CHARACTERIZATION OF LOCALLY INVERSE $*$-SEMIGROUPS

Dedicated to Professor Masami Ito on his 60th birthday

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Abstract. The purpose of this paper is to give a characterization of a locally inverse $*$-semigroup by introducing a new concept of a *locally inductive $*$-groupoid*. Defining a product $\otimes$ in a locally inductive $*$-groupoid G, $G(\otimes)$ becomes a locally inverse $*$-semigroup. Conversely, for a locally inverse $*$-semigroup S, we give a partial product $\cdot$ in S, we show that $S(\cdot, *, \leq)$ is a locally inductive $*$-groupoid and that $S(\cdot, *, \leq)(\otimes) = S$.

1 Introduction Ehresmann introduced a concept of an *inductive groupoid* in [1] and [2], and Schein characterized an inverse semigroup by using the concept (see [8]). One of the authors introduced the concept of *the symmetric locally inverse $*$-semigroup* $\mathcal{LI}_{(X;\sigma)}$ on an *ι-set* $(X;\sigma)$ and obtained a generalization of Preston-Vagner Representation Theorem. That is, $\mathcal{LI}_{(X;\sigma)}$ on an *ι-set* $(X;\sigma)$ is a locally inverse $*$-semigroup and every locally inverse $*$-semigroup can be embedded in the symmetric locally inverse $*$-semigroup on an ι-set (see [6] and [7]). This result leads us a new partial product on a locally inverse $*$-semigroup and another its characterization.

First, we give definitions and basic results. A semigroup S with a unary operation $* : S \to S$ is called a *regular $*$-semigroup* if it satisfies (i) $(x^*)^* = x$; (ii) $(xy)^* = y^*x^*$; (iii) $xx^*x = x$.

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if $e^* = e$. For a subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively. A regular $*$-semigroup S is called a *locally inverse $*$-semigroup* if for any $e \in E(S)$, eSe is an inverse subsemigroup of S. The following results are well-known and are used frequently throughout this paper.

Result 1.1. [3] [6] *Let S be a regular $*$-semigroup.*

(1) $E(S) = P(S)^2$. *In fact, for any $e \in E(S)$, there exist $f, g \in P(S)$ such that $f\mathcal{R}e\mathcal{L}g$ and $e = fg$.*

(2) *For any $a \in S$ and $e \in P(S)$, $a^*ea \in P(S)$.*

(3) *Each $\mathcal{L}$-class and each $\mathcal{R}$-class contain one and only one projection.*

(4) *S is a locally inverse $*$-semigroup if and only if it satisfies that eSe is an inverse subsemigroup of S for any $e \in P(S)$.*

Define a relation $\leq$ on a regular $*$-semigroup S as follows:

$$a \leq b \iff a = eb = bf \quad \text{for some} \quad e, f \in P(S).$$

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Result 1.2. [4] *Let a and b be elements of a regular $*$-semigroup S. Then the following conditions are equivalent:*

(1) $a \leq b$,

(2) $aa^* = ba^*$ *and* $a^*a = b^*a$,

(3) $aa^* = ab^*$ *and* $a^*a = a^*b$,

(4) $a = aa^*b = ba^*a$.

Result 1.3. [4] [5] *The relation $\leq$ on a regular $*$-semigroup, defined above, is a partial order on S which preserves the unary operation. If S is a locally inverse $*$-semigroup, then $\leq$ is compatible.*

We call the partial order $\leq$, defined above, *the natural order* on S.

Let S be a locally inverse $*$-semigroup. In [5], we introdued a new partial product $\cdot$ on S, which is called a *restricted product*, as follows:

$$a \cdot b = \begin{cases} ab & ab \in R_a \cap L_b \\ \text{undefined} & \text{otherwise} \end{cases}$$

where R_a and L_a denote the $\mathcal{R}$-class and the $\mathcal{L}$-class containing a, respectively.

Lemma 1.4. *$ab \in R_a \cap L_b$ if and only if $a^*abb^*a^*a = a^*a$ and $bb^*a^*abb^* = bb^*$.*

Proof. Let $ab \in R_a \cap L_b$. Then there exists an idempotent $e \in L_a \cap R_b$. By the Result 1.1 (1), $e = bb^*a^*a$. Then we have that $a^*abb^*a^*a = a^*a$ and $bb^*a^*abb^* = bb^*$. The converse is clear. □

Result 1.5. [5] *Let S be a regular $*$-semigroup.*

(1) *Let $x \in S$ and $e \in P(S)$ such that $e \leq x^*x$. Then $a = xe$ is the unique element in S such that $a \leq x$ and $a^*a = e$.*

(2) *Let $x \in S$ and $e \in P(S)$ such that $e \leq xx^*$. Then $a = ex$ is the unique element in S such that $a \leq x$ and $aa^* = e$.*

(3) *For any elemants $x, y \in S$, $xy = a \cdot b$ where $a = xe$, $b = fy$, $e = x^*xyy^*x^*x$ and $f = yy^*x^*xyy^*$.*

In Section 2, we define an *ordered $*$-groupoid* and give fundamental properties of ordered $*$-semigroups.

In Section 3, we introduce a *loally inductive $*$-groupoid* and its product $\otimes$, which is called a *pseudoproduct*, and characterize a locally inverse $*$-semigroup.

2 Ordered $*$-groupoids Let G be a non-empty set with a partial product $\cdot$, a unary operation $*$ and a partial order $\leq$. We simply write ab instead of $a \cdot b$. If ab is defined for $a, b \in G$, we sometimes write $\exists ab$. An element $e \in G$ is called an *idempotent* if $\exists ee$ and $ee = e$. If an idempotent e satisfies $e^* = e$, it is called a *projection*. Denote the sets of idempotents and projections of G by $E(G)$ and $P(G)$, respectively.

If G satisfies the following axioms, it is called an *ordered $*$-groupoid.*

(A1) $a(bc)$ exists if and only if $(ab)c$ exists, in which case they are equal.

(A2) $a(bc)$ exists if and only if ab and bc exist.

(A3) $(a^*)^* = a$.

(A4) If ab exists, then b^*a^* exists and $(ab)^* = b^*a^*$.

(A5) For any $a \in G$, a^*a exists and a^*a is the unique projection of G such that $\exists a(a^*a)$ and $a(a^*a) = a$. We write $a^*a = d(a)$ and call it the *domain identity.*

(A6) $a \leq b$ implies $a^* \leq b^*$.

(A7) For $a, b, c, d \in G$, if $a \leq b$, $c \leq d$, $\exists ac$ and $\exists bd$, then $ac \leq bd$.

(A8) Let $a \in G$ and $e \in P(G)$ such that $e \leq d(a)$. Then there exists a unique element $(a|e)$, called the *restriction* of a to e, such that $(a|e) \leq a$ and $d(a|e) = e$.

(A9) $E(G)$ is an order ideal.

Proposition 2.1. *If S is a locally inverse $*$-semigroup, then $S(\cdot, *, \leq)$ is an ordered $*$-groupoid, where $\cdot$ denotes the restricted product of S defined in Section 1.*

Proof. Assume that $a \cdot (b \cdot c)$ exists. By Lemma 1.4, we have $b^*bcc^*b^*b = b^*b$, $cc^*b^*bcc^* = cc^*$, $a^*a(bc)(bc)^*a^*a = a^*a$ and $(bc)(bc)^*a^*a(bc)(bc)^* = (bc)(bc)^*$. Then

$$a^*a = a^*a(b(b^*bcc^*b^*b)b^*)a^*a = a^*ab(b^*b)b^*a^*a = a^*abb^*a^*a$$

$$bb^* = b(b^*b)b^* = b(b^*bcc^*b^*b)b^* = (bc)(bc)^* = (bc)(bc)^*a^*a(bc)(bc)^* = bb^*a^*abb^*$$

$$(ab)^*(ab) = b^*a^*ab = b^*(a^*abcc^*b^*a^*a)b = (ab)^*(ab)cc^*(ab)^*(ab)$$

$$cc^* = cc^*b^*bcc^* = cc^*b^*(bb^*)bcc^* = cc^*b^*(bb^*a^*abb^*)bcc^* = cc^*(ab)^*(ab)cc^*$$

Thus $a \cdot b$ and $(a \cdot b) \cdot c$ exist. It is obvious that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. Thus we have (A1) and (A2).

Since S is a locally inverse $*$-semigroup, (A3), (A4), (A5), (A6) and (A7) holds. Axiom (A8) follows from Result 1.5 (1).

Let $e \in E(S)$ and $a \in S$ such that $a \leq e$. By the definition $\leq$, there exist $p, q \in P(S)$ such that $a = pe = eq$. Then

$$\begin{aligned} (a^*a)(aa^*)(a^*a) &= (pe)^*(pe)(eq)(eq)^*(pe)^*(pe) \\ &= e^*peqe^*ppe \\ &= (pe)^*(pe)(pe)^*(pe) \\ &= a^*a \end{aligned}$$

Similarly, $(aa^*)(a^*a)(aa^*) = aa^*$. Thus $a \cdot a$ exists. Moreover, $a \cdot a = aa = (pe)(eq) = p(eq) = p(pe) = pe = a$, and we have (A9). Hence $S(\cdot, *, \leq)$ is an ordered $*$-groupoid. □

Proposition 2.2. *Let G be an ordered $*$-groupoid. Then we have the following.*

(1) *For any $a \in G$, aa^* exists and aa^* is the unique element of $P(G)$ such that $\exists(aa^*)a$ and $(aa^*)a = a$. We write $aa^* = r(a)$ and call it the* range identity.

(2) *Let $a \in G$ and $e \in P(G)$ such that $e \leq r(a)$. Then there exists a unique element $(e|a)$, called the* corestriction *of a to e, such that $(e|a) \leq a$ and $r(e|a) = e$.*

(3) *$\exists ab$ if and only if $\exists d(a)r(b)$, $\exists r(b)d(a)$ and $d(a)r(b)d(a) = d(a)$ and $r(b)d(a)r(b) = r(b)$.*

(4) *If* $\exists ab$*, then* $d(ab) = d(b)$ *and* $r(ab) = r(a)$.

(5) *If* $\exists ab$*, then* $d(a)r(b)$ *and* $r(b)d(a)$ *are idempotents.*

(6) *For any* $e \in E(S)$*, there exist* $p, q \in P(S)$ *such that* $e = pq$.

(7) *If* $a \leq b$*, then* $d(a) \leq d(b)$ *and* $r(a) \leq r(b)$.

(8) *If* $\exists ab$ *and* e *is a projection such that* $e \leq d(ab)$*, then*

$$(ab|e) = (a|(r(d(a)r(b)|r(b|e)))(b|e).$$

(9) *If* $\exists ab$ *and* e *is a projection such that* $e \leq r(ab)$*, then*

$$(e|ab) = (e|a)(d((d(e|a)|d(a)r(b))|b).$$

(10) *If* $c \leq ab$*, then there exist* a' *and* b' *such that* $\exists a'b'$*,* $a' \leq a$*,* $b' \leq b$ *and* $c = a'b'$.

(11) *If* $e \leq f \leq d(a)$*, then* $(a|e) \leq (a|f) \leq a$.

(12) *If* $e \leq f \leq r(a)$*, then* $(e|a) \leq (f|a) \leq a$.

(13) *Let* $a, b \in G$ *and* $e, f \in P(G)$ *such that* $a \leq b$*,* $e \leq f$*,* $e \leq d(a)$ *and* $f \leq d(b)$*. Then* $(a|e) \leq (b|f)$.

(14) $P(G)$ *is an order ideal.*

Proof. (1) It immediately follows from Axioms (A1), (A3), (A4) and (A5) that $aa^* \in P(S)$ and $(aa^*)a = a$. To show the uniqueness, let $e \in P(S)$ such that $ea = a$. Then $a^*e = (ea)^* = a^*$. On the other hand, by (A5), $aa^* = (a^*)^*a^*$ is the unique projection such that $a^*(aa^*) = a^*$, and hence $e = aa^*$.

(2) Let $a \in G$ and $e \in P(G)$ such that $e \leq r(a)$. Then $e \leq d(a^*)$, and by (A8), there exists $(a^*|e)$ such that $(a^*|e) \leq a^*$ and $d(a^*|e) = e$. Let $(e|a) = (a^*|e)^*$. By (A6), $(e|a) = (a^*|e)^* \leq (a^*)^* = a$. Moreover, $r(e|a) = (e|a)(e|a)^* = (a^*|e)^*((a^*|e)^*)^* = d(a^*|e) = e$. To show the uniqueness, assume that $b \leq a$ and $r(b) = e$. Then $b^* \leq a^*$, by (A6), and $d(b^*) = e$. By the uniqueness of (A8), $b^* = (a^*|e)$, and hence $b = (b^*)^* = (a^*|e)^* = (e|a)$.

(3) Assume that $\exists ab$. By (1) above, $r(ab) = abb^*a^*$ is the unique projection such that $r(ab)ab = ab$. On the other hand, aa^* is a projection such that $(aa^*)ab = ab$. Then $abb^*a^* = aa^*$, and we have

$$d(a) = a^*a = a^*(aa^*)a = a^*(abb^*a^*)a = d(a)r(b)d(a).$$

Similarly, we have $r(b)d(a)r(b) = r(b)$. The converse is obvious.

(4) Let $\exists ab$. By (3), we have

$$d(ab) = (ab)^*ab = b^*(bb^*a^*abb^*)b = b^*(r(b)d(a)r(b))b = b^*r(b)b = d(b).$$

Similarly we have $r(ab) = r(a)$.

(5) This immediately follows from (3).

(6) Let e be any idempotent. Then it is obvious that ee^* and e^*e are projections. Since $e^* = (ee)^* = e^*e^*$, we have $e = ee^*e = (ee^*)(e^*e)$.

(7) This immediately follows from (A6) and (A7).

(8) Let $\exists ab$ and $e \in P(S)$ such that $e \leq d(ab)$. By (4), we have $e \leq d(ab) = d(b)$, and hence the restriction $(b|e)$ is defined. Since $(b|e) \leq b$, we have $r(b|e) \leq r(b) = r(b)d(a)r(b) = d(d(a)r(b))$, and hence $(d(a)r(b)|r(b|e))$ exists. Since $(d(a)r(b)|r(b|e)) \leq d(a)r(b)$, we have $r(d(a)r(b)|r(b|e)) \leq r(d(a)r(b)) = d(a)r(b)d(a) = d(a)$. Thus $(a|r(d(a)r(b)|r(b|e)))$ exists. By (5) and (A9), $(d(a)r(b)|r(b|e)) = f$, say, is an idempotent. Then

$$d(a|r(d(a)r(b)|r(b|e))) = r(d(a)r(b)|r(b|e)) = r(f) = ff^*$$
$$r(b|e) = d(d(a)r(b)|r(b|e)) = d(f) = f^*f$$

Since f is an idempotent, $(ff^*)(f^*f)(ff^*) = ff^*$ and $(f^*f)(ff^*)(f^*f) = f^*f$, and hence $(a|r(d(a)r(b)|r(b|e)))(b|e)$ exists. Since $(a|r(d(a)r(b)|r(b|e))) \leq a$ and $(b|e) \leq b$, we have $(a|r(d(a)r(b)|r(b|e)))(b|e) \leq ab$ by (A7). On the other hand, $d((a|r(d(a)r(b)|r(b|d(c)))) = d(b|e) = e$ by (4). Thus it follows from (A8) that $(ab|e) = (a|r(d(a)r(b)|r(b|e)))(b|e)$.

(9) Similar to the proof of (8).

(10) Let $c \leq ab$. By (7), we have $d(c) \leq d(ab)$, and hence $(ab|d(c))$ exists. Since $d(ab|d(c)) = d(c)$ and $(ab|d(c)) \leq ab$, we have $c = (ab|d(c))$. It follows from (8) that

$$(ab|d(c)) = (a|r(d(a)r(b)|r(b|d(c)))(b|d(c)).$$

Put $a' = (a|(r(d(a)r(b)|r(b|e)))(b|e)$ and $b' = (b|d(c))$. Then it is obvious that $a' \leq a$, $b' \leq b$ and $c = (ab|d(c)) = a'b'$.

(11) Let $e \leq f \leq d(a)$. Obviously, $(a|e)$ and $(a|f)$ exist. Since $e \leq d(a|f)$, $((a|f)|e)$ exists. On the other hand, $d(a|e) = e$ and $(a|e) \leq a$. Then $((a|f)|e) = (a|e)$, and hence $(a|e) \leq (a|f)$.

(12) Similar to the proof of (11).

(13) It follows from (11) that $(b|e) \leq (b|f)$. On the other hand, $(a|e), (b|e) \leq b$ and $d(a|e) = d(b|e)$. By Axiom (A8), we have $(a|e) = (b|e)$, and hence $(a|e) \leq (b|f)$.

(14) Let $a \in G$ and $e \in P(G)$ such that $a \leq e$. By Axioms (A6) and (A7), we have $a^*a \leq e^*e = e$. Since $d(a) = a^*a = d(a^*a)$, it follows from Axiom (A8) that $a = a^*a \in P(G)$. $\square$

3 Locally inductive $*$-groupoids An ordered $*$-groupoid G is called a *locally inductive $*$-groupoid* if it satisfies

(LG) For any $e, f \in P(G)$, there exists the maximum element in $< e, f >= \{(g, h) \in P(G) \times P(G) : g \leq e,\ h \leq f \text{ and } \exists gh\}$.

Proposition 3.1. *If S be a locally inverse $*$-semigroup. then $S(\cdot, *, \leq)$ is a locally inductive $*$-groupoid.*

Proof. Let S be a locally inverse $*$-semigroup. Let $e, f \in P(S)$. Then we can easily see that (efe, fef) is the maximum element in $< e, f >$. $\square$

The locally inductive $*$-groupoid associated with S, above, is denoted by $\mathbf{G}(S)$.

Let $G(\cdot, *, \leq)$ be a locally inductive $*$-groupoid. For any $a, b \in G$, there exists the maximun element (e, f) in $< d(a), r(b) > = \{(g, h) \in P(S) \times P(S) : g \leq d(a),\ h \leq r(b),\ \exists gh\}$. We define a new product $\otimes$ on G as follows:

$$a \otimes b = (a|e)(f|b),$$

and we call it a *pseudoproduct* of a and b.

Proposition 3.2. *For a locally inductive $*$-groupoid G, $G(\otimes, *)$, defined above, is a locally inverse $*$-semigroup, which is denoted by* $\mathbf{S}(G)$.

Proof. First, we show that $(a\otimes b)\otimes c = a\otimes(b\otimes c)$ for any $a, b, c \in G$. By Axiom (LG), there exists the maximum element (e, f) in $< d(a\otimes b), r(c) >$ and $(a\otimes b)\otimes c = (a\otimes b|e)(f|c)$. Moreover, there exist $g, h \in P(S)$ such that (g, h) is the maximum element of $< d(a), r(b) >$, and $a\otimes b = (a|g)(h|b)$. Since $(a\otimes b)|e) \leq a\otimes b = (a|g)(h|b)$, there exist $a' \leq (a|g)$ and $b' \leq (h|b)$ such that $(a\otimes b|e) = a'b'$, by Proposition 2.2(10). Thus

$$(a\otimes b)\otimes c = (a'b')(f|c) = a'(b'(f|c)).$$

It follows from $\exists b'(f|c)$ that $\exists d(b')r(f|c)$. Since $b' \leq (h|b) \leq b$ and $(f|c) \leq c$, we have $(d(b'), r(f|c)) \in< d(b), r(c) >$. Thus $b'(f|c) \leq b\otimes c$. Similarly, $\exists d(a')r(b'(f|c))$, $a' \leq a$ and $b'(f|c) \leq b\otimes c$ imply that $(a\otimes b)\otimes c = a'(b'(f|c)) \leq a\otimes(b\otimes c)$. Similarly, we have $a\otimes(b\otimes c) \leq (a\otimes b)\otimes c$, and hence $G(\otimes)$ is a semigroup.

It is clear that $G(\otimes, *)$ is a regular $*$-semigroup. To show that $G(\otimes, *)$ is a locally inverse $*$-semigroup, it is sufficient to prove that, for any $e \in P(G)$, $P(e\otimes G\otimes e)$ ia a semilattice. Let $f, g \in P(e\otimes G\otimes e)$. Then it is clear that $f \leq e$ and $g \leq e$. There exists the maximum element (i, j) in $< g, h >$ such that $f\otimes g = (f|i)(j|g) = ij$. Since $i \leq f \leq e$ and $j \leq g \leq e$, we have $ij \leq e$. By Proposition 2.2 (14), $ij \in P(G)$, and so $ij = (ij)^* = j^*i^* = ji$. Thus $f\otimes g = ij = ji = g\otimes f$, and hence $P(e\otimes G\otimes e)$ ia a semilattice. □

Theorem 3.3. (1) *For a locally inverse $*$-semigroup S, we have* $\mathbf{S}(\mathbf{G}(S)) = S$.

(2) *For a locally inductive $*$-groupoid $G(\cdot, *, \leq)$, we have* $\mathbf{G}(\mathbf{S}(G(\cdot, *, \leq))) = G(\cdot, *, \leq)$.

Proof. (1) Let a and b be any elements of S. Then

$$a\otimes b = (a|e)\cdot(f|b),$$

where (e, f) is the maximum element of $< a^*\cdot a, b\cdot b^* >$. Then $e \leq a^*a$, $f \leq bb^*$, $efe = e$ and $fef = f$, since $x\cdot y = xy$ if $x\cdot y$ exists. By Result 1.5, $(a|e) = ae$, $(f|b) = fb$, $e = a^*abb^*a^*a$, $f = bb^*a^*abb^*$ and $a\otimes b = ae\cdot fb = a(a^*abb^*a^*a)(bb^*a^*abb^*)b = ab$. Hence we have $\mathbf{S}(\mathbf{G}(S)) = S$.

(2) First, we show that the partial order $\preceq$, say, on $\mathbf{G}(\mathbf{S}(G(\cdot, *, \leq)))$ is equal to $\leq$. We remark that $a\otimes a^* = a\cdot a^*$ and $a^*\otimes a = a^*\cdot a$ for any $a \in G$. Assume that $a \preceq b$. It follows from Result 1.3 that $a^*\cdot a = a^*\otimes a \preceq b^*\otimes b = b^*\cdot b$. By Result 1.2,

$$a^*\cdot a = (a^*\cdot a)\otimes(a^*\cdot a)^*\otimes(b^*\cdot b) = (b^*\cdot b)\otimes(a^*\cdot a)^*\otimes(a^*\cdot a).$$

Then $a^*\cdot a = (a^*\cdot a)\otimes(b^*\cdot b) = (b^*\cdot b)\otimes(a^*\cdot a)$. Let (e, f) be the maximum element of $< a^*\cdot a,\ b^*\cdot b >$. It is clear that (f, e) is the maximum element of $< b^*\cdot b,\ a^*\cdot a >$. Then $e \leq a^*\cdot a$, $f \leq b^*\cdot b$, $e\cdot f\cdot e = e$ and $f\cdot e\cdot f = f$. Moreovere,

$$\begin{aligned} a^*\cdot a &= (a^*\cdot a|e)\cdot(f|b^*\cdot b) = e\cdot f \\ &= (b^*\cdot b|f)\cdot(e|a^*\cdot a) = f\cdot e \end{aligned}$$

Then $e\cdot f = f\cdot e$ and so $a^*\cdot a = e\cdot f = e\cdot f\cdot e = e \leq a^*\cdot a$. Thus we have $a^*\cdot a = e = f$. By using Result 1.2 again, $a = b\otimes a^*\otimes a = b\otimes a^*\cdot a = (b|f)\cdot(e|a^*\cdot a) = (b|e)\cdot e = (b|e)$. Hence we have $a \leq b$.

Conversely, let $a \leq b$ in $G(\cdot, *, \leq)$. Then $d(a) \leq d(b)$, $r(a) \leq r(b)$ and $a = (b|d(a)) = (r(a)|b)$. Since $(d(a), d(a))$ is the maximum element of $< d(b),\ r(d(a)) >$ and $(r(a), r(a))$ is the maximum element of $< d(r(a)),\ r(b) >$, we have

$$\begin{aligned} b\otimes a^*\otimes a &= b\otimes a^*\cdot a = (b|d(a))\cdot d(a) = (b|d(a)) = a \\ a\otimes a^*\otimes b &= a\cdot a^*\otimes b = r(a)\cdot(r(a)|b) = (r(a)|b) = a \end{aligned}$$

Thus we have $a \preceq b$.

Next, we prove that, for $a, b \in G$, $a \odot b$ exists if and only if $a \cdot b$, where $\odot$ denotes the restricted product of $\mathbf{G}(\mathbf{S}(G(\cdot, *, \leq)))$. Assume that $a \odot b$ exists. Then $(a^* \cdot a) \otimes (b \cdot b^*) \otimes (a^* \cdot a) = a^* \cdot a$ and $(b \cdot b^*) \otimes (a^* \cdot a) \otimes (b \cdot b^*) = b \cdot b^*$, since $a^* \otimes a = a^* \cdot a$ and $b \otimes b^* = b \cdot b^*$. Let (e, f) be the maximum element of $< a^*a, bb^* >$. Then $e \leq a^*a$, $f \leq bb^*$, $e \cdot f \cdot e = e$ and $f \cdot e \cdot f = f$. Thus

$$a^* \cdot a = (a^* \cdot a) \otimes (b \cdot b^*) \otimes (a^* \cdot a) = (a^* \cdot a|e) \cdot (f|b \cdot b^*) \cdot (e|a^* \cdot a) = e \cdot f \cdot e = e \leq a^* \cdot a$$

Hence $a^* \cdot a = e$. Similarly, we have $b \cdot b^* = f$. So $(a^* \cdot a) \cdot (b \cdot b^*) \cdot (a^* \cdot a) = a^* \cdot a$ and $(b \cdot b^*) \cdot (a^* \cdot a) \cdot (b \cdot b^*) = b \cdot b^*$, and hence $a \cdot b$ exists. The converse is clear. Now, we have $G(\odot, *, \preceq) = G(\cdot, *, \leq)$. □

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