ROUGHNESS OF IDEALS IN BCK-ALGEBRAS

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Received April 4, 2002

ABSTRACT. As a generalization of ideals in BCK-lagberas, the notion of rough ideals is discussed.

1. INTRODUCTION

In 1982, Pawlak introduced the concept of a rough set (see [5]). This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [6]). Rough set theory is applied to semigroups and groups (see [2, 3]). In this paper, we apply the rough set theory to BCK-algebras, and we introduce the notion of upper/lower rough subalgebras/ideals which is an extended notion of an ideal in a BCK-algebra.

2. PRELIMINARIES

Recall that a *BCK-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$,

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $0 * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$.

For any BCK-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X. A nonempty subset S of a BCK-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A nonempty subset A of a BCK-algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies

- $0 \in A$,
- $x * y \in A$ and $y \in A$ imply $x \in A$ for all $x, y \in X$.

Note that every ideal of a BCK-algebra X is a subalgebra of X.

Let V be a set and E an equivalence relation on V and let $\mathcal{P}(V)$ denote the power set of V. For all $x \in V$, let $[x]_E$ denote the equivalence class of x with respect to E. Define the functions $E_-, E^- : \mathcal{P}(V) \to \mathcal{P}(V)$ as follows: $\forall S \in \mathcal{P}(V)$,

$$E_-(S) = \{x \in V \mid [x]_E \subseteq S\} \text{ and } E^-(S) = \{x \in V \mid [x]_E \cap S \neq \emptyset\}.$$

The pair (V, E) is called an *approximation space*. Let S be a subset of V. Then S is said to be *definable* if $E_-(S) = E^-(S)$ and *rough* otherwise. $E_-(S)$ is called the *lower approximation* of S while $E^-(S)$ is called the *upper approximation*.

2000 *Mathematics Subject Classification.* 06F35, 03G25.

Key words and phrases. lower/upper approximation, definable subset, lower/upper rough subalgebra/ideal.

3. Roughness of ideals

Throughout this section X is a BCK-algebra. Let A be an ideal of X. Define a relation Θ on X by

$$(x, y) \in \Theta \text{ if and only if } x * y \in A \text{ and } y * x \in A.$$

Then Θ is an equivalence relation on X related to an ideal A of X. Moreover Θ satisfies

$$(x, y) \in \Theta \text{ and } (u, v) \in \Theta \text{ imply } (x * u, y * v) \in \Theta.$$

Hence Θ is a congruence relation on X. Let A_x denote the equivalence class of x with respect to the equivalence relation Θ related to the ideal A of X, and X/A denote the collection of all equivalence classes, that is, $X/A = \{A_x \mid x \in X\}$. Then $A_0 = A$. If $A_x * A_y$ is defined as the class containing $x * y$, that is, $A_x * A_y = A_{x*y}$, then $(X/A, *, A_0)$ is a BCK-algebra (see [4]). Let Θ be an equivalence relation on X related to an ideal A of X. For any nonempty subset S of X, the lower and upper approximation of S are denoted by $\underline{\Theta}(A; S)$ and $\overline{\Theta}(A; S)$ respectively, that is,

$$\underline{\Theta}(A; S) = \{x \in X \mid A_x \subseteq S\} \text{ and } \overline{\Theta}(A; S) = \{x \in X \mid A_x \cap S \neq \emptyset\}.$$

If $A = S$, then $\underline{\Theta}(A; S)$ and $\overline{\Theta}(A; S)$ are denoted by $\underline{\Theta}(A)$ and $\overline{\Theta}(A)$, respectively.

Example 3.1. (1) Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the Cayley table as follows (see [4]).

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Let $A = \{0, 1\} \lhd X$ and let Θ be an equivalence relation on X related to A. Then $A_0 = A_1 = A$, $A_2 = \{2\}$, and $A_3 = \{3\}$. Hence $\underline{\Theta}(A; \{0, 2\}) = \{2\} = \underline{\Theta}(A; \{2\})$, $\underline{\Theta}(A; \{0\}) = \emptyset$, $\underline{\Theta}(A; \{0, 3\}) = \{3\}$, $\underline{\Theta}(A; \{0, 1, 3\}) = \{0, 1, 3\} \lhd X$, $\overline{\Theta}(A; \{0, 2\}) = \{0, 1, 2\} \lhd X$, and $\overline{\Theta}(A; \{0, 3\}) = \{0, 1, 3\} \lhd X$.

(2) Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the Cayley table as follows (see [4]).

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Consider $A = \{0, 1, 2\} \lhd X$ and let Θ be an equivalence relation on X related to A. Then the equivalence classes are as follows: $A_0 = A_1 = A_2 = A$, $A_3 = \{3\}$, and $A_4 = \{4\}$. Thus $\underline{\Theta}(A; \{0, 1, 3\}) = \{3\}$, $\underline{\Theta}(A; \{0, 2, 4\}) = \{4\}$, $\underline{\Theta}(A; \{0, 1, 2, 3\}) = \{0, 1, 2, 3\} \lhd X$, $\underline{\Theta}(A; \{0, 1, 2, 4\}) = \{0, 1, 2, 4\} \lhd X$, $\overline{\Theta}(A; \{0, 2\}) = \{0, 1, 2\} \lhd X$, and $\overline{\Theta}(A; \{0, 3\}) = \{0, 1, 2, 3\} \lhd X$.

In Example 3.1, we know that there exists a non-ideal U of X such that $\underline{\Theta}(A; U) \lhd X$; and there exists a non-ideal V of X such that $\overline{\Theta}(A; V) \lhd X$, where Θ is an equivalence relation on X related to $A \lhd X$.

Proposition 3.2. *Let Θ and Ψ be equivalence relations on X related to ideals A and B of X, respectively. If $A \subseteq B$, then $\Theta \subseteq \Psi$.*

Proof. If $(x, y) \in \Theta$, then $x * y \in A \subseteq B$ and $y * x \in A \subseteq B$. Hence $(x, y) \in \Psi$, and so $\Theta \subseteq \Psi$. □

Proposition 3.3. *Let Θ be an equivalence relation on X related to an ideal A of X. Then*

(1) $\underline{\Theta}(A;S) \subseteq S \subseteq \overline{\Theta}(A;S)$, $\forall S \in \mathcal{P}(X)$.
(2) $\overline{\Theta}(A;S\cup T) = \overline{\Theta}(A;S) \cup \overline{\Theta}(A;T)$, $\forall S,T \in \mathcal{P}(X)$.
(3) $\underline{\Theta}(A;S\cap T) = \underline{\Theta}(A;S) \cap \underline{\Theta}(A;T)$, $\forall S,T \in \mathcal{P}(X)$.
(4) $\forall S,T \in \mathcal{P}(X)$, $S \subseteq T \Rightarrow \underline{\Theta}(A;S) \subseteq \underline{\Theta}(A;T)$ *and* $\overline{\Theta}(A;S) \subseteq \overline{\Theta}(A;T)$.
(5) $\underline{\Theta}(A;S\cup T) \supseteq \underline{\Theta}(A;S) \cup \underline{\Theta}(A;T)$, $\forall S,T \in \mathcal{P}(X)$.
(6) $\overline{\Theta}(A;S\cap T) \subseteq \overline{\Theta}(A;S) \cap \overline{\Theta}(A;T)$, $\forall S,T \in \mathcal{P}(X)$.
(7) *If Ψ is an equivalence relation on X related to an ideal B of X and if $A \subseteq B$, then* $\overline{\Theta}(A;S) \subseteq \overline{\Psi}(B;S)$, $\forall S \in \mathcal{P}(X)$.

Proof. (1) is straightforward.

(2) For any subsets S and T of X, we have

$$\begin{aligned} x \in \overline{\Theta}(A;S\cup T) &\Leftrightarrow A_x \cap (S\cup T) \neq \emptyset \\ &\Leftrightarrow (A_x\cap S)\cup(A_x\cap T) \neq \emptyset \\ &\Leftrightarrow A_x\cap S \neq \emptyset \text{ or } A_x \cap T \neq \emptyset \\ &\Leftrightarrow x \in \overline{\Theta}(A;S) \text{ or } x \in \overline{\Theta}(A;T) \\ &\Leftrightarrow x \in \overline{\Theta}(A;S) \cup \overline{\Theta}(A;T), \end{aligned}$$

and hence $\overline{\Theta}(A;S\cup T) = \overline{\Theta}(A;S) \cup \overline{\Theta}(A;T)$.

(3) For any subsets S and T of X we have

$$\begin{aligned} x \in \underline{\Theta}(A;S\cap T) &\Leftrightarrow A_x \subseteq S\cap T \\ &\Leftrightarrow A_x \subseteq S \text{ and } A_x \subseteq T \\ &\Leftrightarrow x \in \underline{\Theta}(A;S) \text{ and } x \in \underline{\Theta}(A;T) \\ &\Leftrightarrow x \in \underline{\Theta}(A;S) \cap \underline{\Theta}(A;T). \end{aligned}$$

Hence $\underline{\Theta}(A;S\cap T) = \underline{\Theta}(A;S) \cap \underline{\Theta}(A;T)$.

(4) Let $S,T \in \mathcal{P}(X)$ be such that $S \subseteq T$. Then $S\cap T = S$ and $S \cup T = T$. It follows from (3) and (2) that

$$\underline{\Theta}(A;S) = \underline{\Theta}(A;S\cap T) = \underline{\Theta}(A;S) \cap \underline{\Theta}(A;T)$$

and

$$\overline{\Theta}(A;T) = \overline{\Theta}(A;S\cup T) = \overline{\Theta}(A;S) \cup \overline{\Theta}(A;T),$$

which yield $\underline{\Theta}(A;S) \subseteq \underline{\Theta}(A;T)$ and $\overline{\Theta}(A;S) \subseteq \overline{\Theta}(A;T)$, respectively.

(5) Since $S \subseteq S\cup T$ and $T \subseteq S \cup T$, it follows from (4) that

$$\underline{\Theta}(A;S) \subseteq \underline{\Theta}(A;S\cup T) \text{ and } \underline{\Theta}(A;T) \subseteq \underline{\Theta}(A;S\cup T).$$

Thus $\underline{\Theta}(A;S) \cup \underline{\Theta}(A;T) \subseteq \underline{\Theta}(A;S\cup T)$.

(6) Since $S\cap T \subseteq S,T$, it follows from (4) that

$$\overline{\Theta}(A;S\cap T) \subseteq \overline{\Theta}(A;S) \text{ and } \overline{\Theta}(A;S\cap T) \subseteq \overline{\Theta}(A;T)$$

so that $\overline{\Theta}(A;S\cap T) \subseteq \overline{\Theta}(A;S) \cap \overline{\Theta}(A;T)$.

(7) If $x \in \overline{\Theta}(A;S)$, then $A_x \cap S \neq \emptyset$, and so there exists $a \in S$ such that $a \in A_x$. Hence $(a,x) \in \Theta$, that is, $a*x \in A$ and $x*a \in A$. Since $A \subseteq B$, it follows that $a*x \in B$ and $x*a \in B$ so that $(a,x) \in \Psi$, that is, $a \in B_x$. Therefore $a \in B_x \cap S$, which means $x \in \overline{\Psi}(B;S)$. This completes the proof. □

Proposition 3.4. *Let Θ be an equivalence relation on X related to any ideal A of X. Then $\underline{\Theta}(A;X) = X = \overline{\Theta}(A;X)$, that is, X is definable.*

Proof. It is straightforward. □

Proposition 3.5. *Let Θ be an equivalence relation on X related to the trivial ideal $\{0\}$ of X. Then $\underline{\Theta}(\{0\};S) = S = \overline{\Theta}(\{0\};S)$ for every nonempty subset S of X, that is, every nonempty subset of X is definable.*

Proof. Note that $\{0\}_x = \{x\}$ for all $x \in X$, since if $a \in \{0\}_x$ then $(a,x) \in \Theta$ and hence $a * x = 0$ and $x * a = 0$. It follows that $a = x$. Hence

$$\underline{\Theta}(\{0\};S) = \{x \in X \mid \{0\}_x \subseteq S\} = S$$

and

$$\overline{\Theta}(\{0\};S) = \{x \in X \mid \{0\}_x \cap S \neq \emptyset\} = S.$$

This completes the proof. □

Remark 3.6. Let Θ be an equivalence relation on X related to an ideal A of X. If B is an ideal of X such that $A \neq B$, then $\underline{\Theta}(A;B)$ is not an ideal of X in general. For, consider a BCK-algebra X in Example 3.1(2) and an equivalence relation Θ on X related to the ideal $A = \{0,1,2\}$. If we take an ideal $B = \{0,1,3\}$ of X, then $A \neq B$ and $\underline{\Theta}(A;B) = \{3\}$ which is not an ideal of X.

Definition 3.7. Let Θ be an equivalence relation on X related to an ideal A of X. A nonempty subset S of X is called an *upper* (resp. a *lower*) *rough subalgebra/ideal* of X if the upper (resp. nonempty lower) approximation of S is a subalgebra/ideal of X. If S is both an upper and a lower rough subalgebra/ideal of X, we say that S is a *rough subalgebra/ideal* of X.

Theorem 3.8. *Let Θ be an equivalence relation on X related to an ideal A of X. Then every subalgebra S of X is a rough subalgebra of X.*

Proof. Let $x, y \in \underline{\Theta}(A;S)$. Then $A_x \subseteq S$ and $A_y \subseteq S$. Since S is a subalgebra of X, it follows that $A_{x*y} = A_x * A_y \subseteq S$ so that $x * y \in \underline{\Theta}(A;S)$. Hence $\underline{\Theta}(A;S)$ is a subalgebra of X. Now if $x, y \in \overline{\Theta}(A;S)$, then $A_x \cap S \neq \emptyset$ and $A_y \cap S \neq \emptyset$, and so there exist $a, b \in S$ such that $a \in A_x$ and $b \in A_y$. It follows that $(a,x) \in \Theta$ and $(b,y) \in \Theta$. Since Θ is a congruence relation on X, we have $(a * b, x * y) \in \Theta$. Hence $a * b \in A_{x*y}$. Since S is a subalgebra of X, we get $a * b \in S$, and therefore $a * b \in A_{x*y} \cap S$, that is, $A_{x*y} \cap S \neq \emptyset$. This shows that $x * y \in \overline{\Theta}(A;S)$, and consequently $\overline{\Theta}(A;S)$ is a subalgebra of X. This completes the proof. □

Corollary 3.9. *Let Θ be an equivalence relation on X related to an ideal A of X. Then $\underline{\Theta}(A)$ $(\neq \emptyset)$ and $\overline{\Theta}(A)$ are subalgebras of X, that is, A is a rough subalgebra of X.*

Proof. It is straightforward. □

Theorem 3.10. *Let Θ be an equivalence relation on X related to an ideal A of X. If U is an ideal of X containing A, then*

(1) *$\underline{\Theta}(A;U)$ $(\neq \emptyset)$ is an ideal of X, that is, U is a lower rough ideal of X.*
(2) *$\overline{\Theta}(A;U)$ is an ideal of X, that is, U is an upper rough ideal of X.*

Proof. Let U be an ideal of X containing A. Let $x \in A_0$. Then $x \in A \subseteq U$, and so $A_0 \subseteq U$. Hence $0 \in \underline{\Theta}(A;U)$. Let $x, y \in X$ be such that $y \in \underline{\Theta}(A;U)$ and $x * y \in \underline{\Theta}(A : U)$. Then $A_y \subseteq U$ and $A_x * A_y = A_{x*y} \subseteq U$. Let $a \in A_x$ and $b \in A_y$. Then $(a,x) \in \Theta$ and $(b,y) \in \Theta$, which implies $(a*b, x*y) \in \Theta$. Hence $a*b \in A_{x*y} \subseteq U$. Since $b \in A_y \subseteq U$ and U is an ideal, it follows that $a \in U$ so that $A_x \subseteq U$. Thus $x \in \underline{\Theta}(A;U)$. This shows that $\underline{\Theta}(A;U)$ is an ideal of X, that is, U is a lower rough ideal of X. Now, obviously $0 \in \overline{\Theta}(A;U)$. Let $x, y \in X$ be such that $y \in \overline{\Theta}(A;U)$ and $x * y \in \overline{\Theta}(A;U)$. Then $A_y \cap U \neq \emptyset$ and $A_{x*y} \cap U \neq \emptyset$, and so there exist $a, b \in U$ such that $a \in A_y$ and $b \in A_{x*y}$. Hence $(a, y) \in \Theta$ and $(b, x * y) \in \Theta$, which implies $y * a \in A \subseteq U$ and $(x * y) * b \in A \subseteq U$. Since $a, b \in U$ and U is an ideal, we get $y \in U$ and $x * y \in U$; hence $x \in U$. Note that $x \in A_x$, thus $x \in A_x \cap U$, that is, $A_x \cap U \neq \emptyset$. Therefore $x \in \overline{\Theta}(A; U)$, and consequently U is an upper rough ideal of X. □

Corollary 3.11. *Let Θ be an equivalence relation on X related to an ideal A of X. Then $\underline{\Theta}(A)$ $(\neq \emptyset)$ and $\overline{\Theta}(A)$ are ideals of X, that is, A is a rough ideal of X.*

Theorem 3.10 shows that the notion of an upper (resp. a lower) rough ideal is an extended notion of an ideal in a BCK-algebra. The following example provides that the converse of Theorem 3.10 may not be true.

Example 3.12. (1) Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the Cayley table as follows (see [4]).

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Consider $A = \{0, 2\} \lhd X$ and a subset $U = \{0, 2, 3\}$ of X which is not an ideal of X. Let Θ be an equivalence relation on X related to A. Then $A_0 = A_2 = A$, $A_1 = \{1\}$, $A_3 = \{3\}$, and $A_4 = \{4\}$. Hence $\underline{\Theta}(A; U) = \{0, 2\} \lhd X$.

(2) Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the Cayley table as follows (see [4]).

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Consider $B = \{0, 2\} \lhd X$ and let Ψ be an equivalence relation on X related to B. Then all equivalence classes are $B_0 = B_2 = \{0, 2\}$, $B_1 = \{1\}$, $B_3 = \{3\}$ and $B_4 = \{4\}$. Note that $V = \{0, 1, 4\}$ is not an ideal of X, but $\overline{\Psi}(B; V) = \{0, 1, 2, 4\} \lhd X$.

Acknowledgements. This work was supported by Korea Research Foundation Grant (KRF-2001-005-D00002).

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