*|^{\delta+r}x\right\| \quad \text{for all } x\in H$$

iff

$$(|T^*|^r|T|^{2p}|T^*|^r x,x)^{\frac{\delta+r}{p+r}}(x,x)^{\frac{p-\delta}{p+r}} \geq (|T^*|^{2(\delta+r)}x,x) \quad \text{for all } x\in H. \tag{4.1}$$

By arithmetic-geometric mean inequality,

$$\begin{aligned} &(|T^*|^r|T|^{2p}|T^*|^r x,x)^{\frac{\delta+r}{p+r}}(x,x)^{\frac{p-\delta}{p+r}} \\ &= \left\{\left(\frac{1}{\lambda}\right)^{p-\delta}(|T^*|^r|T|^{2p}|T^*|^r x,x)\right\}^{\frac{\delta+r}{p+r}}\cdot\left\{\lambda^{\delta+r}(x,x)\right\}^{\frac{p-\delta}{p+r}} \\ &\leq \frac{\delta+r}{p+r}\frac{1}{\lambda^{p-\delta}}(|T^*|^r|T|^{2p}|T^*|^r x,x)+\frac{p-\delta}{p+r}\lambda^{\delta+r}(x,x) \end{aligned} \tag{4.2}$$

for all $x\in H$ and all $\lambda>0$, so (4.1) ensures the following (4.3) by (4.2).

$$\frac{\delta+r}{p+r}\frac{1}{\lambda^{p-\delta}}(|T^*|^r|T|^{2p}|T^*|^r x,x)+\frac{p-\delta}{p+r}\lambda^{\delta+r}(x,x) \geq (|T^*|^{2(\delta+r)}x,x) \quad \text{for all } x\in H \text{ and all } \lambda>0. \tag{4.3}$$

Conversely, (4.1) follows from (4.3) by putting $\lambda=\left\{\frac{(|T^*|^r|T|^{2p}|T^*|^r x,x)}{(x,x)}\right\}^{\frac{1}{p+r}}$. (In case $(|T^*|^r|T|^{2p}|T^*|^r x,x)=0$, let $\lambda\to+0$.) Hence (4.3) holds if and only if

$$(\delta+r)|T^*|^r|T|^{2p}|T^*|^r-(p+r)\lambda^{p-\delta}|T^*|^{2(\delta+r)}+(p-\delta)\lambda^{p+r}\geq 0 \quad \text{for all } \lambda>0,$$

so that the proof is complete. $\square$

Secondly we shall give the following Proposition 9. But we omit to describe these calculation because it is obtained by easy calculation.

Proposition 9. *Let* $K=\bigoplus_{n=-\infty}^{\infty}H_n$ *where* $H_n\cong H$. *For given positive operators* A,B *on* H, *define the operator* $T_{A,B}$ *on* K *as follows:*

$$T_{A,B}=\begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & B^{\frac{1}{2}} & 0 & & & & \\ & & B^{\frac{1}{2}} & \boxed{0} & & & \\ & & & A^{\frac{1}{2}} & 0 & & \\ & & & & A^{\frac{1}{2}} & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{4.4}$$

where $\boxed{}$ *shows the place of the* $(0,0)$ *matrix element.*

(i) *For each* $p>0$, $T_{A,B}$ *is* p*-quasihyponormal if and only if*

$$B^{\frac{1}{2}}A^pB^{\frac{1}{2}}\geq B^{p+1}.$$

(ii) *For each* $p>0$, $r>0$ *and* $-r<\delta\le p$, $T_{A,B}$ *is* $(p,r,\frac{p+r}{\delta+r})$*-paranormal if and only if*

$$(\delta+r)B^{\frac{r}{2}}A^pB^{\frac{r}{2}}-(p+r)\lambda^{p-\delta}B^{\delta+r}+(p-\delta)\lambda^{p+r}I\ge 0 \quad \textit{for all } \lambda>0.$$

Proof of Theorem 5. Let

$$\text{(4.5)}\quad \begin{aligned} &A=U\Lambda U^* \text{ and } B=\begin{pmatrix}1&0\\0&0\end{pmatrix}\\ &\text{where } U=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} \text{ and } \Lambda=\begin{pmatrix}(2-e^{-p_0})^{\frac{1}{p_0}}&0\\0&e^{-2}\end{pmatrix},\end{aligned}$$

and also let $K=\bigoplus_{n=-\infty}^{\infty}H_n$ where $H_n\cong\mathbb{R}^2$. For positive matrices A,B on $\mathbb{R}^2$ given in (4.5), define the operator $T_{A,B}$ on K as (4.4) in Proposition 9. By (i) of Proposition 9, $T_{A,B}$ is p-quasihyponormal for $p>0$ if and only if

$$B^{\frac12}A^pB^{\frac12}-B^{p+1}=\begin{pmatrix}\frac12\{(2-e^{-p_0})^{\frac{p}{p_0}}+e^{-2p}\}-1&0\\0&0\end{pmatrix}\ge 0$$

if and only if

$$f(p)\equiv\frac12\{(2-e^{-p_0})^{\frac{p}{p_0}}+e^{-2p}\}-1\ge 0.$$

On the other hand, let $X_p(\lambda)$ as

$$\begin{aligned}X_p(\lambda)&\equiv(\delta+r)B^{\frac{r}{2}}A^pB^{\frac{r}{2}}-(p+r)\lambda^{p-\delta}B^{\delta+r}+(p-\delta)\lambda^{p+r}I\\ &=\begin{pmatrix}\frac12(\delta+r)\{(2-e^{-p_0})^{\frac{p}{p_0}}+e^{-2p}\}-(p+r)\lambda^{p-\delta}+(p-\delta)\lambda^{p+r}&0\\0&(p-\delta)\lambda^{p+r}\end{pmatrix}.\end{aligned}$$

By (ii) of Proposition 9, $T_{A,B}$ is $(p,r,\frac{p+r}{\delta+r})$-paranormal for $p>0$, $r>0$ and $-r<\delta\le p$ if and only if $X_p(\lambda)\ge 0$ for all $\lambda>0$ if and only if

$$\text{(4.6)}\quad g_p(\lambda)\equiv\frac12(\delta+r)\{(2-e^{-p_0})^{\frac{p}{p_0}}+e^{-2p}\}-(p+r)\lambda^{p-\delta}+(p-\delta)\lambda^{p+r}\ge 0 \quad \text{for all } \lambda>0$$

since $(p-\delta)\lambda^{p+r}\ge 0$ for all $\lambda>0$. Since $g_p'(\lambda)=(p+r)(p-\delta)\lambda^{p-\delta-1}(-1+\lambda^{\delta+r})$, we get that

$$\min_{\lambda>0}g_p(\lambda)=g_p(1)=\frac12(\delta+r)\{(2-e^{-p_0})^{\frac{p}{p_0}}+e^{-2p}\}-(\delta+r)=(\delta+r)f(p),$$

so that (4.6) holds if and only if $f(p)\ge 0$.

$f(p)$ is a convex function for $p>0$ since

$$f''(p)=\frac12[(2-e^{-p_0})^{\frac{p}{p_0}}\{\log(2-e^{-p_0})^{\frac{1}{p_0}}\}^2+4e^{-2p}]>0 \quad \text{for all } p>0,$$

and also $f(p)=0$ if $p=0,p_0$. So we have $f(p_0)=0$ but $f(p)<0$ for $0<p<p_0$. Therefore $g_p(1)<0$, that is $X_p(1)\not\ge 0$ for any $p>0$, $r>0$ and $\delta>-r$ such that $\delta\le p<p_0$.

Hence $T_{A,B}$ is p_0-quasihyponormal but non-$(p,r,\frac{p+r}{\delta+r})$-paranormal for any $p>0$, $r>0$ and $\delta>-r$ such that $\delta\le p<p_0$, so the proof is complete. □

Proof of Theorem 6. Let $p_0>0$, $r_0>0$ and $-r_0<\delta\le p_0$.

Proof of (i). By (i-1) of Theorem 2.C, T is p-quasihyponormal if and only if T is $(p,r,1)$-paranormal for some $p>0$ and $r>0$. Therefore there exists a p_0-quasihyponormal operator T such that T is not p-quasihyponormal for any $0<p<p_0$ by putting $\delta=p$ in Theorem 5.

Proof of (ii). By (i-1) of Theorem 2.C and (ii) of Theorem 2.B, every p_0-quasihyponormal operator belongs to class $\mathrm{F}(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$. And also, by (iv) of Theorem 2.B, T does not belong to class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ if T is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for each $p > 0$, $r > 0$ and $-r < \delta \le p$. Therefore there exists a class $\mathrm{F}(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator T such that T does not belong to class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ for any $p > 0$ and $r > 0$ such that $-r < \delta \le p < p_0$ by Theorem 5.

Proof of (iii). By (i-1) of Theorem 2.C and (iii) of Theorem 2.B, every p_0-quasihyponormal operator is $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal. Therefore there exists a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal operator T such that T is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p > 0$ and $r > 0$ such that $-r < \delta \le p < p_0$ by Theorem 5. □

Remark. In [10], we introduced two families of classes of operators based on class A and paranormality as follows: An operator T belongs to class $\mathrm{A}(k)$ for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \ge |T|^2$, and also an operator T is absolute-k-paranormal for $k > 0$ if $\||T|^k Tx\| \ge \|Tx\|^{k+1}$ for every unit vector $x \in H$. In [5], Fujii-Izumino-Nakamoto introduced p-paranormality for $p > 0$ defined by $\||T|^p U |T|^p x\| \ge \||T|^p x\|^2$ for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T. It was pointed out in [20] that class $\mathrm{A}(k)$ equals class $\mathrm{A}(k, 1)$, and also it was shown in [21] that absolute-k-paranormality equals absolute-$(k, 1)$-paranormality and p-paranormality equals absolute-(p, p)-paranormality. We shall also get the results on inclusion relation among the families of these classes.

Corollary 10.

(i) *For each $k_0 > 0$, there exists a class $A(k_0)$ operator T such that T does not belong to class $A(k)$ for any $0 < k < k_0$.*

(ii) *For each $k_0 > 0$, there exists an absolute-k_0-paranormal operator T such that T is not absolute-k-paranormal for any $0 < k < k_0$.*

(iii) *For each $p_0 > 0$, there exists a p_0-paranormal operator T such that T is not p-paranormal for any $0 < p < p_0$.*

Proof of Corollary 10.

Proofs of (i) *and* (ii). By putting $p_0 = k_0$, $r = 1$, $\delta = 0$ and $p = k$ in Corollary 7, we have (i) and (ii) since class $\mathrm{A}(k)$ equals class $\mathrm{F}(k, 1, k+1)$ and absolute-k-paranormality equals $(k, 1, k+1)$-paranormality.

Proof of (iii). By putting $p_0 = r_0$, $\delta = 0$ and $p = r$ in (iii) of Theorem 6, we have (iii) since p-paranormality equals $(p, p, 2)$-paranormality. □

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Department of Mathematical Information Science, Faculty of Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

E-mail address: m-ito@boat.zero.ad.jp