THE FREENESS OF MODULES OF MIXED SPLINES

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Received December 20, 2001

Dedicated to Professor Masami Ito on his 60th birthday

Abstract. For a d-dimensional simplicial complex $\Delta \subset \mathbb{R}^d$ such that Δ and all its links are pseudomanifolds, we consider the module $C^\alpha(\Delta)$ of mixed splines. In particular, we study the freeness of the module $C^\alpha(\widehat{\Delta})$ for a triangulation $\Delta \subset \mathbb{R}^2$ of a topological disk and for a non-negative integer vector α of length $f_1^0(\Delta)$, where $\widehat{\Delta} \subset \mathbb{R}^3$ is the join of Δ with the origin in $\mathbb{R}^3$ and $f_1^0(\Delta)$ is the number of interior edges in Δ. We completely characterize Δ for which $C^\alpha(\widehat{\Delta})$ is free for any non-negative integer vector α. Moreover, we obtain a method for determining whether $C^\alpha(\widehat{\Delta})$ is free for a triangulation $\Delta \subset \mathbb{R}^2$ of a topological disk which has a totally interior edge, and for a generic non-negative integer vector α.

Introduction. Let $\Delta \subset \mathbb{R}^d$ be a d-dimensional simplicial complex such that Δ and all its links are pseudomanifolds. We define $C^r(\Delta)$ to be the set of piecewise polynomial functions on Δ which are continuously differentiable of order r. The elements of $C^r(\Delta)$ are also known as C^r-splines. Such functions are used in the finite element method for solving partial differential equations, and play an important role in computer-aided design and computer graphics.

Fundamental problems in spline theory are to determine the dimension of the vector space $C_k^r(\Delta)$ over $\mathbb{R}$ which consists of C^r-splines of degree at most k, to determine whether the module $C^r(\Delta)$ is free, and to determine whether the module $C^r(\widehat{\Delta})$ is free, where $\widehat{\Delta} \subset \mathbb{R}^{d+1}$ is the join of Δ with the origin in $\mathbb{R}^{d+1}$. The algebraic structure of $C^r(\Delta)$, including these problems, has been studied by [1], [2], [3], [4], [7], [8], [9], and [10]. In this paper, we consider the set $C^\alpha(\Delta)$ of mixed splines, which are obtained by extending C^r-splines.

We denote the set of i-faces of Δ by Δ_i, the set of interior i-faces of Δ by Δ_i^0 (all d-faces are considered interior), and the set of interior faces of Δ by Δ^0. Moreover, $f_i(\Delta)$ denotes the number of i-faces of Δ, and $f_i^0(\Delta)$ denotes the number of interior i-faces of Δ. Let $t = f_d(\Delta)$. We fix an ordering $\sigma_1, \dots, \sigma_t$ of the elements in Δ_d. For this ordering, we can represent F in $C^\alpha(\Delta)$ as a t-tuple $F = (f_1, \dots, f_t)$ of polynomials, where $f_i = F|_{\sigma_i}$ for each $i = 1, \dots, t$, and we can view $C^\alpha(\Delta)$ as a module over the polynomial ring in d variables. Similarly, $C^\alpha(\widehat{\Delta})$ is a module over the polynomial ring in $(d+1)$ variables. It is natural to consider the above fundamental problems for mixed splines. One of these problems is the determination of the dimension of $C_k^\alpha(\Delta)$ as a vector space over $\mathbb{R}$, where $C_k^\alpha(\Delta)$ is the set of $F = (f_1, \dots, f_t)$ in $C^\alpha(\Delta)$ such that, for each i, f_i has degree at most k. In [3], Billera and Rose showed how the theory of Gröbner bases can be used to compute the dimension of $C_k^r(\Delta)$ as a vector space over $\mathbb{R}$ as well as the explicit basis for this vector space (see also [5] for the theory of Gröbner bases). In the same way, we can use the theory of Gröbner bases to compute the dimension of $C_k^\alpha(\Delta)$ as a vector space over $\mathbb{R}$ as well as the explicit basis for this vector space. Moreover, in [6], Geramita and Schenck derived a formula for the dimension of $C_k^\alpha(\Delta)$ in high degree in the case $d = 2$.

2000 *Mathematics Subject Classification.* 65D07,13C10,55N35.
Key words and phrases. mixed spline, free module.

We focus on the problem of the freeness of the module $C^{\alpha}(\widehat{\Delta})$. This problem is useful, since if $C^{\alpha}(\widehat{\Delta})$ is free, then $C^{\alpha}(\Delta)$ has a reduced basis (see [4] for the materials on reduced bases). In this paper, we study the freeness of the module $C^{\alpha}(\widehat{\Delta})$ for a triangulation $\Delta \subset \mathbb{R}^2$ of a topological disk.

We call an edge $\tau \in \Delta$ *totally interior*, if both vertices of τ are interior vertices. We say that $\alpha = (\alpha_{\tau_1}, \ldots, \alpha_{\tau_e}) \in \mathbb{Z}^e_{\geq 0}$ $(e = f^0_1(\Delta) = f^0_2(\widehat{\Delta}))$ is *generic* if $\alpha_{\tau_i} \neq \alpha_{\tau_j}$ for any $v \in \Delta^0_0$ and for every pair $\tau_i, \tau_j \in \Delta^0_1$ such that $v \in \tau_i$ and $v \in \tau_j$. For each $\tau_i \in \Delta^0_1$, let $l_{\tau_i} \in R = \mathbb{R}[x, y, z]$ be the homogeneous linear polynomial defining the plane containing $\widehat{\tau}_i \subset \mathbb{R}^3$, where $\widehat{\tau}_i$ is the convex hull of τ_i and the origin in $\mathbb{R}^3$. Moreover, for each $v \in \Delta^0_0$, we set $H_v := \{l_{\tau_j}^{\alpha_{\tau_j}+1} : v \in \tau_j\}$ and construct the set $L_v \subset H_v$ in the following manner. If there are $l_{\tau_p}^{\alpha_{\tau_p}+1}, l_{\tau_q}^{\alpha_{\tau_q}+1} \in H_v$ such that $l_{\tau_p} = l_{\tau_q}$ and $\alpha_{\tau_p} \leq \alpha_{\tau_q}$, then we remove $l_{\tau_q}^{\alpha_{\tau_q}+1}$ from H_v. Moreover, for each totally interior edge $\tau \in \Delta^0_1$ and for each vertex $v_\tau \in \Delta^0_0$ of τ, we set $K_{v_\tau} := \{ \tau_j \in \Delta^0_1 : l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau}, \alpha_{\tau_j} < \alpha_\tau \}$ and $m_{v_\tau} := |K_{v_\tau}|$.

The main results in this paper are as follows:

Theorem 3.4. *The module $C^{\alpha}(\widehat{\Delta})$ over R is free for all $\alpha \in \mathbb{Z}^e_{\geq 0}$ if and only if Δ possesses no totally interior edge.*

Theorem 3.11. *Let $\Delta \subset \mathbb{R}^2$ be a triangulation of a topological disk which has at least one totally interior edge, and let $\alpha \in \mathbb{Z}^e_{\geq 0}$ be generic. Then, $C^{\alpha}(\widehat{\Delta})$ is a free R-module if and only if, for any totally interior edge $\tau \in \Delta^0_1$, there exists a vertex v_τ of τ such that either* (i) *or* (ii) *below is satisfied:*

(i) $@l_{\tau}^{\alpha_\tau+1} \notin L_{v_\tau}$;

(ii) $@l_{\tau}^{\alpha_\tau+1} \in L_{v_\tau}$, $m_{v_\tau} \geq 2$, *and*

$$\alpha_\tau + 1 > \frac{\sum_{\tau_j \in K_{v_\tau}} (\alpha_{\tau_j} + 1) - m_{v_\tau}}{m_{v_\tau} - 1}.$$

This paper is organized as follows. First, in Section 1, we introduce some preliminary notions on simplicial complexes. Second, in Section 2, we define the set $C^{\alpha}(\Delta)$ of mixed splines and describe some algebraic properties of $C^{\alpha}(\Delta)$ and $C^{\alpha}(\widehat{\Delta})$. Finally, in Section 3, we focus on the freeness of $C^{\alpha}(\widehat{\Delta})$ for a triangulation $\Delta \subset \mathbb{R}^2$ of a topological disk. In particular, we prove our main results above.

1 Preliminaries. A *simplicial complex* in $\mathbb{R}^d$ is a finite set Δ of simplices in $\mathbb{R}^d$ such that

(i) if $\sigma \in \Delta$, then each face of σ is in Δ;

(ii) if $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a face of σ and of τ.

If Δ is a simplicial complex in $\mathbb{R}^d$, each simplex of Δ is called a *face* of Δ. Moreover, the *dimension* of Δ is defined to be

$$\dim \Delta := \max\{\dim \sigma : \sigma \in \Delta\}.$$

Let Δ be a simplicial complex in $\mathbb{R}^d$ and let σ be a face of Δ. Then, the *link* of σ in Δ is defined by

$$\text{link}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \text{ and } \text{CONV}(\sigma \cup \tau) \in \Delta\}.$$

Moreover, we set $\text{link}_\Delta(\emptyset) = \Delta$.

We say that a d-dimensional simplicial complex Δ in $\mathbb{R}^d$ is a *pseudomanifold* if the following conditions are satisfied:

(i) each face in Δ such that its dimension is less than or equal to $d-1$ is a face of some d-face in Δ;

(ii) for any two d-faces σ, $\sigma' \in \Delta$, there is a sequence of d-faces

$$\sigma = \sigma_1, \sigma_2, \ldots, \sigma_m = \sigma'$$

such that each $\sigma_i \cap \sigma_{i+1}$ is a $(d-1)$-face of Δ for each i, $1 \le i \le m-1$.

2 The module $C^\alpha(\Delta)$ and its algebraic properties. In this section, let Δ be a d-dimensional simplicial complex in $\mathbb{R}^d$ such that Δ and all its links are pseudomanifolds. Let $R = \mathbb{R}[x_1, \ldots, x_d]$. We now define $C^r(\Delta)$ more explicitly.

Definition 2.1. For $r \in \mathbb{Z}_{\ge 0}$ and $\Delta \subset \mathbb{R}^d$, $C^r(\Delta)$ is the set of functions $F : |\Delta| \longrightarrow \mathbb{R}$ such that

(i) $F|_\sigma$ is given by a polynomial in R for all $\sigma \in \Delta_d$;

(ii) F is continuously differentiable of order r.

Let $t = f_d(\Delta)$. Given an ordering $\sigma_1, \ldots, \sigma_t$ of Δ_d, $G \in C^r(\Delta)$ can be represented as a t-tuple of polynomials in R, i.e., $G = (g_1, \ldots, g_t)$, where each g_i is just $G|_{\sigma_i}$. If $\sigma_i, \sigma_j \in \Delta_d$ are adjacent (i.e., $\sigma_i \cap \sigma_j \in \Delta^0_{d-1}$), let $l_\tau \in R$ be the linear polynomial defining the affine hyperplane containing $\tau = \sigma_i \cap \sigma_j \in \Delta^0_{d-1}$.

Proposition 2.2 ([3, Corollary 1.3])**.** *Let F be a piecewise polynomial function on $\Delta \subset \mathbb{R}^d$, and for each i, $1 \le i \le t$, let $f_i = F|_{\sigma_i} \in R$. Then $F = (f_1, \ldots, f_t) \in C^r(\Delta)$ if and only if, for every adjacent pair $\sigma_i, \sigma_j \in \Delta_d$, $f_i - f_j \in (l_\tau^{r+1})$, where $\tau = \sigma_i \cap \sigma_j \in \Delta^0_{d-1}$.*

By Proposition 2.2, the elements of $C^r(\Delta)$ are piecewise polynomial functions $F = (f_1, \ldots, f_t)$ such that, for every adjacent pair $\sigma_i, \sigma_j \in \Delta_d$, the partial derivatives up to order r of f_i and f_j agree at every point in $\tau = \sigma_i \cap \sigma_j \in \Delta^0_{d-1}$.

Mixed splines are obtained by extending C^r-splines. Let $e = f^0_{d-1}(\Delta)$. We fix an ordering $\tau_1, \ldots, \tau_e$ of Δ^0_{d-1}. We now define mixed splines.

Definition 2.3. For $\Delta \subset \mathbb{R}^d$ and $\alpha = (\alpha_{\tau_1}, \ldots, \alpha_{\tau_e}) \in \mathbb{Z}^e_{\ge 0}$, $C^\alpha(\Delta)$ is the set of functions $F : |\Delta| \longrightarrow \mathbb{R}$ such that

(i) $F|_{\sigma_i}$ is given by a polynomial in R for all $\sigma_i \in \Delta_d$;

(ii) for every adjacent pair $\sigma_i, \sigma_j \in \Delta_d$, the partial derivatives up to order α_{τ_s} of $F|_{\sigma_i}$ and $F|_{\sigma_j}$ agree at every point in $\tau_s = \sigma_i \cap \sigma_j \in \Delta^0_{d-1}$, that is, $F|_{\sigma_i} - F|_{\sigma_j} \in (l_{\tau_s}^{\alpha_{\tau_s}+1})$.

We call the elements of $C^\alpha(\Delta)$ *mixed splines.*

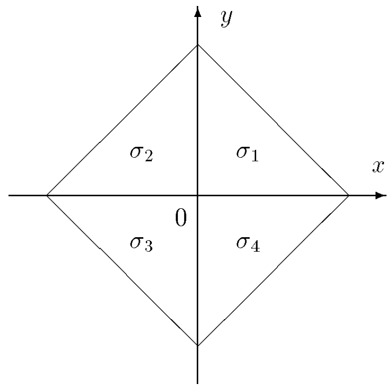

Figure 1:

Note that if $\alpha_{\tau_s} = r$ for every s, $1 \leq s \leq e$, then $C^\alpha(\Delta)$ is the set of C^r-splines, that is, $C^r(\Delta)$.

Example 2.4. Let $\Delta \subset \mathbb{R}^2$ be the simplicial complex shown in Figure 1. Let $\tau_1 = \sigma_1 \cap \sigma_2$, $\tau_2 = \sigma_2 \cap \sigma_3$, $\tau_3 = \sigma_3 \cap \sigma_4$, and $\tau_4 = \sigma_1 \cap \sigma_4$. Then $l_{\tau_1} = l_{\tau_3} = x$, and $l_{\tau_2} = l_{\tau_4} = y$. Let $\alpha = (1, 2, 3, 4)$. Then, for example, $(y^5, x^4+y^5, x^4, 0) \in C^\alpha(\Delta)$, and $(y^5, x^4+y^5, x^4, x^4) \notin C^\alpha(\Delta)$.

We now describe some important properties of $C^\alpha(\Delta)$. Let $t = f_d(\Delta)$. Fixing an ordering $\sigma_1, \ldots, \sigma_t$ of Δ_d, we can represent $F \in C^\alpha(\Delta)$ as a t-tuple of polynomials in R, i.e., $F = (f_1, \ldots, f_t)$, where $f_i = F|_{\sigma_i} \in R$ for each i. In this way, we can view $C^\alpha(\Delta)$ as a submodule of R^t. Moreover, we can easily see that $C^\alpha(\Delta)$ is a finitely generated R-module of rank t.

We say that Δ is *central* if there is some vertex $v \in \Delta$ such that every $\sigma_i \in \Delta_d$ contains v. For example, the simplicial complex in Figure 1 is central. If Δ is central, then $C^\alpha(\Delta)$ is a graded R-module.

Let $\Delta \subset \mathbb{R}^d$ and $R = \mathbb{R}[x_1, \ldots, x_d]$. We define $\widehat{\Delta} \subset \mathbb{R}^{d+1}$ in the following manner. We think of Δ as a subset of the hyperplane $x_{d+1} = 1 \subset \mathbb{R}^{d+1}$. Let $\widehat{\Delta}$ be the join of Δ with the origin in $\mathbb{R}^{d+1}$, which we define to be the complex $\Delta \cup \{\hat{\sigma} : \sigma \in \Delta\}$, where $\hat{\sigma}$ denotes the convex hull of σ and the origin in $\mathbb{R}^{d+1}$. Then, $\widehat{\Delta}$ is a $(d+1)$-dimensional simplicial complex in $\mathbb{R}^{d+1}$ such that $\widehat{\Delta}$ and all its links are pseudomanifolds. Therefore, for $\Delta \subset \mathbb{R}^d$, we can consider the set $C^\alpha(\widehat{\Delta})$. Since $\widehat{\Delta}$ is central, $C^\alpha(\widehat{\Delta})$ is a finitely generated graded $\widehat{R}$-module of rank $f_{d+1}(\widehat{\Delta}) = f_d(\Delta)$, where $\widehat{R} = \mathbb{R}[x_1, \ldots, x_{d+1}]$.

In the next section, we will focus on the problem of the freeness of the module $C^\alpha(\widehat{\Delta})$ in the case $d = 2$.

3 Conditions for $C^\alpha(\widehat{\Delta})$ to be free when $d = 2$.

Let $d = 2$ and $R = \mathbb{R}[x, y, z]$. The module $C^\alpha(\widehat{\Delta})$ over R can be free only if $\Delta \subset \mathbb{R}^2$ has genus zero. So, let $\Delta \subset \mathbb{R}^2$ be a triangulation of a topological disk. We fix an ordering $\tau_1, \ldots, \tau_e$ of Δ_1^0, where $e = f_1^0(\Delta) = f_2^0(\widehat{\Delta})$. Let $l_{\tau_j} \in R$ be the homogeneous linear polynomial defining the plane containing $\widehat{\tau}_j \subset \mathbb{R}^3$. We define a complex $\mathcal{J}$ as

$$\mathcal{J} : 0 \longrightarrow \bigoplus_{\sigma_k \in \Delta_2} \mathcal{J}(\sigma_k) \xrightarrow{\partial_2} \bigoplus_{\tau_j \in \Delta_1^0} \mathcal{J}(\tau_j) \xrightarrow{\partial_1} \bigoplus_{v_i \in \Delta_0^0} \mathcal{J}(v_i) \longrightarrow 0,$$

where $\mathcal{J}(\sigma_k) := 0$ for $\sigma_k \in \Delta_2$, $\mathcal{J}(\tau_j) := (\, l_{\tau_j}^{\alpha_{\tau_j}+1}\,)$ for $\tau_j \in \Delta_1^0$, $\mathcal{J}(v_i) := (\, l_{\tau_j}^{\alpha_{\tau_j}+1} \,:\, \tau_j \in \Delta_1^0,\, v_i \in \tau_j\,)$ for $v_i \in \Delta_0^0$, and ∂_i is the usual (relative to $\partial\Delta$) simplicial boundary map, and we define $H_*(\mathcal{J})$ to be the homology of this complex. Moreover, we define a complex $\mathcal{R}$ as

$$\mathcal{R}:\ 0 \longrightarrow R^{f_2} \xrightarrow{\partial_2} R^{f_1^0} \xrightarrow{\partial_1} R^{f_0^0} \longrightarrow 0,$$

where $\mathcal{R}(\sigma) := R = \mathbb{R}[x,y,z]$ for any $\sigma \in \Delta^0$, and we define $H_*(\mathcal{R})$ to be the homology of this complex. Let $\mathcal{R}/\mathcal{J}$ be the quotient of $\mathcal{R}$ by $\mathcal{J}$, and let $H_*(\mathcal{R}/\mathcal{J})$ be the homology of this complex. From the short exact sequence of complexes $0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}/\mathcal{J} \longrightarrow 0$, we get a long exact sequence in homology:

$$0 \to H_2(\mathcal{R}) \to H_2(\mathcal{R}/\mathcal{J}) \to H_1(\mathcal{J}) \to H_1(\mathcal{R}) \to H_1(\mathcal{R}/\mathcal{J}) \to H_0(\mathcal{J}) \to 0.$$

By the same argument as [9, Theorem 4.1], it follows that $C^\alpha(\widehat{\Delta})$ is a free R-module if and only if $H_1(\mathcal{R}/\mathcal{J}) = 0$. Moreover, since $H_1(\mathcal{R}) = 0$ if $\Delta \subset \mathbb{R}^2$ is a triangulation of a topological disk, $H_1(\mathcal{R}/\mathcal{J}) \cong H_0(\mathcal{J})$. So, it follows that $C^\alpha(\widehat{\Delta})$ is a free R-module if and only if $H_0(\mathcal{J}) = 0$. In this section, we characterize Δ and α such that $C^\alpha(\widehat{\Delta})$ can be free.

We call an edge $\tau \in \Delta$ *totally interior*, if both vertices of τ are interior vertices in Δ. For example, none of the edges in the simplicial complex in Figure 1 is totally interior. We define $K^\alpha \subset \bigoplus_{\tau\in\Delta_1^0} R\mathbf{e}_\tau$ to be the submodule generated by

$$\{\mathbf{e}_\tau : \tau \in \Delta_1^0 \text{ is not totally interior}\}$$

and

$$\left\{ \sum_{v\in\tau} a_\tau \mathbf{e}_\tau : \sum_{v\in\tau} a_\tau l_\tau^{\alpha_\tau+1} = 0,\ a_\tau \in R \right\}$$

for each $v \in \Delta_0^0$, where $\mathbf{e}_\tau \in \mathbb{R}^e$ is the vector such that the component corresponding to τ is 1 and all the other components are 0. Then, there exists an exact sequence

$$0 \longrightarrow K^\alpha \longrightarrow \bigoplus_{\tau\in\Delta_1^0} R\mathbf{e}_\tau \longrightarrow H_0(\mathcal{J}) \longrightarrow 0.$$

Putting the above argument together, we get the following result.

Proposition 3.1. *The module $C^\alpha(\widehat{\Delta})$ over R is free if and only if $\mathbf{e}_\tau \in K^\alpha$ for any $\tau \in \Delta_1^0$.*

Proof. By the above argument, $C^\alpha(\widehat{\Delta})$ is a free R-module if and only if $H_0(\mathcal{J}) = 0$. Moreover, by the above exact sequence, it follows that $H_0(\mathcal{J}) = 0$ if and only if $K^\alpha = \bigoplus_{\tau\in\Delta_1^0} R\mathbf{e}_\tau$. ∎

Let $\Delta \subset \mathbb{R}^2$ be a triangulation of a topological disk which has at least one totally interior edge. For each $v \in \Delta_0^0$, we set $H_v := \{l_{\tau_j}^{\alpha_{\tau_j}+1} \,:\, v \in \tau_j\}$ and construct the set $L_v \subset H_v$ in the following manner. If there are $l_{\tau_p}^{\alpha_{\tau_p}+1}, l_{\tau_q}^{\alpha_{\tau_q}+1} \in H_v$ such that $l_{\tau_p} = l_{\tau_q}$ and $\alpha_{\tau_p} < \alpha_{\tau_q}$, then we remove $l_{\tau_q}^{\alpha_{\tau_q}+1}$ from H_v. If there are $l_{\tau_p}^{\alpha_{\tau_p}+1}, l_{\tau_q}^{\alpha_{\tau_q}+1} \in H_v$ such that $l_{\tau_p} = l_{\tau_q}$ and $\alpha_{\tau_p} = \alpha_{\tau_q}$, then we may remove either of $l_{\tau_p}^{\alpha_{\tau_p}+1}$ and $l_{\tau_q}^{\alpha_{\tau_q}+1}$ from H_v since $l_{\tau_p}^{\alpha_{\tau_p}+1} = l_{\tau_q}^{\alpha_{\tau_q}+1}$, but we consider $l_{\tau_p}^{\alpha_{\tau_p}+1}, l_{\tau_q}^{\alpha_{\tau_q}+1}$ as distinct polynomials from a viewpoint that $l_{\tau_p}^{\alpha_{\tau_p}+1}$ is the polynomial corresponding to τ_p and $l_{\tau_q}^{\alpha_{\tau_q}+1}$ is the polynomial corresponding to τ_q. Thus, we can get distinct L_v from H_v by removing $l_{\tau_p}^{\alpha_{\tau_p}+1}$ or removing $l_{\tau_q}^{\alpha_{\tau_q}+1}$.

Proposition 3.2. *Let $\Delta \subset \mathbb{R}^2$ be a triangulation of a topological disk which has at least one totally interior edge. If the following condition holds, then $C^\alpha(\widehat{\Delta})$ is not a free R-module:*

Whatever L_v $(v \in \Delta_0^0)$ we construct, we have a totally interior edge $\tau \in \Delta_1^0$ such that

$$(1)\qquad \begin{cases} l_\tau^{\alpha_\tau+1} \notin \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \;:\; \tau_j \neq \tau,\ \alpha_{\tau_j} \leq \alpha_\tau \right), \\ l_\tau^{\alpha_\tau+1} \notin \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{w_\tau} \;:\; \tau_j \neq \tau,\ \alpha_{\tau_j} \leq \alpha_\tau \right), \end{cases}$$

where v_τ, w_τ are the vertices of τ.

Proof. By assumption, there are totally interior edges $\tau_1, \ldots, \tau_s$ which satisfy the following condition:

$\alpha_{\tau_1} = \cdots = \alpha_{\tau_s}$, $l_{\tau_1} = \cdots = l_{\tau_s}$ and for each $i = 1, \ldots, s-1$, τ_i and τ_{i+1} share a common vertex, and furthermore

$$(2)\qquad \begin{array}{rcl} l_{\tau_1}^{\alpha_{\tau_1}+1} & \notin & \left(l_\tau^{\alpha_\tau+1} : \tau \in \Delta_1^0,\ v_0 \in \tau, \tau \neq \tau_1,\ \alpha_\tau \leq \alpha_{\tau_1} \right), \\ l_{\tau_1}^{\alpha_{\tau_1}+1} = l_{\tau_2}^{\alpha_{\tau_2}+1} & \notin & \left(l_\tau^{\alpha_\tau+1} : \tau \in \Delta_1^0,\ v_1 \in \tau, \tau \neq \tau_1, \tau_2,\ \alpha_\tau \leq \alpha_{\tau_1} \right), \\ l_{\tau_2}^{\alpha_{\tau_2}+1} = l_{\tau_3}^{\alpha_{\tau_3}+1} & \notin & \left(l_\tau^{\alpha_\tau+1} : \tau \in \Delta_1^0,\ v_2 \in \tau, \tau \neq \tau_2, \tau_3,\ \alpha_\tau \leq \alpha_{\tau_2} \right), \\ & \vdots & \\ l_{\tau_{s-1}}^{\alpha_{\tau_{s-1}}+1} = l_{\tau_s}^{\alpha_{\tau_s}+1} & \notin & \left(l_\tau^{\alpha_\tau+1} : \tau \in \Delta_1^0,\ v_{s-1} \in \tau, \tau \neq \tau_{s-1}, \tau_s,\ \alpha_\tau \leq \alpha_{\tau_{s-1}} \right), \\ l_{\tau_s}^{\alpha_{\tau_s}+1} & \notin & \left(l_\tau^{\alpha_\tau+1} : \tau \in \Delta_1^0,\ v_s \in \tau, \tau \neq \tau_s,\ \alpha_\tau \leq \alpha_{\tau_s} \right), \end{array}$$

where, for $i = 1, \ldots, s-1$, v_i is the vertex which τ_i and τ_{i+1} share, v_0 is the vertex of τ_1 which is different from v_1, and v_s is the vertex of τ_s which is different from v_{s-1}.

In fact, we assume that the condition (2) does not hold for some vertex v_i. If we construct

$$\begin{array}{c} L_{v_0} = \{l_{\tau_1}^{\alpha_{\tau_1}+1}, \ldots\}, \quad \ldots \quad , L_{v_{i-1}} = \{l_{\tau_i}^{\alpha_{\tau_i}+1}, \ldots\}, \\ L_{v_{i+1}} = \{l_{\tau_{i+1}}^{\alpha_{\tau_{i+1}}+1}, \ldots\}, \quad \ldots \quad , L_{v_s} = \{l_{\tau_s}^{\alpha_{\tau_s}+1}, \ldots\}, \end{array}$$

then none of the edges $\tau_1, \ldots, \tau_s$ satisfies the condition (1) in Proposition 3.2 whatever L_{v_i} we construct. Hence, if there are not $\tau_1, \ldots, \tau_s$ as above, then we can construct the sets L_v $(v \in \Delta_0^0)$ such that the condition (1) in Proposition 3.2 does not hold for any totally interior edge. This contradicts the assumption.

For each $v_i \in \Delta_0^0$, we set

$$K_{v_i}^\alpha := \left\{ \sum_{v_i \in \tau} a_\tau \mathbf{e}_\tau : \sum_{v_i \in \tau} a_\tau l_\tau^{\alpha_\tau+1} = 0,\ a_\tau \in R \right\}.$$

For any element $\sum_{v_0 \in \tau} a_\tau \mathbf{e}_\tau$ in $K_{v_0}^\alpha$, the constant term a'_{τ_1} of $a_{\tau_1} \in R$ is 0. In fact, we assume that $a'_{\tau_1} \neq 0$. Since $\sum_{v_0 \in \tau} a_\tau \mathbf{e}_\tau \in K_{v_0}^\alpha$,

$$a_{\tau_1} l_{\tau_1}^{\alpha_{\tau_1}+1} + \sum_{v_0 \in \tau,\, \tau \neq \tau_1} a_\tau l_\tau^{\alpha_\tau+1} = 0.$$

Comparing the homogeneous parts of degree $\alpha_{\tau_1} + 1$ on both sides, we get

$$a'_{\tau_1} l_{\tau_1}^{\alpha_{\tau_1}+1} + \sum_{v_0 \in \tau,\, \tau \neq \tau_1} a'_\tau l_\tau^{\alpha_\tau+1} = 0,$$

where $a'_\tau \in R$. This contradicts the condition (2). Hence, it follows that $a'_{\tau_1} = 0$. Similarly, for any element $\sum_{v_s \in \tau} a_\tau \mathbf{e}_\tau$ in $K^\alpha_{v_s}$, the constant term a'_{τ_s} of $a_{\tau_s} \in R$ is 0.

Moreover, for any element $\sum_{v_i \in \tau} a_\tau \mathbf{e}_\tau$ in $K^\alpha_{v_i}$ ($i = 1, \ldots, s-1$), let a'_{τ_i} (resp. $a'_{\tau_{i+1}}$) be the constant term in $a_{\tau_i} \in R$ (resp. $a_{\tau_{i+1}} \in R$). Then, it holds that $a'_{\tau_i} + a'_{\tau_{i+1}} = 0$. In fact, we assume that $a'_{\tau_i} + a'_{\tau_{i+1}} \neq 0$. Since $\sum_{v_i \in \tau} a_\tau \mathbf{e}_\tau \in K^\alpha_{v_i}$, it follows that

$$a_{\tau_i} l_{\tau_i}^{\alpha_{\tau_i}+1} + a_{\tau_{i+1}} l_{\tau_{i+1}}^{\alpha_{\tau_{i+1}}+1} + \sum_{\substack{v_i \in \tau \\ \tau \neq \tau_i,\, \tau_{i+1}}} a_\tau l_\tau^{\alpha_\tau + 1} = 0.$$

Since $l_{\tau_i}^{\alpha_{\tau_i}+1} = l_{\tau_{i+1}}^{\alpha_{\tau_{i+1}}+1}$, we get

$$\left(a_{\tau_i} + a_{\tau_{i+1}}\right) l_{\tau_i}^{\alpha_{\tau_i}+1} + \sum_{\substack{v_i \in \tau \\ \tau \neq \tau_i,\, \tau_{i+1}}} a_\tau l_\tau^{\alpha_\tau + 1} = 0.$$

Comparing the homogeneous parts of degree $\alpha_{\tau_i} + 1$ on both sides,

$$\left(a'_{\tau_i} + a'_{\tau_{i+1}}\right) l_{\tau_i}^{\alpha_{\tau_i}+1} + \sum_{\substack{v_i \in \tau \\ \tau \neq \tau_i,\, \tau_{i+1}}} a'_\tau l_\tau^{\alpha_\tau + 1} = 0,$$

where $a'_\tau \in R$. This contradicts the condition (2). Hence, it follows that $a'_{\tau_i} + a'_{\tau_{i+1}} = 0$.

In this way, if, for any element in the submodule generated by $\bigcup_{i=0}^{s} K^\alpha_{v_i}$, we denote the constant term in the coefficient of $\mathbf{e}_{\tau_i}$ by a_{τ_i}, then it follows that $\sum_{i=1}^{s} a_{\tau_i} = 0$. Hence, for any element in K^α, the sum of the constant terms in the coefficients of $\mathbf{e}_{\tau_1}, \ldots, \mathbf{e}_{\tau_s}$ is also 0. Therefore, it follows that $\mathbf{e}_{\tau_1} \notin K^\alpha$. This implies that $K^\alpha \neq \bigoplus_{\tau \in \Delta_1^0} R\mathbf{e}_\tau$. Hence, by Proposition 3.1, $C^\alpha(\widehat{\Delta})$ is not a free R-module. ∎

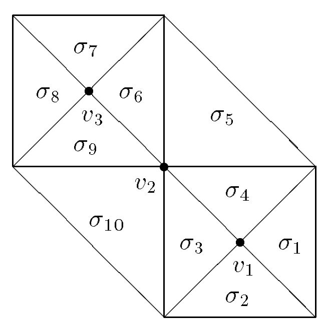

Figure 2:

Example 3.3. Let $\Delta \subset \mathbb{R}^2$ be the simplicial complex shown in Figure 2. We order the elements in Δ_1^0 as

$$\begin{aligned}
\tau_1 &= \sigma_1 \cap \sigma_2, & \tau_2 &= \sigma_2 \cap \sigma_3, & \tau_3 &= \sigma_3 \cap \sigma_4,\\
\tau_4 &= \sigma_1 \cap \sigma_4, & \tau_5 &= \sigma_4 \cap \sigma_5, & \tau_6 &= \sigma_5 \cap \sigma_6,\\
\tau_7 &= \sigma_6 \cap \sigma_7, & \tau_8 &= \sigma_7 \cap \sigma_8, & \tau_9 &= \sigma_8 \cap \sigma_9,\\
\tau_{10} &= \sigma_6 \cap \sigma_9, & \tau_{11} &= \sigma_9 \cap \sigma_{10}, & \tau_{12} &= \sigma_3 \cap \sigma_{10}.
\end{aligned}$$

If $\alpha = (2,3,1,3,2,2,3,1,2,0,3,3)$, then

$$\begin{aligned} L_{v_1} &= \{l_{\tau_3}^2, l_{\tau_2}^4\} \text{ or } \{l_{\tau_3}^2, l_{\tau_4}^4\}, \\ L_{v_2} &= \{l_{\tau_{10}}, l_{\tau_5}^3, l_{\tau_6}^3\}, \\ L_{v_3} &= \{l_{\tau_{10}}, l_{\tau_9}^3\}. \end{aligned}$$

Hence, whatever L_{v_i} $(i = 1,2,3)$ we construct, the condition in Proposition 3.2 holds for τ_{10}. Therefore, by Proposition 3.2, $C^\alpha(\widehat{\Delta})$ is not a free R-module.

If $\alpha = (2,1,1,1,2,2,1,3,1,1,3,3)$, then

$$\begin{aligned} L_{v_1} &= \{l_{\tau_2}^2, l_{\tau_3}^2\} \text{ or } \{l_{\tau_3}^2, l_{\tau_4}^2\}, \\ L_{v_2} &= \{l_{\tau_3}^2, l_{\tau_5}^3, l_{\tau_6}^3\} \text{ or } \{l_{\tau_{10}}^2, l_{\tau_5}^3, l_{\tau_6}^3\}, \\ L_{v_3} &= \{l_{\tau_7}^2, l_{\tau_{10}}^2\} \text{ or } \{l_{\tau_9}^2, l_{\tau_{10}}^2\}. \end{aligned}$$

Hence, whatever L_{v_i} $(i = 1,3)$ we construct, the condition in Proposition 3.2 holds for τ_3 if L_{v_2} is the former, and the condition in Proposition 3.2 holds for τ_{10} if L_{v_2} is the latter. Therefore, by Proposition 3.2, $C^\alpha(\widehat{\Delta})$ is not a free R-module.

We now come to the first main result in this paper.

Theorem 3.4. *The module $C^\alpha(\widehat{\Delta})$ over R is free for all $\alpha \in \mathbb{Z}_{\geq 0}^e$ if and only if Δ possesses no totally interior edge.*

Proof. If Δ does not have a totally interior edge, none of the edges in Δ is totally interior. Hence, for all $\alpha \in \mathbb{Z}_{\geq 0}^e$, it follows that $\mathbf{e}_\tau \in K^\alpha$ for any $\tau \in \Delta_1^0$. Therefore, by Proposition 3.1, $C^\alpha(\widehat{\Delta})$ is a free R-module for all $\alpha \in \mathbb{Z}_{\geq 0}^e$.

On the other hand, by Proposition 3.2, it follows that if Δ has a totally interior edge, there exists $\alpha \in \mathbb{Z}_{\geq 0}^e$ such that $C^\alpha(\widehat{\Delta})$ is not a free R-module. ∎

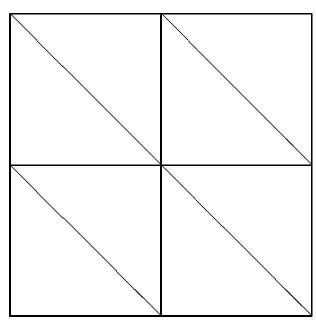

Figure 3:

Example 3.5. Let $\Delta \subset \mathbb{R}^2$ be the simplicial complex in Figure 3. By Theorem 3.4, $C^\alpha(\widehat{\Delta})$ is free for all $\alpha \in \mathbb{Z}_{\geq 0}^8$, since Δ does not have a totally interior edge.

We next consider the freeness of $C^\alpha(\widehat{\Delta})$ for a triangulation $\Delta \subset \mathbb{R}^2$ of a topological disk which has at least one totally interior edge.

Proposition 3.6. *Let $\Delta \subset \mathbb{R}^2$ be a triangulation of a topological disk which has at least one totally interior edge. If the following condition holds, then $C^\alpha(\widehat{\Delta})$ is a free R-module:*

We can construct the sets L_v ($v \in \Delta_0^0$) such that, for any totally interior edge $\tau \in \Delta_1^0$, there is a vertex $v_\tau \in \Delta_0^0$ of τ such that

$$l_\tau^{\alpha_\tau+1} \in \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \;:\; \alpha_{\tau_j} < \alpha_\tau \right).$$

Proof. By Proposition 3.1, in order to prove that $C^\alpha(\widehat{\Delta})$ is a free R-module, it suffices to show that $\mathbf{e}_\tau \in K^\alpha$ for any $\tau \in \Delta_1^0$. Since $\mathbf{e}_\tau \in K^\alpha$ for any edge $\tau \in \Delta_1^0$ which is not totally interior, we have only to show that $\mathbf{e}_\tau \in K^\alpha$ for any totally interior edge $\tau \in \Delta_1^0$.

First, we set

$$r_1 := \min\{\alpha_\tau \,:\, \tau \in \Delta_1^0 \text{ is totally interior}\},$$

and take any totally interior edge $\tau_1 \in \Delta_1^0$ such that $\alpha_{\tau_1} = r_1$. By assumption, there is a vertex $v_1 \in \Delta_0^0$ of τ_1 such that

$$l_{\tau_1}^{\alpha_{\tau_1}+1} \in \left(l_\tau^{\alpha_\tau+1} \in L_{v_1} \;:\; \alpha_\tau < \alpha_{\tau_1} \right).$$

By the choice of r_1, the edge $\tau \in \Delta_1^0$ satisfying $\alpha_\tau < \alpha_{\tau_1}$ is not totally interior. Hence, if

$$l_{\tau_1}^{\alpha_{\tau_1}+1} = \sum_{v_1 \in \tau,\, \alpha_\tau < \alpha_{\tau_1}} a_\tau \, l_\tau^{\alpha_\tau+1},$$

where $a_\tau \in R$, then it follows that

$$\mathbf{e}_{\tau_1} - \sum_{v_1 \in \tau,\, \alpha_\tau < \alpha_{\tau_1}} a_\tau \, \mathbf{e}_\tau \in K^\alpha.$$

Since τ is not totally interior, $\mathbf{e}_\tau \in K^\alpha$. Therefore, it follows that $\mathbf{e}_{\tau_1} \in K^\alpha$.

We next set

$$r_2 := \min\{\alpha_\tau \,:\, \tau \in \Delta_1^0 \text{ is a totally interior edge such that } \alpha_\tau \neq r_1\},$$

and take any totally interior edge $\tau_2 \in \Delta_1^0$ such that $\alpha_{\tau_2} = r_2$. By assumption, there is a vertex $v_2 \in \Delta_0^0$ of τ_2 such that

$$l_{\tau_2}^{\alpha_{\tau_2}+1} \in \left(l_\tau^{\alpha_\tau+1} \in L_{v_2} \;:\; \alpha_\tau < \alpha_{\tau_2} \right).$$

By the choice of r_2, the edge $\tau \in \Delta_1^0$ satisfying $\alpha_\tau < \alpha_{\tau_2}$ is not a totally interior edge or is a totally interior edge such that $\alpha_\tau = r_1$. In either case, it follows that $\mathbf{e}_\tau \in K^\alpha$. Hence, if

$$l_{\tau_2}^{\alpha_{\tau_2}+1} = \sum_{v_2 \in \tau,\, \alpha_\tau < \alpha_{\tau_2}} a_\tau \, l_\tau^{\alpha_\tau+1},$$

where $a_\tau \in R$, then it follows that

$$\mathbf{e}_{\tau_2} - \sum_{v_2 \in \tau,\, \alpha_\tau < \alpha_{\tau_2}} a_\tau \, \mathbf{e}_\tau \in K^\alpha.$$

Since $\mathbf{e}_\tau \in K^\alpha$, it follows that $\mathbf{e}_{\tau_2} \in K^\alpha$.

Since the number of totally interior edges in Δ_1^0 is finite, by the repeat of this process, it follows that $\mathbf{e}_\tau \in K^\alpha$ for any totally interior edge $\tau \in \Delta_1^0$. This implies that $C^\alpha(\widehat{\Delta})$ is a free R-module. ∎

Example 3.7. Let $\Delta \subset \mathbb{R}^2$ be the same simplicial complex as in Example 3.3. If $\alpha = (1,3,2,3,2,2,2,4,2,3,3,3)$, then

$$\begin{aligned} L_{v_1} &= \{l_{\tau_1}^2, l_{\tau_2}^4\} \text{ or } \{l_{\tau_1}^2, l_{\tau_4}^4\}, \\ L_{v_2} &= \{l_{\tau_3}^3, l_{\tau_5}^3, l_{\tau_6}^3\}, \\ L_{v_3} &= \{l_{\tau_7}^3, l_{\tau_{10}}^4\} \text{ or } \{l_{\tau_9}^3, l_{\tau_{10}}^4\}. \end{aligned}$$

Since

$$l_{\tau_3}^3 \in (\, l_{\tau_1}^2 \,), \quad l_{\tau_{10}}^4 \in (\, l_{\tau_3}^3 \,) \subset (\, l_{\tau_3}^3, l_{\tau_5}^3, l_{\tau_6}^3 \,),$$

the condition in Proposition 3.6 holds. Hence, by Proposition 3.6, $C^\alpha(\widehat{\Delta})$ is a free R-module.

We say that $\alpha = (\alpha_{\tau_1}, \ldots, \alpha_{\tau_e}) \in \mathbb{Z}_{\geq 0}^e$ is *generic* if $\alpha_{\tau_i} \neq \alpha_{\tau_j}$ for any $v \in \Delta_0^0$ and for every pair $\tau_i, \tau_j \in \Delta_1^0$ such that $v \in \tau_i$ and $v \in \tau_j$. By Proposition 3.2 and Proposition 3.6, we get the following result.

Proposition 3.8. *Let $\Delta \subset \mathbb{R}^2$ be a triangulation of a topological disk which has at least one totally interior edge, and let $\alpha \in \mathbb{Z}_{\geq 0}^e$ be generic. Then, the following conditions are equivalent:*

(i) *$C^\alpha(\widehat{\Delta})$ is a free R-module;*

(ii) *for any totally interior edge $\tau \in \Delta_1^0$, there is a vertex $v_\tau \in \Delta_0^0$ of τ such that*

$$l_\tau^{\alpha_\tau+1} \in \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \,:\, \alpha_{\tau_j} < \alpha_\tau \right).$$

Proof. First, by Proposition 3.6, it follows immediately that (ii) $\Rightarrow$ (i). Thus, we must prove that (i) $\Rightarrow$ (ii). We now assume that there is a totally interior edge $\tau \in \Delta_1^0$ such that

$$\begin{aligned} l_\tau^{\alpha_\tau+1} &\notin \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \,:\, \alpha_{\tau_j} < \alpha_\tau \right), \\ l_\tau^{\alpha_\tau+1} &\notin \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{w_\tau} \,:\, \alpha_{\tau_j} < \alpha_\tau \right), \end{aligned}$$

where v_τ, w_τ are the vertices of τ. Then, since α is generic, it follows that

$$\begin{aligned} l_\tau^{\alpha_\tau+1} &\notin \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \,:\, \tau_j \neq \tau,\, \alpha_{\tau_j} \leq \alpha_\tau \right), \\ l_\tau^{\alpha_\tau+1} &\notin \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{w_\tau} \,:\, \tau_j \neq \tau,\, \alpha_{\tau_j} \leq \alpha_\tau \right). \end{aligned}$$

Hence, by Proposition 3.2, $C^\alpha(\widehat{\Delta})$ is not a free R-module, which contradicts (i). ∎

By the following lemma, we can determine whether $l_\tau^{\alpha_\tau+1} \in (\, l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \,:\, \alpha_{\tau_j} < \alpha_\tau \,)$ or not for each totally interior edge $\tau \in \Delta_1^0$ and for each vertex $v_\tau \in \Delta_0^0$ of τ.

Lemma 3.9 ([6, Corollary 2.5])**.** *Let $f_1, \ldots, f_s \in S = \mathbb{R}[x,y]$ be homogeneous linear polynomials which are pairwise linearly independent, and let $0 < c_1 \leq c_2 \leq \cdots \leq c_s$ be integers. Then, for $m \geq 2$,*

$$f_{m+1}^{c_{m+1}} \notin (f_1^{c_1}, \ldots, f_m^{c_m}) \iff c_{m+1} \leq \frac{\sum_{i=1}^m c_i - m}{m-1}.$$

Remark 3.10. Let $S = \mathbb{R}[x, y]$ and $R = \mathbb{R}[x, y, z]$. For each $\tau_i \in \Delta_1^0$, $i = 1, \dots, s$, containing the vertex $v \in \Delta_0^0$, let $l_{\tau_j} \in R$ be the homogeneous linear polynomial defining the plane containing $\widehat{\tau}_j \subset \mathbb{R}^3$. Suppose that the set $\{l_{\tau_1}, \dots, l_{\tau_s}\}$ is pairwise linearly independent. Let $0 < c_1 \le c_2 \le \cdots \le c_s$ be integers. Moreover, let $f_{\tau_i} = a_i x + b_i y + d_i \in S$ be the linear polynomial defining the line containing $\tau_i \subset \mathbb{R}^2$ and let $f'_{\tau_i} = a_i x + b_i y \in S$. Then, for $m \ge 2$,

$$f'^{\,c_m+1}_{\tau_{m+1}} \in (f'^{\,c_1}_{\tau_1}, \dots, f'^{\,c_m}_{\tau_m}) \iff l^{c_m+1}_{\tau_{m+1}} \in (l^{c_1}_{\tau_1}, \dots, l^{c_m}_{\tau_m}).$$

Hence, we can determine whether $l^{c_m+1}_{\tau_{m+1}} \in (l^{c_1}_{\tau_1}, \dots, l^{c_m}_{\tau_m})$ or not by using the inequality in Lemma 3.9.

For each totally interior edge $\tau \in \Delta_1^0$, and for each vertex v_τ of τ, we set

$$\begin{aligned} K_{v_\tau} &:= \{\, \tau_j \in \Delta_1^0 : l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau},\ \alpha_{\tau_j} < \alpha_\tau \,\}, \\ m_{v_\tau} &:= |K_{v_\tau}|. \end{aligned}$$

By Proposition 3.8 and Lemma 3.9, we obtain a method for determining whether $C^\alpha(\widehat{\Delta})$ is a free R-module if $\Delta \subset \mathbb{R}^2$ is a triangulation of a topological disk which has at least one totally interior edge and $\alpha \in \mathbb{Z}^e_{\ge 0}$ is generic.

Theorem 3.11. *Let $\Delta \subset \mathbb{R}^2$ be a triangulation of a topological disk which has at least one totally interior edge, and let $\alpha \in \mathbb{Z}^e_{\ge 0}$ be generic. Then, $C^\alpha(\widehat{\Delta})$ is a free R-module if and only if, for any totally interior edge $\tau \in \Delta_1^0$, there exists a vertex v_τ of τ such that either* (i) *or* (ii) *below is satisfied:*

(i) $@ l_\tau^{\alpha_\tau+1} \notin L_{v_\tau}$;

(ii) $@ l_\tau^{\alpha_\tau+1} \in L_{v_\tau}$, $m_{v_\tau} \ge 2$, *and*

$$\alpha_\tau + 1 > \frac{\displaystyle\sum_{\tau_j \in K_{v_\tau}} (\alpha_{\tau_j} + 1) - m_{v_\tau}}{m_{v_\tau} - 1}.$$

Proof. Let $\tau \in \Delta_1^0$ be any totally interior edge, and let $v_\tau \in \Delta_0^0$ be a vertex of τ. If $l_\tau^{\alpha_\tau+1} \notin L_{v_\tau}$, then there is $l_{\tau'}^{\alpha_{\tau'}+1} \in L_{v_\tau}$ such that $l_\tau = l_{\tau'}$, $\alpha_\tau > \alpha_{\tau'}$. Hence,

$$l_\tau^{\alpha_\tau+1} \in \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \,:\, \alpha_{\tau_j} < \alpha_\tau \right).$$

If $l_\tau^{\alpha_\tau+1} \in L_{v_\tau}$ and $m_{v_\tau} \ge 2$, then it follows from Lemma 3.9 and Remark 3.10 that

$$l_\tau^{\alpha_\tau+1} \in \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \,:\, \alpha_{\tau_j} < \alpha_\tau \right) \iff \alpha_\tau + 1 > \frac{\displaystyle\sum_{\tau_j \in K_{v_\tau}} (\alpha_{\tau_j} + 1) - m_{v_\tau}}{m_{v_\tau} - 1}.$$

If $l_\tau^{\alpha_\tau+1} \in L_{v_\tau}$ and $m_{v_\tau} \le 1$, then

$$l_\tau^{\alpha_\tau+1} \notin \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \,:\, \alpha_{\tau_j} < \alpha_\tau \right).$$

In this way, for the totally interior edge $\tau \in \Delta_1^0$ and for the vertex $v_\tau \in \Delta_0^0$ of τ, the condition (i) or (ii) holds if and only if

$$l_\tau^{\alpha_\tau+1} \in \left(l_{\tau_j}^{\alpha_{\tau_j}+1} \in L_{v_\tau} \,:\, \alpha_{\tau_j} < \alpha_\tau \right).$$

Hence, we obtain the desired result by Proposition 3.8. ∎

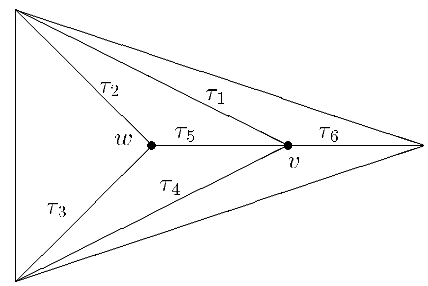

Figure 4:

Example 3.12. Let $\Delta \subset \mathbb{R}^2$ be the simplicial complex shown in Figure 4. Then, τ_5 is the only totally interior edge of Δ. For example, let $\alpha = (0, 4, 2, 1, 3, 4) \in \mathbb{Z}^6_{\geq 0}$. Then α is generic. In this case,

$$\begin{aligned} H_v &= \{l_{\tau_1}, l_{\tau_4}^2, l_{\tau_5}^4, l_{\tau_6}^5\}, \\ L_v &= \{l_{\tau_1}, l_{\tau_4}^2, l_{\tau_5}^4\}, \\ K_v &= \{\tau_1, \tau_4\}, \end{aligned}$$

and

$$3 + 1 = 4 > \frac{(0+1) + (1+1) - 2}{2 - 1} = 1.$$

Therefore, by Theorem 3.11, $C^\alpha(\widehat{\Delta})$ is free.

Moreover, let $\alpha = (0, 2, 3, 2, 1, 4) \in \mathbb{Z}^6_{\geq 0}$, which is also generic. In this case, for the vertex v,

$$\begin{aligned} H_v &= \{l_{\tau_1}, l_{\tau_4}^3, l_{\tau_5}^2, l_{\tau_6}^5\}, \\ L_v &= \{l_{\tau_1}, l_{\tau_4}^3, l_{\tau_5}^2\}, \\ K_v &= \{\tau_1\}, \end{aligned}$$

and for the vertex w,

$$\begin{aligned} H_w &= L_w = \{l_{\tau_2}^3, l_{\tau_3}^4, l_{\tau_5}^2\}, \\ K_w &= \emptyset. \end{aligned}$$

Therefore, by Theorem 3.11, $C^\alpha(\widehat{\Delta})$ is not free.

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