METRIZABILITY OF CERTAIN POINT-COUNTABLE UNIONS

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Abstract. A collection $\mathcal{P}$ of subsets of X is *point-countable* if every point of X is in at most countably many elements of $\mathcal{P}$. Every first countable space having a countable open (or closed) cover of metric subsets need not be metrizable. For a space X having a (not necessarily open or closed) point-countable cover of metric subsets, we shall consider conditions for X to be metrizable in terms of weak topology.

1. Introduction

Following [3], a space X *is determined by* a cover $\mathcal{P}$, if $F \subset X$ is closed in X if and only if $F \cap P$ is closed in P for every $P \in \mathcal{P}$. Here, we can replace " closed " by " open ". In this paper, we assume that all spaces are regular, T_1, and we shall use " X is determined by $\mathcal{P}$ " instead of the usual " X has the weak topology with respect to $\mathcal{P}$ ". Obviously, every space X is determined by any open (or hereditarily closure-preserving closed) cover of X.

A space X is *sequential* ([1]) if, $F \subset X$ is closed in X whenever any convergent sequence in F has the limit point in F. Every Fréchet space is sequential. We note that a space is sequential if and only if it is determined by a cover of metric subsets. We recall that every sequential space is characterized as a quotient image of a metric space ([1]).

A space X is called *strongly Fréchet* ([6]) (i.e., *countably bi-sequential* in the sense of E. Michael [5]) if, whenever $\{A_n : n \in N\}$ is a decreasing sequence with $x \in cl(A_n - \{x\})$ for every $n \in N$, then there exist $x_n \in A_n$ such that the sequence $\{x_n : n \in N\}$ converges to x. When the A_n are the same sets, then such a space X is called *Fréchet*. Every first countable space is strongly Fréchet, and every strongly Fréchet space is Fréchet.

Let us recall two canonical countable spaces S_ω and S_2.

The *sequential fan* S_ω is the quotient space obtained from the topological sum of countably many convergent sequences by identifying all limit points.

The *Arens' space* S_2 is defined as follows: $S_2 = \{x_0\} \cup \{x_n : n \in N\} \bigcup \{x_{nm} : n, m \in N\}$, where $x_n \to x_0, x_{nm} \to x_n$ $(m \to \infty)$. Also, a basic nbd at $\{x_0\}$ has the form $\{x_0\} \cup \{x_n : n \geq i\} \bigcup \{x_{nm}; n \geq i, m \geq j(n)\}$ $(i, j(n) \in N)$, and the points x_{nm} are isolated, and each point x_n has the obvious basic nbds.

We note that S_ω is a Fréchet space which is not strongly Fréchet, and S_2 is a sequential space which is not Fréchet. S_ω and S_2 are spaces determined by the obvious increasing countable cover of compact metric subsets, but they are not metrizable. On the other hand, every space having a countable (or point-finite), closed (or open) cover of metric subsets need not be metrizable even if X is first countable (or strongly Fréchet) by *Examples* below, where (1) is well-known, (2); (3) & (4) are shown in [4]; [7] respectively.

It is a natural question to consider conditions for spaces having certain point-countable covers to be metrizable. In this paper, we shall give metrizability of these spaces by whether

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or not they contain the canonical spaces S_ω and S_2, also apply the results to complexes, inductive limits, and spaces dominated by metric subsets.

Examples. (1) A first countable space X having a countable closed cover of separable metric subsets, but X is not metrizable.

(2) A strongly Fréchet space X having a countable cover of singletones (only one non-isolated point), but X is not first countable, hence not metrizable.

(3) A first countable space X having a point-finite cover of closed and open metric subsets, but X is not normal, hence not metrizable

(4) A first countable space X having an increasing open countable cover of metric subsets, but X is not normal, hence not metrizable.

2. Metrization theorem

For a space X having a point-countable cover of metric subsets, we will consider conditions for X to be metrizable. First we give some Lemmas.

For a cover $\mathcal{P}$ of a space X, let $\mathcal{P}^*$ be the collection of all finite unions of elements of $\mathcal{P}$. Obviously, if a space is determined by $\mathcal{P}$, then so is by $\mathcal{P}^*$. The converse need not hold, but the converse holds if $\mathcal{P}$ is closed.

Lemma 1. Let X be a sequential space, and let $\mathcal{P}$ be a cover of X. Then the following are equivalent.

(1) X is determined by $\mathcal{P}$ (resp. $\mathcal{P}^*$).

(2) For every infinite sequence $L = \{x_n : n \in n\}$ converging to x, some $P \in \mathcal{P}$ contains x and x_n frequently (resp. x_n frequently).

Proof. For (1) $\rightarrow$ (2), note that $L - \{x\}$ is not closed in X. For (2) $\rightarrow$ (1), if F is not closed in X, then there exists a sequence L in F converging to x not in F, hence, $F \cap P$ is not closed in P for some $P \in \mathcal{P}$.

In Lemmas below, Lemma 2 is shown as in the proof of [3; Proposition 3.2], using Lemma 1. Lemma 3; Lemma 4 holds by [8; Corollary 1.5]; [3; Corollary 3.6] respectively. For Lemma 7, see [7] or [9], for example.

Lemma 2. Let X be a strongly Fréchet space, and let $\mathcal{P}$ be a point-countable cover of X. If X is determined by $\mathcal{P}^*$, then each point of X has a nbd which is contained in some element of $\mathcal{P}^*$.

Let $\mathcal{P}$ be a cover of a space X. Then, $\mathcal{P}$ is a *k-network* for X, if whenever $K \subset U$ with K compact and U open in X, $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. When the K is a singleton, then such a cover $\mathcal{P}$ is called a *network*. Clearly, every open base is a k-network.

Lemma 3. Let X be a sequential space, and let $\mathcal{P}$ be a point-countable cover of X. Then $\mathcal{P}$ is a k-network if and only if, for a sequence $L = \{x_n : n \in N\}$ converging to x, and a nbd V of x, there exists $P \in \mathcal{P}$ such that $P \subset V$ and P contains x_n frequently.

Lemma 4. Every strongly Fréchet space with a point-countable k-network has a point-countable base.

As is well-known, every space having a locally finite closed (resp. point-countable open) cover of metric (resp. separable metric) subsets is metrizable. For a more general case where a space is a strongly Fréchet space determined by a point-countable cover of (locally separable) metric subsets, the following metrizability holds.

Lemma 5. Let X be a strongly Fréchet space, and let $\mathcal{P}$ be a point-countable cover of X. Then X is metrizable if the following case (a), (b), or (c) holds.

(a) X is determined by $\mathcal{P}$, and each $P \in \mathcal{P}$ is locally separable, metric.

(b) X is determined by $\mathcal{P}^*$, and for each $P \in \mathcal{P}$, clP is locally separable, metric.

(c) X is a paracompact space, X is determined by $\mathcal{P}^*$, and each $P^* \in \mathcal{P}^*$ is locally metric.

Proof. For case (a), let $\mathcal{P} = \{X_\alpha : \alpha\}$. Since each X_α is locally separable, metric, X_α is determined by the obvious point-countable (open) cover $\{X_{\alpha\beta} : \beta\}$ of separable metric subsets. While, X is determined by the point-countable cover $\{X_\alpha : \alpha\}$. Thus it is routinely shown that X is determined by a point-countable cover $\{X_{\alpha\beta} : \alpha, \beta\}$. So, we can assume that X_α are separable metric. Since X is strongly Fréchet, by Lemma 2, each $x \in X$ has a nbd $V(x)$ such that $V(x) \subset \bigcup \mathcal{F}$ for some finite $\mathcal{F} \subset \mathcal{P}$. Then, $V(x)$ is separable. This shows that X is locally separable. On the other hand, let $\mathcal{B}_\alpha$ be a countable base for X_α. Let $\mathcal{C} = \bigcup\{\mathcal{B}_\alpha : \alpha\}$. Then $\mathcal{C}$ is a point-countable cover of X. For $x \in X$, let V be a nbd of x, and let $L = \{x_n : n \in N\}$ be a sequence converging to x. Since X is determined by $\mathcal{P}$, some X_α contains the point x, and contains x_n frequently by Lemma 1. Thus, there exists $B \in \mathcal{B}_{\alpha_0}$ such that $x \in B \subset V$, and B contains x_n frequently. Then, $\mathcal{C}$ is a point-countable k-network by Lemma 3. Since, X is strongly Fréchet, X has a point-countable base by Lemma 4. Therefore, X is a locally separable space with a point-countable base. Then, as is well known, X is the topological sum of locally separable metric subsets. Thus X is metrizable. For case (b), each point of X has a nbd which is contained in some $P^* \in \mathcal{P}^*$ by Lemma 2. Thus, X is locally separable. Let $\mathcal{P} = \{X_\alpha : \alpha\}$. Let $\mathcal{G}_\alpha$ be a point-countable base for clX_α, and let $\mathcal{H}_\alpha = \{B \cap X_\alpha : B \in \mathcal{G}_\alpha\}$. Since X is determined by $\mathcal{P}^*$, we show that $\bigcup\{\mathcal{H}_\alpha : \alpha\}$ is a point-countable k-network for X by means of Lemmas 1 and 3. Thus X is metrizable by means of Lemma 4. For case (c), X is locally metric by Lemma 2. Since X is paracompact, as is well-known, X is metrizable.

In the previous lemma, the separability of the metric subsets is essential in cases (a) & (b), and the paracompactness is essential in case (c); see *Examples*. However, in view of cases (a) & (b), the author has the following question: Let X be a space having a point-countable cover $\mathcal{P}$ of X such that X is determined by $\mathcal{P}^*$, and each element of $\mathcal{P}^*$ is locally separable, metric. If X is strongly Fréchet, then is X metrizable ?

Lemma 6. For a point-countable cover $\mathcal{P}$ of a space X, suppose that the following case (a) or (b) holds. Then X is Fréchet if and only if X contains no (closed) copy of S_2.

(a) X is determined by $\mathcal{P}^*$, and each $P^* \in \mathcal{P}^*$ is metric.

(b) X is sequential, X is determined by $\mathcal{P}$, and each $P \in \mathcal{P}$ is locally separable, metric.

Proof. Since the " only if " part is obvious, we shall show the " if " by refering to the proof of [7; Theorem 2.1]. To show that X is Fréchet, suppose not. For case (a), X is sequential since it is determined by a cover of sequential subspaces. Then, by [2; Proposition 7.3], X contains a subset $S = \{x_0\} \cup \{x_n : n \in N\} \bigcup\{L_n : n \in N\}$, where $x_n \to x_0, x_{nm} \to x_n$ $(m \to \infty)$, but no points $p_n \in L_n$ $(n \in N)$ converges to x_0. Now, let $\{P \in \mathcal{P} : P \cap S \neq \emptyset\} = \{P_n : n \in N\}$, and let $X_n = \bigcup\{P_i : i \leq n\}$ for each $n \in N$. Since X is determined by $\mathcal{P}^*$ and $x_n \to x_0$, by Lemma 1, some X_{n_1} contains x_0 and some subsequence $\{x_{n_i} : i \in N\}$. But, since X_{n_1} is metric, X_{n_1} doesn't contain

S, because if X_{n_1} contains a copy of S, then there exists a sequence $\{p_k : k \in N\}$ with $p_k \in L_{n_k}$ $(k \in N)$ converging to x_0, a contradiction. Then, we can assume that X_{n_1} is disjoint from some $L_{n_{i(1)}}$. But, $L_{n_{i(1)}}$ converges to $x_{n_{i(1)}}$, then some X_{n_2} $(n_2 > n_1)$ contains some subsequence T_1 of $L_{n_{i(2)}}$ by Lemma 1. Then, $T_1 \subset X_{n_2} - X_{n_1}$. By induction, we can choose a sequence $\{T_k : k \in N\}$, where T_k is a subsequence of $L_{n_{i(k)}}$ $(i(k) < i(k+1))$ such that $T_k \subset X_{n_k} - X_{n_{k-1}}$ $(n_{k-1} < n_k)$. Let $T = \{x_0\} \cup \{x_{n_{i(k)}} : k \in N\} \bigcup \{T_k : k \in N\}$. Let $P^* \in \mathcal{P}^*$. Then $T \cap P^* \subset T \cap X_{n_k}$ for some X_{n_k}, but $T \cap X_{n_k}$ is compact. Thus $T \cap P^*$ is closed in P^*. Hence, T is closed in X, so T is sequential. Then T is a copy of S_2. Thus, X contains a closed copy of S_2. This is a contradiction. Hence X is Fréchet. For case (b), we can assume that the point-countable cover $\mathcal{P}$ consists of separable metric subsets. Let $\{P \in \mathcal{P} : P \cap S \neq \emptyset\} = \{P_n : n \in N\}$. Let $A = \bigcup\{P_n : n \in N\}$. Then $S \subset A$, and A has a countable network by closed subsets. Thus, there exist open subsets G_n $(n \in N)$ of A such that $\{x_0\} = \bigcap\{G_n : n \in N\}$, and $clG_{n+1} \subset G_n$ in A. We can assume that $L_n \cup \{x_n\} \subset G_n$ for each $n \in N$. Then S is closed in A. But, X is determined by $\mathcal{P}$, so X is determined by a cover $\{A\} \cup \{P \in \mathcal{P} : S \cap P = \emptyset\}$. Then S is closed in X. Thus X contains a closed copy of $S = S_2$, a contradiction. Hence X is also Fréchet.

Lemma 7. Let X be a Fréchet space. Then X is strongly Fréchet if and only if it contains no (closed) copy of S_ω.

We have the following metrization theorem on space having certain point-countable covers of metric subsets. The result for case (a) is due to [7; Theorem 4.6].

Theorem 8. For a point-countable cover $\mathcal{P}$ of a space X, suppose that the case (a), (b), or (c) in Lemma 5 holds. Then the following (1), (2), and (3) are equivalent.

(1) X is metrizable.

(2) X is strongly Fréchet.

(3) X contains no (closed) copy of S_ω, and no S_2.

Proof. (1) $\to$ (3) is obvious. (3) $\to$ (2) holds by Lemmas 6 and 7. (2) $\to$ (1) holds by Lemma 5.

Related to Theorem 8, (3) need not imply (2) under spaces being sequential (indeed, there exists a compact sequential space which contains no copy of S_ω, and S_2, but it is not even Fréchet ([9; Example 1.21 & Corollary 1.10]).

Corollary 9. Let X be a sequential space, and let X be a complex having the cover $\mathcal{E} = \{e_\lambda : \lambda\}$ of cells in X such that, for each convergent sequence $\{x_n : n \in N\}$, some e_λ contains x_n frequently (in particular, let X be a CW-complex). Then X is metrizable if and only if X contains no (closed) copy of S_ω, and no S_2 (equivalently, X is strongly Fréchet).

Proof. The cover $\mathcal{E}$ is disjoint, and each cle_λ is compact metric. Also, X is determined by $\mathcal{E}^*$ by Lemma 1. Thus, the corollary holds by Theorem 8 (for case (b)).

Let X be a space, and let $\mathcal{F}$ be a closed cover of X. Then X *is dominated by* $\mathcal{F}$ if, for any $\mathcal{A} \subset \mathcal{F}$, $A = \bigcup \mathcal{A}$ is closed in X, and A is determined by $\mathcal{A}$. Every space is dominated by its hereditarily closure-preserving closed cover. It is well-known that every space dominated by metric (or paracompact) subsets is paracompact.

Corollary 10 ([11]). Let X be a space dominated by a closed cover $\{X_\alpha : \alpha \leq \gamma\}$ of metric subsets. Then X is metrizable if and only if X contains no (closed) copy of S_ω, and no S_2 (equivalently, X is strongly Fréchet).

Proof. For each $\alpha \leq \gamma$, let $Y_\alpha = X_\alpha - \bigcup\{X_\beta : \beta < \alpha\}$. Let $\mathcal{P} = \{Y_\alpha : \alpha \leq \gamma\}$. Then $\mathcal{P}$ is a disjoint cover of X. Also, every convergent sequence $L = \{x_n : n \in N\}$ meets only finitely many Y_α (in fact, assume $x_{n_i} \in Y_{\alpha_i}$ with $\alpha_i < \alpha_{i+1}$, and let $D = \{x_{n_i} : i \in N\}$. Then each $D \cap X_{\alpha_i}$ is finite, hence closed in X_{α_i}. Thus D is closed discrete in X, a contradiction). Then, by Lemma 1, X is determined by $\mathcal{P}^*$. While, X is paracompact, and each element of $\mathcal{P}^*$ is metric. Thus, the corollary holds by Theorem 8.

Let X be a space determined by a countable cover $\mathcal{P} = \{X_n : n \in N\}$ such that $X_n \subset X_{n+1}$ for each $n \in N$ (hence, if all X_n are closed in X, X is dominated by $\mathcal{P}$). Then X is called the *inductive limit* (or *direct limit*) of $\{X_n : n \in N\}$, and it is denoted by $X = \varinjlim X_n$. For a space X determined by a countable cover $\{C_n : n \in N\}$, putting $X_n = \cup\{C_m : m \geq n\}$ for each $n \in N$, $X = \varinjlim X_n$.

For a metric space M having a non-isolated point p, let $X_n = M^n \times \{p\} \times \{p\} \times \ldots$ for each $n \in N$. Then, each X_n is metric, but $T = \varinjlim X_n$ is not Fréchet, hence not metrizable, because T contains a closed copy of S_2 (and S_ω) ([10]). But, the following metrizability of the inductive limits holds.

Corollary 11. Let $X = \varinjlim X_n$ such that each X_n is metric. Suppose that (a) X is paracompact, (b) each X_n is locally separable, or (c) each X_n is closed in X. Then X is metrizable if and only if X contains no (closed) copy of S_ω, and no S_2 (equivalently, X is strongly Fréchet).

In the previous corollary, the condition (a), (b), or (c) is essential even if the metric spaces X_n are open in X (*Example* (4)). Also, not every normal space X determined by an *increasing* closed (or open) cover of separable metric subsets is metrizable even if X is first countable, hence X contains no copy of S_ω, and no S_2 (by the ordinal space $X = [0, \omega_1)$).

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