Scientiae Mathematicae Japonicae Online, Vol.**6** (2002), 511–530

A ONE-NOISY-VERSUS-TWO-SILENT DUEL WITH ARBITRARY ACCURACY FUNCTIONS UNDER ARBITRARY MOTION

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Received January 25, 2002

Abstract. This paper deals with a two-person zero-sum timing game with the following structure: Player I has a gun with one bullet and player II has a gun with two bullets and they fight a duel. Player I's gun is noisy and player II's gun is silent, and hence player II hears the shot of player I as soon as player I fires, whereas player I does not hear the shot of player II. Player I is at the place 0 at the beginning of the duel and he can move as he likes and player II is always at the place 1. The accuracy functions, which denote the probability of hitting the opponent when a player fires his bullet, are arbitrary. If player I hits player II without being hit himself first, then the payoff is +1; if player I is hit by player II without hitting player II first, the payoff is -1; if they hit each other at the same time or both survive, the payoff is 0.

The objective of this paper is to obtain the game value and the optimal strategies for this timing game. In the final section, some examples are given.

1. Introduction

A duel under arbitrary motion is a two-person zero-sum timing game with the following structure: Each of two competitors, denoted by player I and player II, has a gun and he can fire his bullets aiming at his opponent. At time $t = 0$, these two players are one distance apart on a line and each player can move on the line as he likes. The maximum speed of player I is v_1, the maximum speed of player II is v_2 and we assume $v_1 > v_2 \geqq 0$. Without loss of generality, we can suppose that $v_1 = 1$ and $v_2 = 0$, and hence, player II is motionless. Thus we assume that player II is at the place 1 all the time and player I is at the place 0 at time $t = 0$ and he can move towards player II, he can move away from player II, and he can stay in one place. If player I or player II fires his bullet when player I is at a place x, he hits his opponent with probability $p(x)$ or $q(x)$, respectively. The functions $p(x)$ and $q(x)$ are called accuracy functions for players I and II, respectively, and they are continuous and strictly increasing on $[0, 1]$ with $p(0) = q(0) = 0$ and $p(1) = q(1) = 1$. The duel ends when at least one player is hit or both players fire all of their bullets; otherwise it continues indefinitely. If player I hits player II without being hit himself first, then the payoff of the duel is +1; if player I is hit by player II without hitting player II first, the payoff is -1; if they hit each other at the same time or both survive, the payoff is 0. The objective of player I is to maximize the expected payoff and the objective of player II is to minimize it. A gun is said to be silent if the shot of the owner is not heard by his opponent and a gun is said to be noisy if the shot of the owner is heard by his opponent as soon as the owner of the gun fires the bullet. Thus if a player has a silent gun, then his opponent does not know whether the owner of the gun has fired or not. On the other hand, if a player has a noisy

2000 *Mathematics Subject Classification.* 91A55, 91A05.

Key words and phrases. Games of timing, two-person zero-sum games, game value, optimal strategies.

gun, then his opponent always knows whether the owner has fired or not. If each player has a silent gun, the duel is said to be silent and if each player has a noisy gun, the duel is said to be noisy.

Trybula [7, 8] solved the silent duel with arbitrary accuracy functions under arbitrary motion under the assumption that each player has a silent gun with one bullet and that $p(x)$ and $q(x)$ increase with a continuous second derivative each. Trybula [4-6] also solved the noisy duel under arbitrary motion.

The author [2] dealt with the silent-versus-noisy duel under arbitrary motion in which player I has a silent gun with one bullet and player II has a noisy gun with one bullet and the accuracy functions are $p(x)$ and $q(x)$ for players I and II, respectively. The author [3] also dealt with the noisy-versus-silent duel under arbitrary motion in which player I has a noisy gun with one bullet and player II has a silent gun with one bullet and the accuracy functions are $p(x)$ and $q(x)$ for players I and II, respectively.

Further researches on duels under arbitrary motion have been done by Trybula [9-11] and general researches on games of timing are summarized by Karlin [1].

In this paper, we deal with the duel under arbitrary motion in which player I has a noisy gun with one bullet and player II has a silent gun with two bullets and each player's accuracy function is arbitrary.

2. Problem

In this paper, we deal with the one-noisy-versus-two-silent duel with arbitrary accuracy functions under arbitrary motion. Player I has a noisy gun with one bullet and he is at the place 0 at time $t = 0$. He can move as he likes. On the other hand, player II has a silent gun with two bullets and he is always at the place 1. The accuracy functions are $p(x)$ and $q(x)$ for player I and player II, respectively. We suppose that $p(x)$ and $q(x)$ have positive and continuous derivatives $p'(x)$ and $q'(x)$, respectively, with $p(0) = q(0) = 0$ and $p(1) = q(1) = 1$.

Suppose that player I fires his bullet and fails in hitting player II. Then player I will go in the opposite direction from the place where player II is, since the probability of being hit increases if he approaches player II. Obviously, this going back to the place 0 by player I directly after firing is necessary in any optimal strategy and it will be assumed in all of player I's strategies for the rest of this paper. In addition, player II's chance of hitting the opponent decreases as the distance between the players becomes large. Thus if player II has bullets after player I has fired his bullet, then player II will fire these bullets as soon as player I has fired since player I will escape from player II directly after player I has fired. These firings by player II directly after player I's firing are necessary in any optimal strategy and it will also be assumed in all of player II's strategies for the rest of this paper.

Assume that player I goes to x', he then goes back to x $(x < x')$ and then he fires his bullet at x. Then player I may be hit by his opponent during his movements from x to x' and back to x, also his chance of hitting the opponent does not increase. Thus if player I fires his bullet at a place, he should fire his bullet when he is at the place for the first time. Therefore we can confine the strategy for player I to those going directly to the place where he fires his bullet and then going back to the place 0 immediately after the firing. Similarly, we can confine the strategies for player II to those where he fires each of his bullets when player I is at a place for the first time.

Let $M(x, y, z)$ be the expected payoff of the duel when player I fires his bullet at the place x $(0 \leqq x \leqq 1)$, and player II fires his first and second bullet at the moments when player I is at the place y and z $(0 \leqq y \leqq z \leqq 1)$, respectively. The function $M(x, y, z)$, called the payoff kernel, is of the form

$$M(x,y,z)=\begin{cases} p(x)-\{1-p(x)\}\{2-q(x)\}q(x), & \text{if } x<y\leqq z,\\ -q(y)+\{1-q(y)\}p(x)-\{1-q(y)\}\{1-p(x)\}q(x), & \text{if } y<x<z,\\ -q(y)-\{1-q(y)\}q(z)+\{1-q(y)\}\{1-q(z)\}p(x), & \text{if } y\leqq z<x,\\ p(x)-q(x)-\{1-p(x)\}\{1-q(x)\}q(z), & \text{if } x=y<z,\\ -q(y)+\{1-q(y)\}\{p(x)-q(x)\}, & \text{if } y<x=z,!\,!\\ p(x)-\{2-q(x)\}q(x), & \text{if } x=y=z. \end{cases}$$

For the duel described above, we shall search for an optimal strategy $\{f(x),\alpha\}$ for player I consisting of density part $f(x)$ on an interval $[a,1]$ with a mass α at $x=1$ and an optimal strategy $\{g(y),h(z)\}$ for player II consisting of two densities $g(y)$ and $h(z)$, where

$$\int_a^1 f(x)\,dx+\alpha=1,$$

$$\int_a^b g(y)\,dy=1$$

and

$$\int_b^1 h(z)\,dz=1$$

for some b in $[a,1]$. In fact, optimal strategies such as $\{f(x),\alpha\}$ and $\{g(y),h(z)\}$ may not exist, but as we shall see, in some cases, such optimal strategies can exist for this timing game.

3. Preliminary lemmas

The following lemma was proved by the author [3].

Lemma 1. *If $p(x)-q(x)+p(x)q(x)$ is increasing over $[0,1]$, then there exists a unique root in $(0,1)$ for the following equation:*

$$\text{(3.1)}\qquad \exp\left\{-\frac{1}{2}\int_x^1\frac{q'(t)}{p(t)q(t)}\,dt\right\}+\int_x^1\frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}}\exp\left\{-\frac{1}{2}\int_x^t\frac{q'(u)}{p(u)q(u)}\,du\right\}dt =\frac{2\{1-q(x)\}}{1+p(x)-q(x)+p(x)q(x)}\,q(x)^{-\frac{1}{2}}.$$

In this paper, we assume $p(x)-q(x)+p(x)q(x)$ is increasing. We can see that $p(x)-q(x)+p(x)q(x)$ is increasing when $p(x)=x$ and $q(x)=x^n$ with $n\geqq 1$. We denote by b the unique root in $(0,1)$ for the equation (3.1), and thus b satisfies the following equation

$$\text{(3.2)}\qquad \exp\left\{-\frac{1}{2}\int_b^1\frac{q'(x)}{p(x)q(x)}\,dx\right\}+\int_b^1\frac{q'(x)}{p(x)q(x)^{\frac{3}{2}}}\exp\left\{-\frac{1}{2}\int_b^x\frac{q'(t)}{p(t)q(t)}\,dt\right\}dx =\frac{2\{1-q(b)\}}{1+p(b)-q(b)+p(b)q(b)}\,q(b)^{-\frac{1}{2}}.$$

We set

$$A_1(x)=1+p(x)-\{1-p(x)\}\{2-q(x)\}q(x),$$

$$A_2(x) = \frac{1-q(x)}{q(x)^{\frac{1}{2}} A_1(x)},$$

$$A_3(x) = \exp\left\{ -\frac{1}{2} q(x) + \frac{1}{2} \int_x^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)} \, dt \right\}$$

and

$$A_4(x) = A_2(x)A_3(x) - \frac{1}{2} \int_x^b \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}} A_3(t) \, dt.$$

In what follows, we assume that $A_1(x)$ is unimodal and thus $A_1(x)$ is decreasing over $[0, \tau]$ and increasing over $[\tau, 1]$ for some τ in $[0, 1]$.

Lemma 2. *If $A_1'(x) > 0$ for x in $[0, b]$, then $A_4'(x) < 0$ for the x.*

Proof. We directly get

$$\begin{aligned} A_4'(x) &= A_2'(x)A_3(x) + A_2(x)A_3'(x) + \frac{q'(x)}{2p(x)q(x)^{\frac{3}{2}}} A_3(x) \\ &= \left\{ A_2'(x) - \frac{p(x)q(x) + 1 - q(x)}{2p(x)q(x)} q'(x)A_2(x) + \frac{q'(x)}{2p(x)q(x)^{\frac{3}{2}}} \right\} A_3(x). \end{aligned}$$

Further we have

$$A_2'(x) = -\frac{\{1+q(x)\}q'(x)}{2q(x)^{\frac{3}{2}} A_1(x)} - \frac{\{1-q(x)\}A_1'(x)}{q(x)^{\frac{1}{2}} \{A_1(x)\}^2}$$

and

$$\frac{q'(x)}{2p(x)q(x)^{\frac{3}{2}}} - \frac{p(x)q(x) + 1 - q(x)}{2p(x)q(x)} q'(x)A_2(x) = \frac{\{1+q(x)\}q'(x)}{2q(x)^{\frac{3}{2}} A_1(x)}.$$

Thus we get

$$A_4'(x) = -\frac{\{1-q(x)\}A_1'(x)A_3(x)}{q(x)^{\frac{1}{2}} \{A_1(x)\}^2}.$$

Therefore if $A_1'(x) > 0$, then $A_4'(x) < 0$. This completes our proof.

Lemma 3. *If $A_1(x)$ is increasing over $[0, b]$, then there exists a unique root in $[0, b)$ for the following equation:*

$$A_4(x) = \frac{1}{\{1 + p(b) - q(b) + p(b)q(b)\}q(b)^{\frac{1}{2}}} e^{-\frac{1}{2}q(b)}. \tag{3.3}$$

Proof. From lemma 2, there is at most one root for the equation (3. 3). We set

$$\begin{aligned} \varphi(x) = &\frac{2\{1 + p(x) - q(x) + p(x)q(x)\}A_4(x)}{A_3(x)} \\ &- \frac{2\{1 + p(x) - q(x) + p(x)q(x)\}}{\{1 + p(b) - q(b) + p(b)q(b)\}q(b)^{\frac{1}{2}} A_3(x)} e^{-\frac{1}{2}q(b)}. \end{aligned}$$

It suffices to show that there is an x in $[0,b)$ with $\varphi(x)=0$. Since

$$\begin{aligned}\varphi(x) = {}& \frac{2\{1+p(x)-q(x)+p(x)q(x)\}\{1-q(x)\}}{[1+p(x)-\{1-p(x)\}\{2-q(x)\}q(x)]q(x)^{\frac{1}{2}}} \\ & - \frac{1+p(x)-q(x)+p(x)q(x)}{A_3(x)}\int_x^b \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}}A_3(t)\,dt \\ & - \frac{2\{1+p(x)-q(x)+p(x)q(x)\}}{\{1+p(b)-q(b)+p(b)q(b)\}q(b)^{\frac{1}{2}}}\exp\left\{\frac{1}{2}q(x)-\frac{1}{2}q(b)-\frac{1}{2}\int_x^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\},\end{aligned}$$

we get

$$\varphi(b) = \frac{2}{q(b)^{\frac{1}{2}}}\left[\frac{\{1+p(b)-q(b)+p(b)q(b)\}\{1-q(b)\}}{1+p(b)-\{1-p(b)\}\{2-q(b)\}q(b)}-1\right] < 0.$$

Further, since $1+p(x)-q(x)+p(x)q(x)$ is increasing, we get

$$\begin{aligned}& \frac{1+p(x)-q(x)+p(x)q(x)}{A_3(x)}\int_x^b \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}}A_3(t)\,dt \\ & \quad \leqq \frac{1}{A_3(x)}\int_x^b \frac{1+p(t)-q(t)+p(t)q(t)}{p(t)q(t)^{\frac{3}{2}}}q'(t)A_3(t)\,dt \\ & \quad = \frac{2}{q(x)^{\frac{1}{2}}} - \frac{2}{q(b)^{\frac{1}{2}}}\exp\left\{\frac{1}{2}q(x)-\frac{1}{2}q(b)-\frac{1}{2}\int_x^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}\end{aligned}$$

for any x in $(0,b]$, and thus

$$\begin{aligned}\varphi(x) \geqq {}& \frac{2}{q(b)^{\frac{1}{2}}}\left\{1-\frac{1+p(x)-q(x)+p(x)q(x)}{1+p(b)-q(b)+p(b)q(b)}\right\} \\ & \times \exp\left\{\frac{1}{2}q(x)-\frac{1}{2}q(b)-\frac{1}{2}\int_x^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \\ & - \frac{4p(x)q(x)^{\frac{1}{2}}}{1+p(x)-\{1-p(x)\}\{2-q(x)\}q(x)}.\end{aligned}$$

Therefore we get $\varphi(0) \geqq 0$. This completes our proof.

In the following sections, we suppose that there is a root a for the equation (3.3) in (τ, b) and thus a satisfies the following equation:

(3.4)

$$\begin{aligned}& \frac{1-q(a)}{[1+p(a)-\{1-p(a)\}\{2-q(a)\}q(a)]q(a)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}q(a)+\frac{1}{2}\int_a^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \\ & \quad - \frac{1}{2}\int_a^b \frac{q'(x)}{p(x)q(x)^{\frac{3}{2}}}\exp\left\{-\frac{1}{2}q(x)+\frac{1}{2}\int_x^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}dx \\ & \quad = \frac{1}{\{1+p(b)-q(b)+p(b)q(b)\}q(b)^{\frac{1}{2}}}e^{-\frac{1}{2}q(b)}.\end{aligned}$$

4. A strategy for player I

As was stated in section 2, player II fires his bullets directly after he hears the shot by his opponent if player II still has his bullets when he hears the shot. Suppose that player II fires his first and second bullets at the moments when player I is at the places y and z, respectively, under the assumption that player I has not fired his bullet until these moments. Further suppose that player I applies the strategy $\{f(x),\alpha\}$. In such a circumstance, we denote the expected payoff by $v_1(y,z)$. The function $v_1(y,z)$ is represented as

$$\begin{aligned} v_1(y,z) = &\int_a^y [p(x) - \{1-p(x)\}\{2-q(x)\}q(x)]f(x)\,dx \\ &+ \int_y^z [-q(y) + \{1-q(y)\}p(x) - \{1-q(y)\}\{1-p(x)\}q(x)]f(x)\,dx \\ &+ \int_z^1 [-q(y) - \{1-q(y)\}q(z) + \{1-q(y)\}\{1-q(z)\}p(x)]f(x)\,dx \\ &+ \alpha[-q(y) - \{1-q(y)\}q(z) + \{1-q(y)\}\{1-q(z)\}] \end{aligned} \tag{4.1}$$

for all y and z with $a \leqq y \leqq z < 1$. We set

$$\begin{aligned} \rho_1(z) = &\int_b^z [1 + p(x) - \{1-p(x)\}q(x)]f(x)\,dx \\ &+ \{1-q(z)\}\int_z^1 \{1+p(x)\}f(x)\,dx + 2\alpha\{1-q(z)\} \end{aligned} \tag{4.2}$$

for z in $[b,1)$. Then we get

$$\begin{aligned} v_1(y,z) = &\int_a^y [1 + p(x) - \{1-p(x)\}\{2-q(x)\}q(x)]f(x)\,dx \\ &+ \{1-q(y)\}\int_y^b [1+p(x) - \{1-p(x)\}q(x)]f(x)\,dx \\ &- \int_a^1 f(x)\,dx - \alpha + \{1-q(y)\}\rho_1(z) \end{aligned}$$

for all y and z with $a \leqq y \leqq b \leqq z < 1$.

Lemma 4. *Set*

$$f(x) = \begin{cases} \dfrac{c_1 q'(x)}{p(x)q(x)^{\frac{3}{2}}} \exp\left\{-\dfrac{1}{2}q(x) - \dfrac{1}{2}\displaystyle\int_a^x \dfrac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}, & a \leqq x < b, \\ \dfrac{c_2 q'(x)}{p(x)q(x)^{\frac{3}{2}}} \exp\left\{-\dfrac{1}{2}\displaystyle\int_b^x \dfrac{q'(t)}{p(t)q(t)}\,dt\right\}, & b \leqq x \leqq 1, \end{cases} \tag{4.3}$$

and

$$\alpha = c_2 \exp\left\{-\frac{1}{2}\int_b^1 \frac{q'(t)}{p(t)q(t)}\,dt\right\}, \tag{4.4}$$

where

$$c_1 = \frac{q(a)^{\frac{1}{2}}}{2\{1-q(a)\}}[1+p(a)-\{1-p(a)\}\{2-q(a)\}q(a)]e^{\frac{1}{2}q(a)} \tag{4.5}$$

and

$$c_2 = \frac{c_1}{1-q(b)}\exp\left\{-\frac{1}{2}q(b) - \frac{1}{2}\int_a^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}. \tag{4.6}$$

Then the following statements hold;

(*i*) $\int_a^1 f(x)\,dx + \alpha = 1.$

(*ii*) *For all* y *and* z *with* $a \leqq y \leqq b \leqq z < 1$,

$$v_1(y,z) = p(a) - \{1-p(a)\}\{2-q(a)\}q(a).$$

(*iii*) *For all* y *and* z *with* $a \leqq y \leqq z \leqq b$,

$$v_1(y,z) \geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a).$$

(*iv*) *For all* y *and* z *with* $b \leqq y \leqq z < 1$,

$$v_1(y,z) \geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a).$$

Proof. (*i*) From (3.4) and (4.3), we directly get

$$\begin{aligned}\int_a^b f(x)\,dx &= c_1\int_a^b \frac{q'(x)}{p(x)q(x)^{\frac{3}{2}}}\exp\left\{-\frac{1}{2}q(x) - \frac{1}{2}\int_a^x \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}dx \\ &= \frac{2c_1\{1-q(a)\}}{[1+p(a)-\{1-p(a)\}\{2-q(a)\}q(a)]q(a)^{\frac{1}{2}}}e^{-\frac{1}{2}q(a)} \\ &\quad - \frac{2c_1}{\{1+p(b)-q(b)+p(b)q(b)\}q(b)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}q(b) - \frac{1}{2}\int_a^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\},\end{aligned}$$

and hence, by (3.2), (4.3), (4.4), (4.5) and (4.6)

$$\begin{aligned}\int_a^b f(x)\,dx &= 1 - \frac{c_1}{1-q(b)}\exp\left\{-\frac{1}{2}q(b) - \frac{1}{2}\int_a^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt - \frac{1}{2}\int_b^1 \frac{q'(t)}{p(t)q(t)}\,dt\right\} \\ &\quad - \frac{c_1}{1-q(b)}\exp\left\{-\frac{1}{2}q(b) - \frac{1}{2}\int_a^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \\ &\quad \times \int_b^1 \frac{q'(x)}{p(x)q(x)^{\frac{3}{2}}}\exp\left\{-\frac{1}{2}\int_b^x \frac{q'(t)}{p(t)q(t)}\,dt\right\}dx \\ &= 1 - \alpha - c_2\int_b^1 \frac{q'(x)}{p(x)q(x)^{\frac{3}{2}}}\exp\left\{-\frac{1}{2}\int_b^x \frac{q'(t)}{p(t)q(t)}\,dt\right\}dx \\ &= 1 - \alpha - \int_b^1 f(x)\,dx.\end{aligned}$$

Accordingly we have

$$\int_a^1 f(x)\,dx + \alpha = 1$$

(ii) For any z in $[b,1]$, we get

(4.7)

$$\begin{aligned}&\int_z^1 \{1+p(x)\}f(x)\,dx\\ &\quad = 2c_2 q(z)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\int_b^z \frac{q'(t)}{p(t)q(t)}\,dt\right\} - 2c_2 \exp\left\{-\frac{1}{2}\int_b^1 \frac{q'(t)}{p(t)q(t)}\,dt\right\},\end{aligned}$$

and thus,

$$\rho_1'(z) = 2p(z)q(z)f(z) - q'(z)\int_z^1 \{1+p(x)\}f(x)\,dx - 2\alpha q'(z) = 0.$$

By (4.4) and (4.7), we further get

$$\begin{aligned}\rho_1(b) &= \{1-q(b)\}\int_b^1 \{1+p(x)\}f(x)\,dx + 2\alpha\{1-q(b)\}\\ &= 2c_2\{1-q(b)\}q(b)^{-\frac{1}{2}}.\end{aligned}$$

Therefore

$$\rho_1(z) = 2c_2\{1-q(b)\}q(b)^{-\frac{1}{2}} \tag{4.8}$$

for all z in $[b,1)$. Accordingly, we have

$$\begin{aligned}v_1(y,z) = -1 &+ \int_a^y [1+p(x) - \{1-p(x)\}\{2-q(x)\}q(x)]f(x)\,dx\\ &+ \{1-q(y)\}\int_y^b [1+p(x)-\{1-p(x)\}q(x)]f(x)\,dx\\ &+ 2c_2\{1-q(b)\}\{1-q(y)\}q(b)^{-\frac{1}{2}}\end{aligned} \tag{4.9}$$

for all y and z with $a \leqq y \leqq b \leqq z < 1$. For any y in $[a,b]$, we get

$$\begin{aligned}&\int_y^b [1+p(x)-\{1-p(x)\}q(x)]f(x)\,dx\\ &\quad = 2c_1 q(y)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}q(y) - \frac{1}{2}\int_a^y \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}\\ &\qquad - 2c_1 q(b)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}q(b) - \frac{1}{2}\int_a^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}\end{aligned} \tag{4.10}$$

and

(4.11)

$$\begin{aligned}&\int_a^y [1+p(x)-\{1-p(x)\}\{2-q(x)\}q(x)]f(x)\,dx\\ &\quad = 2c_1 q(a)^{-\frac{1}{2}}\{1-q(a)\}e^{-\frac{1}{2}q(a)}\\ &\qquad - 2c_1 q(y)^{-\frac{1}{2}}\{1-q(y)\}\exp\left\{-\frac{1}{2}q(y) - \frac{1}{2}\int_a^y \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}.\end{aligned}$$

It follows, from (4.5), (4,6), (4.9), (4.10) and (4.11), that

$$v_1(y,z) = p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for all y and z with $a \leqq y \leqq b \leqq z < 1$.

(*iii*) By (4.1), we get

$$\begin{aligned}
\frac{1}{1-q(y)}\frac{\partial v_1(y,z)}{\partial z} &= 2p(z)q(z)f(z) - q'(z)\int_z^1 \{1+p(x)\}f(x)\,dx - 2\alpha q'(z) \\
&= 2p(z)q(z)f(z) - q'(z)\int_z^b [1+p(x) - \{1-p(x)\}q(x)]f(x)\,dx \\
&\quad - q'(z)\int_z^b \{1-p(x)\}q(x)f(x)\,dx - q'(z)\int_b^1 \{1+p(x)\}f(x)\,dx - 2\alpha q'(z)
\end{aligned}$$

for all y and z with $a \leqq y \leqq z \leqq b$. Thus, from (4.3), (4.4), (4.6), (4.7) and (4.10), it follows that

$$\begin{aligned}
\frac{1}{1-q(y)}\frac{\partial v_1(y,z)}{\partial z} &= 2c_1 q(b)^{-\frac{1}{2}} q'(z)\exp\left\{-\frac{1}{2}q(b) - \frac{1}{2}\int_a^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \\
&\quad - q'(z)\int_z^b \{1-p(x)\}q(x)f(x)\,dx - 2c_2 q(b)^{-\frac{1}{2}}q'(z) \\
&= -2c_2 q(b)^{\frac{1}{2}}q'(z) - q'(z)\int_z^b \{1-p(x)\}q(x)f(x)\,dx < 0.
\end{aligned}$$

Therefore we have

$$v_1(y,z) \geqq v_1(y,b) = p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for any y and z with $a \leqq y \leqq z \leqq b$.

(*iv*) By (4.1), (4.2) and (4.8), we get

$$\begin{aligned}
\frac{\partial v_1(y,z)}{\partial y} &= 2p(y)q(y)f(y) - q'(y)\int_y^z \{1+p(x)-q(x)+p(x)q(x)\}f(x)\,dx \\
&\quad - q'(y)\{1-q(z)\}\int_z^1 \{1+p(x)\}f(x)\,dx - 2\alpha q'(y)\{1-q(z)\} \\
&= 2p(y)q(y)f(y) - q'(y)\rho_1(z) \\
&\quad + q'(y)\int_b^y [1+p(x)-\{1-p(x)\}q(x)]f(x)\,dx \\
&= 2p(y)q(y)f(y) - q'(y)\rho_1(z) \\
&\quad + q'(y)\left[\rho_1(y) - \{1-q(y)\}\int_y^1 \{1+p(x)\}f(x)\,dx - 2\alpha\{1-q(y)\}\right] \\
&= 2p(y)q(y)f(y) - \{1-q(y)\}q'(y)\left[\int_y^1 \{1+p(x)\}f(x)\,dx + 2\alpha\right]
\end{aligned}$$

for all y and z with $b \leqq y \leqq z < 1$. Thus, it follows, from (4.3), (4.4) and (4.7), that

$$\frac{\partial v_1(y,z)}{\partial y} = 2c_2 q(y)^{\frac{1}{2}} q'(y)\exp\left\{-\frac{1}{2}\int_b^y \frac{q'(t)}{p(t)q(t)}\,dx\right\} > 0.$$

Hence we have

$$v_1(y,z) \geqq v_1(b,z) = p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for all y and z with $b \leqq y \leqq z < 1$. This completes our proof.

5. A Strategies for Player II

We suppose that player II applies a strategy $\{g(y), h(z)\}$ satisfying (1.2) and (1.3) and that player I fires his bullet at the first moment when he is at a place x in $[0,1]$. In this circumstance, we denote the expected payoff by $v_2(x)$. We get

$$v_2(x) = \int_a^x [-q(y) + \{1-q(y)\}p(x) - \{1-q(y)\}\{1-p(x)\}q(x)]g(y)\,dy + \int_x^b [p(x) - \{1-p(x)\}\{2-q(x)\}q(x)]g(y)\,dy \tag{5.1}$$

for all x in $[a,b]$ and

$$v_2(x) = \int_a^b \int_b^x [-q(y) - \{1-q(y)\}q(z) + \{1-q(y)\}\{1-q(z)\}p(x)]g(y)h(z)\,dz\,dy + \int_a^b \int_x^1 [-q(y) + \{1-q(y)\}p(x) - \{1-q(y)\}\{1-p(x)\}q(x)]g(y)h(z)\,dz\,dy \tag{5.2}$$

for all x in $[b,1]$. We set

$$B_1(x) = \frac{2}{\{1+p(b)-q(b)+p(b)q(b)\}q(b)^{\frac{1}{2}}} \exp\left\{\frac{1}{2}q(x) - \frac{1}{2}q(b) - \frac{1}{2}\int_x^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} + e^{\frac{1}{2}q(x)} \int_x^b \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}} \exp\left\{-\frac{1}{2}q(t) - \frac{1}{2}\int_x^t \frac{\{1-q(u)\}q'(u)}{p(u)q(u)}\,du\right\} dt,$$

$$B_2(x) = 1 + p(x) - q(x) + p(x)q(x),$$

$$B_3(x) = \frac{B_2'(x)}{p(x)q(x)^{\frac{1}{2}}} + \frac{-2q(x)p'(x) + \{1-p(x)\}\{1-q(x)\}q'(x)}{2p(x)^2q(x)^{\frac{3}{2}}}B_2(x)$$

and

$$B_4(x) = \frac{2q(x)p'(x) - \{1-p(x)\}\{1-q(x)\}q'(x)}{p(x)^2q(x)^2}.$$

Then we directly get

$$B_1'(x) = \frac{1-q(x)+p(x)q(x)}{2p(x)q(x)}q'(x)B_1(x) - \frac{q'(x)}{p(x)q(x)^{\frac{3}{2}}} \tag{5.3}$$

and by (3.4), we have

$$B_1(a) = \frac{2\{1-q(a)\}}{[1+p(a)-\{1-p(a)\}\{2-q(a)\}q(a)]q(a)^{\frac{1}{2}}}. \tag{5.4}$$

Lemma 5. *Set*

$$g(y) = c_3B_1(y)B_3(y) + c_3B_4(y), \tag{5.5}$$

for y *in* $[a,b]$, *where*

$$c_3 = \frac{1+p(a)-\{1-p(a)\}\{2-q(a)\}q(a)}{4}. \tag{5.6}$$

Then

$$\int_a^b g(y)\,dy = 1 \tag{5.7}$$

and

$$v_2(x) = p(a)-\{1-p(a)\}\{2-q(a)\}q(a)$$

for all x *in* $[a,b]$.

Proof. We put

$$B_5(x) = 1 - \frac{2c_3}{p(x)q(x)} + \frac{c_3 B_1(x)B_2(x)}{p(x)q(x)^{\frac{1}{2}}}$$

and

$$B_6(x) = 1 + \frac{c_3\{1-p(x)\}\{1-q(x)\}q(x)^{\frac{1}{2}}}{p(x)}B_1(x) - \frac{2c_3}{p(x)}.$$

Then (5.4) and (5.6) yield

$$B_5(a) = B_6(a) = 0.$$

Furthermore, by (5.3), we get

$$B_5'(x) = g(x)$$

and

$$B_6'(x) = q(x)g(x).$$

Thus we have

$$\int_a^x g(y)\,dy = B_5(x) \tag{5.8}$$

and

$$\int_a^x q(y)g(y)\,dy = B_6(x), \tag{5.9}$$

and hence

$$\int_a^b g(y)\,dy = B_5(b) = 1.$$

Accordingly, from (5.1), (5.8) and (5.9), it follows that

$$\begin{aligned} v_2(x) &= p(x) - \{1-p(x)\}\{2-q(x)\}q(x) + \{1-p(x)\}\{1-q(x)\}q(x)\int_a^x g(y)\,dy \\ &\quad - \{1+p(x)-q(x)+p(x)q(x)\}\int_a^x q(y)g(y)\,dy \\ &= -1+4c_3 = p(a)-\{1-p(a)\}\{2-q(a)\}q(a) \end{aligned}$$

for all x in $[a,b]$. This completes our proof.

We set

$$B_7(x) = \exp\left\{-\frac{1}{2}\int_x^1 \frac{q'(t)}{p(t)q(t)}\,dt\right\} + \int_x^1 \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}}\exp\left\{-\frac{1}{2}\int_x^t \frac{q'(u)}{p(u)q(u)}\,du\right\}dt$$

for all x in $[b,1]$. We directly get

$$B_7'(x) = \frac{q'(x)}{2p(x)q(x)}B_7(x) - \frac{q'(x)}{p(x)q(x)^{\frac{3}{2}}}. \tag{5.10}$$

Lemma 6. *Set*

$$h(z) = \frac{c_4[2q(z)p'(z) - \{1-p(z)\}q'(z)]}{p(z)^2 q(z)^2} + \frac{c_4[\{1-p(z)^2\}q'(z) - 2q(z)p'(z)]}{2p(z)^2 q(z)^{\frac{3}{2}}} B_7(z)$$

for z in $[b,1]$, where

$$c_4 = \frac{1+p(b)-q(b)+p(b)q(b)}{4}.$$

Then

$$\int_b^1 h(z)\,dz = 1$$

and

$$v_2(x) = p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for all x in $[b,1]$.

Proof. By (3.2) and (5.10), we get

$$\int_x^1 h(z)\,dz = \frac{2c_4}{p(x)q(x)} - \frac{c_4\{1+p(x)\}}{p(x)q(x)^{\frac{1}{2}}} B_7(x) \tag{5.11}$$

and

$$\int_b^x q(z)h(z)\,dz = 1 - \frac{2c_4}{p(x)} + \frac{c_4\{1-p(x)\}q(x)^{\frac{1}{2}}}{p(x)} B_7(x). \tag{5.12}$$

Thus it follows from (3.2) that

$$\int_b^1 h(z)\,dz = 1. \tag{5.13}$$

We set

$$\rho_2(x) = \int_b^x [-q(z) + \{1-q(z)\}p(x)]h(z)\,dz + \int_x^1 [\,p(x) - \{1-p(x)\}q(x)]h(z)\,dz$$

for x in $[b,1)$. By (5.2) and (5.13), we have

$$v_2(x) = -\int_a^b q(y)g(y)\,dy + \rho_2(x)\int_a^b \{1-q(y)\}g(y)\,dy$$

for any x in $[b,1]$. From (5.11), (5.12) and (5.13), it follows that

$$\begin{aligned}\rho_2(x) &= p(x)\int_b^1 h(z)\,dz - \{1+p(x)\}\int_b^x q(z)h(z)\,dz - \{1-p(x)\}q(x)\int_x^1 h(z)\,dz \\ &= -1 + 4c_4 = p(b) - q(b) + p(b)q(b)\end{aligned}$$

for all x in $[b,1)$. Thus

$$\begin{aligned}v_2(x) = \{p(b) - q(b) + p(b)q(b)\}&\int_a^b g(y)\,dy \\ - \{1+p(b)-q(b)+p(b)q(b)\}&\int_a^b q(y)g(y)\,dy\end{aligned}$$

and hence, from (5.7) and (5.9), we obtain

$$v_2(x) = -1 + 4c_3 = p(a) - \{1 - p(a)\}\{2 - q(a)\}q(a)$$

for all x in $[b, 1)$. This completes our proof.

We set

$$B_8(y) = 1 - \frac{1}{2}q(y)^{\frac{1}{2}}B_1(y)B_2(y)$$

Then we get

$$\frac{g(y)}{c_3} = \frac{B_1(y)B_2'(y)}{p(y)q(y)^{\frac{1}{2}}} + B_4(y)B_8(y). \tag{5.14}$$

Lemma 7. *$g(y)$ given in Lemma* 5 is non-negative for all y in $[a, b]$.

Proof. First we suppose that $B_4(y) \geqq 0$. By (5.14) and $B_2'(y) \geqq 0$, it suffices to show $B_8(y) \geqq 0$. We set

$$B_9(y) = \left\{\frac{2}{q(y)^{\frac{1}{2}}} - B_1(y)B_2(y)\right\} \exp\left\{-\frac{1}{2}q(y) + \frac{1}{2}\int_y^b \frac{\{1 - q(t)\}q'(t)}{p(t)q(t)}\,dt\right\}.$$

Then, by (5.3) and $B_2'(y) \geqq 0$, we get

$$B_9'(y) = -B_1(y)B_2'(y) \exp\left\{-\frac{1}{2}q(y) + \frac{1}{2}\int_y^b \frac{\{1 - q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \leqq 0$$

for all y in $[a, b]$. Further, we have $B_9(b) = 0$, and thus $B_9(y) \geqq 0$ for the y in $[a, b]$. Accordingly, we obtain

$$B_8(y) = \frac{q(y)^{\frac{1}{2}}}{2}B_9(y) \exp\left\{\frac{1}{2}q(y) - \frac{1}{2}\int_y^b \frac{\{1 - q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \geqq 0$$

Now we suppose that $B_4(y) < 0$. From the definition of a, we get

$$B_1(y) > \frac{2\{1 - q(y)\}}{A_1(y)q(y)^{\frac{1}{2}}} \tag{5.15}$$

for all y in $(a, b]$, and we have $B_3(y) > 0$ since $B_4(y) < 0$. Therefore, from (5.5) and (5.15), it follows that

$$\begin{aligned}\frac{g(y)}{c_3} &> \frac{2\{1 - q(y)\}B_3(y)}{A_1(y)q(y)^{\frac{1}{2}}} + B_4(y) \\ &= \frac{\{1 - q(y)\}\{2B_2'(y) - p(y)q(y)B_2(y)B_4(y)\} + p(y)q(y)A_1(y)B_4(y)}{p(y)q(y)A_1(y)}.\end{aligned}$$

By assumption, $A_1(y) = 1 + p(y) - \{1 - p(y)\}\{2 - q(y)\}q(y)$ is increasing over $[\tau, b]$, and thus,

$$2p^2(y)q^2(y)B_4(y) + p'(y)\{1 - 2q(y) - q^2(y)\} = A_1'(y) > 0.$$

Hence we have

$$(5.16)\quad \{1-q(y)\}B_2'(y) = p'(y)\{1-2q(y)-q^2(y)\} + p^2(y)q^2(y)B_4(y) > -p^2(y)q^2(y)B_4(y).$$

Further, it is easily seen that

$$(5.17)\qquad A_1(y) = \{1-q(y)\}B_2(y) + 2p(y)q(y)$$

Thus, from (5.16) and (5.17), it follows that

$$\begin{aligned}
&\{1-q(y)\}\{2B_2'(y) - p(y)q(y)B_2(y)B_4(y)\} + p(y)q(y)A_1(y)B_4(y)\\
&\quad = 2\{1-q(y)\}B_2'(y) + p(y)q(y)B_4(y)[A_1(y) - \{1-q(y)\}B_2(y)]\\
&\quad > -2p^2(y)q^2(y)B_4(y) + 2p^2(y)q^2(y)B_4(y) = 0
\end{aligned}$$

Therefore we obtain $g(y) \geqq 0$ for all x in $[a,b]$. This completes our proof.

Lemma 8. *The function* $h(z)$ *defined in Lemma* 6 *is positive over* $[b,1]$.

Proof. We directly get

$$\begin{aligned}
&\int_z^1 \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}} \exp\left\{-\frac{1}{2}\int_z^t \frac{q'(u)}{p(u)q(u)}\,du\right\} dt\\
&\quad = -2\int_z^1 q(t)^{-\frac{1}{2}} \frac{d}{dt}\left[\exp\left\{-\frac{1}{2}\int_z^t \frac{q'(u)}{p(u)q(u)}\,du\right\}\right] dt\\
&\quad = 2q(z)^{-\frac{1}{2}} - 2\exp\left\{-\frac{1}{2}\int_z^1 \frac{q'(t)}{p(t)q(t)}\,dt\right\}\\
&\qquad - \int_z^1 q(t)^{-\frac{3}{2}} q'(t) \exp\left\{-\frac{1}{2}\int_z^t \frac{q'(u)}{p(u)q(u)}\,du\right\} dt
\end{aligned}$$

and hence

$$\begin{aligned}
&2 - q(z)^{\frac{1}{2}} B_7(z)\\
&\quad = q(z)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\int_z^1 \frac{q'(t)}{p(t)q(t)}\,dt\right\} + q(z)^{\frac{1}{2}}\int_z^1 q(t)^{-\frac{3}{2}} q'(t)\exp\left\{-\frac{1}{2}\int_z^t \frac{q'(u)}{p(u)q(u)}\,du\right\} dt\\
&\quad > 0
\end{aligned}$$

for all z in $[b,1]$. Furthermore, since $p(z) - q(z) + p(z)q(z)$ is increasing we have

$$p'(z) > \frac{\{1-p(z)\}q'(z)}{1+q(z)}.$$

Thus,

$$\begin{aligned}
\frac{h(z)}{c_4} &= \frac{2q(z)p'(z)\{2-q(z)^{\frac{1}{2}}B_7(z)\} + \{1-p(z)\}q'(z)\left[\{1+p(z)\}q(z)^{\frac{1}{2}}B_7(z) - 2\right]}{2p(z)^2 q(z)^2}\\
&> \frac{2q(z)\{1-p(z)\}\left\{2-q(z)^{\frac{1}{2}}B_7(z)\right\}}{2p(z)^2q(z)^2\{1+q(z)\}}\, q'(z) + \frac{\{1-p(z)\}\left[\{1+p(z)\}q(z)^{\frac{1}{2}}B_7(z)-2\right]}{2p(z)^2q(z)^2}\, q'(z)\\
&= \frac{\{1-p(z)\}q'(z)}{2p(z)^2q(z)^2\{1+q(z)\}}\left[\{1+p(z)-q(z)+p(z)q(z)\}q(z)^{\frac{1}{2}}B_7(z) - 2\{1-q(z)\}\right].
\end{aligned}$$

From the definition of b, we have

$$B_7(z) \geqq \frac{2\{1-q(z)\}}{1+p(z)-q(z)+p(z)q(z)} q(z)^{-\frac{1}{2}}$$

for all z in $[b,1]$, and thus, we obtain $h(z)>0$ for all z in $[b,1]$. This completes our proof.

So far, we have assumed that the equaton (3.3) has a root a in $[\tau, b]$. From now on, in this section, we assume that the equation (3.3) does not have a root in $[\tau, b]$. We set, for x in $(0,b]$,

$$\begin{aligned}\rho_3(x) = &\frac{2}{\{1+p(x)-q(x)+p(x)q(x)\}q(x)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}q(x)+\frac{1}{2}\int_x^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \\ &-\int_x^b \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}} \exp\left\{-\frac{1}{2}q(t)+\frac{1}{2}\int_t^b \frac{\{1-q(u)\}q'(u)}{p(u)q(u)}\,du\right\} dt.\end{aligned}$$

It is easily seen that $\rho_3'(x)<0$ for all x in $(0,b)$, and hence

$$\begin{aligned}&\frac{2}{\{1+p(b)-q(b)+p(b)q(b)\}q(b)^{\frac{1}{2}}} e^{-\frac{1}{2}q(b)} \\ &< \frac{2}{\{1+p(x)-q(x)+p(x)q(x)\}q(x)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}q(x)+\frac{1}{2}\int_x^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \\ &\quad -\int_x^b \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}} \exp\left\{-\frac{1}{2}q(t)+\frac{1}{2}\int_t^b \frac{\{1-q(u)\}q'(u)}{p(u)q(u)}\,du\right\} dt\end{aligned}$$

for any x in $(0,b)$. Consequently we get

$$2-q(x)^{\frac{1}{2}}B_1(x)B_2(x)>0$$

for all x in $(0,b)$.

Lemma 9. *Set*

$$g^*(y)=c_3^*\{B_1(y)B_3(y)+B_4(y)\},$$

where

$$c_3^* = \frac{p(\tau)q(\tau)}{2-q(\tau)^{\frac{1}{2}}B_1(\tau)B_2(\tau)}.$$

Then

$$\int_\tau^b g^*(y)\,dy = 1$$

and

$$v_2^*(x) \leqq p(\tau)-\{1-p(\tau)\}\{2-q(\tau)\}q(\tau)$$

for all x in $[\tau,1]$, where $v_2^*(x)$ denotes the expected payoff when player II applies the strategy $\{g^*(y),h(z)\}$ and player I fires his bullet at the first moment when he is at the place x.

Proof. We set

$$B_5^*(x) = 1-\frac{2c_3^*}{p(x)q(x)}+\frac{c_3^*B_1(x)B_2(x)}{p(x)q(x)^{\frac{1}{2}}}$$

and

$$B_6^*(x) = q(\tau) + 2c_3^* q(\tau)^{\frac{1}{2}} B_1(\tau) - \frac{2c_3^*}{p(x)} + \frac{c_3^*\{1-p(x)\}\{1-q(x)\}q(x)^{\frac{1}{2}}}{p(x)} B_1(x).$$

It is easy to see that

$$B_5^*(\tau) = B_6^*(\tau) = 0,$$

$$\int_\tau^x g^*(y)\,dy = B_5^*(x)$$

and

$$\int_\tau^x q(y)g^*(y)\,dy = B_6^*(x).$$

Thus we have

$$\int_\tau^b g^*(y)\,dy = B_5^*(b) = 1$$

Further, for any x in $[\tau, b]$, we get

$$\begin{aligned} v_2^*(x) &= \int_\tau^x [-q(y) + \{1-q(y)\}p(x) - \{1-q(y)\}\{1-p(x)\}q(x)] g^*(y)\,dy \\ &\quad + \int_x^b [p(x) - \{1-p(x)\}\{2-q(x)\}q(x)] g^*(y)\,dy \\ &= p(\tau) - \{1-p(\tau)\}\{2-q(\tau)\}q(\tau) \\ &\quad + \left\{1 - \frac{B_2(x)}{B_2(\tau)}\right\}[4c_3^* - 1 - p(\tau) + \{1-p(\tau)\}\{2-q(\tau)\}q(\tau)]\,. \end{aligned}$$

Since the equation (3.3) does not have a root in $[\tau, b]$, we have

$$\begin{aligned} &\frac{1}{\{1+p(b)-q(b)+p(b)q(b)\}q(b)^{\frac{1}{2}}} e^{-\frac{1}{2}q(b)} \\ &\quad > \frac{1-q(\tau)}{[1+p(\tau)-\{1-q(\tau)\}\{2-q(\tau)\}q(\tau)]q(\tau)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}q(\tau) + \frac{1}{2}\int_\tau^b \frac{\{1-q(t)\}q'(t)}{p(t)q(t)}\,dt\right\} \\ &\quad\; - \frac{1}{2}\int_\tau^b \frac{q'(t)}{p(t)q(t)^{\frac{3}{2}}} \exp\left\{-\frac{1}{2}q(t) + \frac{1}{2}\int_t^b \frac{\{1-q(u)\}q'(u)}{p(u)q(u)}\,du\right\} dt, \end{aligned}$$

and thus

$$B_1(\tau) > \frac{2\{1-q(\tau)\}}{[1+p(\tau)-\{1-p(\tau)\}\{2-q(\tau)\}q(\tau)]q(\tau)^{\frac{1}{2}}}.$$

Therefore we have

$$4c_3^* > 1 + p(\tau) - \{1-p(\tau)\}\{2-q(\tau)\}q(\tau)$$

and hence

$$v_2^*(x) \leqq p(\tau) - \{1-p(\tau)\}\{2-q(\tau)\}q(\tau)$$

for all x in $[\tau, b]$. Further, for x in $[b, 1]$, we get

$$\begin{aligned} v_2^*(x) = & \int_\tau^b \int_b^x [-q(y) - \{1-q(y)\}q(z) + \{1-q(y)\}\{1-q(z)\}p(x)]g^*(y)h(z)\,dz\,dy \\ & + \int_\tau^b \int_x^1 [-q(y) + \{1-q(y)\}p(x) - \{1-q(y)\}\{1-p(x)\}q(x)]g^*(y)h(z)\,dz\,dy \\ & = \rho_2(x) \int_\tau^b \{1-q(y)\}g^*(y)\,dy - \int_\tau^b q(y)g^*(y)\,dy \\ & = p(b) - q(b) + p(b)q(b) - \{1 + p(b) - q(b) + p(b)q(b)\} \int_\tau^b q(y)g^*(y)\,dy \\ & = v_2^*(b) \leqq p(\tau) - \{1-p(\tau)\}\{2-q(\tau)\}q(\tau). \end{aligned}$$

This completes our proof.

6. Optimal Strategies

In this section, we give the game value and the optimal strategies for the duel.

Theorem 1. *If there is a root a with $p(a) - \{1-p(a)\}\{2-q(a)\}q(a) > 0$ for the equation (3.3), then the strategy $\{f(x), \alpha\}$ given in Lemma 4 is optimal for player* I *and the strategy $\{g(y), h(z)\}$ given in Lemma 5 and Lemma 6 is optimal for player* II. *Furthermore, the game value v is $p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$.*

Proof. It suffices to show that

$$v_1(y,z) \geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for all y and z with $0 \leqq y \leqq z \leqq 1$ and

$$v_2(x) \leqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for all x in $[0,1]$. From Lemma 4,

$$v_1(y,z) \geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for all y and z with $a \leqq y \leqq z < 1$. For y in $[0,a]$ and z in $[a,1)$, we get

$$\begin{aligned} v_1(y,z) = & -q(y) + \{1-q(y)\} \int_a^z [\,p(x) - \{1-p(x)\}q(x)]f(x)\,dx \\ & - \{1-q(y)\} \int_z^1 [\,q(z) - \{1-q(z)\}p(x)]f(x)\,dx + \alpha\{1-q(y)\}\{1-2q(z)\} \\ & = -1 + \{1-q(y)\} \int_a^z [\,1 + p(x) - \{1-p(x)\}q(x)]f(x)\,dx \\ & + \{1-q(y)\}\{1-q(z)\} \int_z^1 \{1+p(x)\}f(x)\,dx + 2\alpha\{1-q(y)\}\{1-q(z)\} \\ & \geqq v_1(a,z) \geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a), \end{aligned}$$

and for y and z with $0 \leqq y \leqq z \leqq a$, we have

$$\begin{aligned} v_1(y,z) = & -q(y) - \{1-q(y)\}q(z) + \{1-q(y)\}\{1-q(z)\} \int_a^1 p(x)f(x)\,dx \\ & + \alpha\{1-q(y)\}\{1-q(z)\} \\ & = -1 + \{1-q(y)\}\{1-q(z)\} \left\{1 + \int_a^1 p(x)f(x)\,dx + \alpha\right\} \\ & \geqq v_1(a,a) \geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a). \end{aligned}$$

Further, for all y in $[a,1]$

$$\begin{aligned} v_1(y,1) &= \int_a^y [\,p(x) - \{1-p(x)\}\{2-q(x)\}q(x)]f(x)\,dx \\ &\quad + \int_y^1 [\,-q(y) + \{1-q(y)\}p(x) - \{1-q(y)\}\{1-p(x)\}q(x)]f(x)\,dx \\ &\geqq \int_a^y [\,p(x) - \{1-p(x)\}\{2-q(x)\}q(x)]f(x)\,dx \\ &\quad + \int_y^1 [\,-q(y) + \{1-q(y)\}p(x) - \{1-q(y)\}\{1-p(x)\}q(x)]f(x)\,dx - \alpha \\ &\geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a), \end{aligned}$$

and for any y in $[0,a]$

$$\begin{aligned} v_1(y,1) &= -q(y) + \{1-q(y)\}\int_a^1 [\,p(x) - \{1-p(x)\}q(x)]f(x)\,dx \\ &\geqq -q(y) + \{1-q(y)\}\int_a^1 [\,p(x) - \{1-p(x)\}q(x)]f(x)\,dx - \alpha\{1-q(y)\} \\ &\geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a). \end{aligned}$$

Thus we obtain

$$v_1(y,z) \geqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for all y and z with $0 \leqq y \leqq z \leqq 1$.

By Lemma 5 and Lemma 6,

$$v_2(x) = p(a) - \{1-p(a)\}\{2-q(a)\}q(a).$$

for any x in $[a,1]$. Further, for x in $[0,a]$, we have

$$\begin{aligned} v_2(x) &= p(x) - \{1-p(x)\}\{2-q(x)\}q(x) \\ &\leqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a) \end{aligned}$$

since $p(x) - \{1-p(x)\}\{2-q(x)\}q(x)$ is unimodal and $p(a) - \{1-p(a)\}\{2-q(a)\}q(a) > 0$. Thus we get

$$v_2(x) \leqq p(a) - \{1-p(a)\}\{2-q(a)\}q(a)$$

for all x in $[0,1]$. This completes our proof.

In the following theorem, we assume that there is not a root for the equation (3.3) in the interval $[\tau, b)$. In this case, $p(\tau) - \{1-p(\tau)\}\{2-q(\tau)\}q(\tau) < 0$ since $A_1(x)$ is unimodal. We denote by I_0 the strategy of player I where player I stays indefinitely at the place 0.

Theorem 2. *If there is not a root for the equation* (3.3) *in the interval* $[\tau, b)$*, then the strategy* I_0 *is optimal for player* I *and the strategy* $\{g^*(y), h(z)\}$ *given in Lemma* 9 *and Lemma* 6 *is optimal for player* II*. Furthermore, the game value* v *is* 0*.*

Proof. From Lemma 9, it follows that

$$v_2^*(x) \leqq p(\tau) - \{1-p(\tau)\}\{2-q(\tau)\}q(\tau) \leqq 0$$

for any x in $[\tau, 1]$ and since $p(x) - \{1-p(x)\}\{2-q(x)\}q(x)$ is unimodal,

$$v_2^*(x) = p(x) - \{1-p(x)\}\{2-q(x)\}q(x) \leqq 0$$

for any x in $[0, \tau]$. Thus we have $v_2^*(x) \leqq 0$ for all x in $[0,1]$. Further, if player I stays at 0 indefinitely, then the expected payoff is obviously 0. This completes the proof.

7. Examples

In this section, we give examples which illustrate some of the results in Theorem 1 and Theorem 2.

Example 1. If $p(x) = x$ and $q(x) = x^2$, then $p(x) - q(x) + p(x)q(x)$ and $1 + p(x) - \{1 - p(x)\}\{2 - q(x)\}q(x)$ are increasing over $[0, 1]$. In this case we have

$$f(x) = \begin{cases} \dfrac{c_1^*}{x^3} \exp\left\{-\dfrac{x^2}{2} + \dfrac{1}{x} + x\right\}, & a \leqq x < b, \\ \dfrac{c_2^*}{x^3} \exp\left\{\dfrac{1}{x}\right\}, & b \leqq x \leqq 1, \end{cases}$$

$$\alpha = \frac{a\{1 + a - (1-a)(2-a^2)a^2\}}{2(1-a^2)(1-b^2)} \exp\left\{\frac{a^2}{2} - \frac{1}{a} - a - \frac{b^2}{2} + 1 + b\right\} = 0.0135,$$

$$\begin{aligned} g(y) = {} & \frac{2c_3(1-x)(1-3x^2-2x^3+x^4-x^5)}{x^4} \exp\left\{\frac{x^2}{2} - \frac{1}{x} - x\right\} \\ & \times \left[\frac{1}{(1+b-b^2+b^3)b} \exp\left\{-\frac{b^2}{2} + \frac{1}{b} + b\right\} + \int_x^b t^{-3} \exp\left\{-\frac{t^2}{2} + t + \frac{1}{t}\right\} dt\right] \\ & - \frac{2c_3(1-2x-x^2+x^3)}{x^5}, \end{aligned}$$

$$h(z) = \frac{2c_4}{x^2} + \frac{c_4(1-x-x^2)}{x^4} \exp\left\{1 - \frac{1}{x}\right\}$$

and

$$v = a - (1-a)(2-a^2)a^2 (= 0.1655),$$

where b $(= 0.3987)$ is the unique root in $(0, 1)$ of the equation

$$\frac{2x(1-x)^2}{1+x-x^2+x^3} = \exp\left\{1 - \frac{1}{x}\right\},$$

a $(= 0.2652)$ is the unique root in $(0, b)$ of the equation

$$\begin{aligned} & \frac{1-x^2}{x\{1+x-(1-x)(2-x^2)x^2\}} \exp\left\{-\frac{x^2}{2} + \frac{1}{x} + x\right\} - \int_x^b t^{-3} \exp\left\{-\frac{t^2}{2} + \frac{1}{t} + t\right\} dt \\ & = \frac{1}{(1+b-b^2+b^3)b} \exp\left\{-\frac{b^2}{2} + \frac{1}{b} + b\right\}, \end{aligned}$$

$$c_1^* = \frac{a\{1+a-(1-a)(2-a^2)a^2\}}{1-a^2} \exp\left\{\frac{a^2}{2} - \frac{1}{a} - a\right\},$$

$$c_2^* = \frac{a\{1+a-(1-a)(2-a^2)a^2\}}{(1-a^2)(1-b^2)} \exp\left\{\frac{a^2}{2} - \frac{1}{a} - a - \frac{b^2}{2} + b\right\},$$

$$c_3 = \frac{1+a-(1-a)(2-a^2)a^2}{4}$$

and

$$c_4 = \frac{1+b-b^2+b^3}{4}.$$

Example 2. If $p(x) = q(x) = x$, then $p(x) - q(x) + p(x)q(x) = x^2$ is increasing over $[0, 1]$. Further $A_1(x) = 1 - x + 3x^2 - x^3$ is decreasing over $[0, \tau)$ and increasing over $(\tau, 1]$, where $\tau = (3 - \sqrt{6})/3 = 0.1835$. The unique root b in $(0, 1)$ of the equation

$$e^{\frac{1}{2}} + \int_x^1 t^{-\frac{5}{2}} e^{\frac{1}{2t}}\, dt = \frac{2(1-x)}{x^2+1}\, x^{-\frac{1}{2}} e^{\frac{1}{2x}}$$

is 0.2524 and there is no root in (τ, b) for the equation

$$\frac{1-x}{1+x-(1-x)(2-x)x} \exp\left\{-\frac{x}{2} + \frac{1}{2x}\right\} - \frac{1}{2}\int_x^b t^{-2} \exp\left\{\frac{1}{2t} - \frac{t}{2}\right\} dt$$
$$= \frac{1}{1+b^2} \exp\left\{\frac{1}{2b} - \frac{b}{2}\right\}$$

which corresponds to (3.3). Thus, by Theorem 2, the optimal strategy for player I is staying at 0 indefinitely and the optimal strategy $\{g^*(y), h(z)\}$ for player II is

$$g^*(y) = \frac{c_3^*(y^4 + 2y^2 - 4y + 1)}{(1+b^2)y^4} \exp\left\{\frac{1}{2b} - \frac{b}{2} + \frac{y}{2} - \frac{1}{2y}\right\}$$
$$+ \frac{c_3^*(y^4 + 2y^2 - 4y + 1)}{2y^4} \exp\left\{\frac{y}{2} - \frac{1}{2y}\right\} \int_y^b t^{-2} \exp\left\{\frac{1}{2t} - \frac{t}{2}\right\} dt$$
$$- \frac{c_3^*(y^2 - 4y + 1)}{y^4}$$

$$h(z) = \frac{(1+b^2)(3z-1)}{4z^4} + \frac{(1+b^2)(1-2z-z^2)}{8z^{\frac{7}{2}}} e^{-\frac{1}{2z}} \left\{e^{\frac{1}{2}} + \int_z^1 t^{-\frac{5}{2}} e^{\frac{1}{2t}}\, dt\right\},$$

where $c_3^* = 0.9078$. Further, the game value is 0.

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