ON THE KP-SEMISIMPLE PART IN BCI-ALGEBRAS

ZHAN JIANMING & TAN ZHISONG

Received November 2, 2001

ABSTRACT. In this paper, we introduce the concept of kp-semisimple part in BCI-algebras and give some characterization of such algebras

1. Introduction and Preliminaries

A BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ with the following conditions:

(1) $((x*y)*(x*z))*(z*y) = 0$

(2) $(x*(x*y))*y = 0$

(3) $x*x = 0$

(4) $x*y = y*x = 0$ implies $x = y$.

A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x*y = 0$.

The following identities hold for any BCI-algebra X:

(1) $x*0 = x$,

(2) $(x*y^k)*z^k = (x*z^k)*y^k$,

(3) $0*(x*y)^k = (0*x^k)*(0*y^k)$,

(4) $0*(0*x)^k = 0*(0*x^k)$.

where k is any positive integer.

A nonempty subset I of a BCI-algebra X is called an ideal if $0 \in I$ and if $x*y, y \in I$ then $x \in I$. For any BCI-algebra X, the set $P(X) = \{x | 0*x = 0\}$ is called the BCK-part of X. If $P(X) = 0$, then we say that X is a p-semisimple BCI-algebra.

Definition 1.1([1]). A nonempty subset I of a BCI-algebra X is called a k-ideal of X if

(1) $0 \in I$

(2) $x*y^k \in I$ and $y \in I$ imply $x \in I$.

Definition 1.2. Let X be a BCI-algebra and k a positive integer, we define

$$SP_k(X) = \{x \in X \mid 0*(0*x)^k = x\}$$

We say that $SP_k(X)$ is the kp-semisimple part of X. In particular, if $k = 1$. the $SP(X)$ is called the p-semisimple part of X([2])

Proposition 1.3. $SP_k(X) \cap P(X) = 0$

Proof. If $x \in SP_k(X) \cap P(X)$, then $0*x = 0$ and $0*(0*x)^k = x$. Hence $x = 0$ and that $SP_k(X) \cap P(X) = 0$.

Proposition 1.4. For any BCI-algebra $X, SP_k(X)$ is a subalgebra of X.

2000 *Mathematics Subject Classification.* 03G25, 06F35.

Key words and phrases. kp-semisimple part, BCK-part, k-ideal.

Proof. Let $x, y \in SP_k(X)$, then $0 * (0 * x)^k = x$ and $0 * (0 * y)^k = y$.

$$0 * (0 * (x * y)^k) = 0 * (0 * (x * y)^k) = (0 * (0 * x^k)) * (0 * (0 * y^k)) = x * y$$

Hence $x * y \in SP_k(X)$.

2. Main Results

Theorem 2.1. For any BCI-algebra X. $SP_k(X)$ is a k-ideal if and only if for $x, y \in P(X)$ and $u, v \in SP_k(X)$, then $x * u^k = y * u^k$ implies $x = y$ and $u = v$.

Proof. If $SP_k(X)$ is a k-ideal of X and $x * u^k = y * v^k$ for any $x, y \in P(X)$ and $u, v \in SP_k(X)$, then $0 * (x * u^k) = 0 * (y * v^k)$ and thus $(0 * x) * (0 * u^k) = (0 * y) * (0 * v^k)$. Hence $0 * (0 * u^k) = 0 * (0 * v^k)$ since $x, y \in P(X)$. It follows that $u = v$ since $u, v \in SP_k(X)$. From this, we have $x * u^k = y * u^k$ and thus $(x * y) * u^k = (x * u^k) * y = (y * u^k) * y = (y * y) * u^k = 0 * u^k \in SP_k(X)$ by proposition 1.4. Hence $x * y \in SP_k(X)$ since $SP_k(X)$ is a k-ideal. Therefore $x * y = 0$ since $x * y \in SP_k(X) \cap P(X)$. Similarly, we have $y * x = 0$ and thus $x = y$.

Conversely, if $y, x * y^k \in SP_k(X)$, then

$x * y^k = 0 * (0 * (x * y^k)^k) = 0 * ((0 * x^k) * (0 * y^k)^k) = 0 * ((0 * x^k) * (0 * y^k)^k) = (0 * (0 * x^k)) * (0 * (0 * y^k)^k) = (0 * (0 * x^k)) * y^k$

By hypothesis, $x = 0 * (0 * x^k)$. Hence $x \in SP_k(X)$ and consequently $SP_k(X)$ is a k-ideal of X.

For any BCI-algebra X and any element a in X, we use a_r^k denote the k-selfmap of X defined by $a_r^k(x) = x * a^k$.

Theorem 2.2. For any BCI-algebra X, then $SP_k(X)$ is a k-ideal of X if and only if a_r^k is bijective for any $SP_k(X)$.

Proof. At first we assume that $SP_k(X)$ is a k-ideal of X and $a \in SP_k(X)$. If $a_r^k(x) = a_r^k(y)$, then $x * a^k = y * a^k$ for any $x, y \in X$. $(x * y) * a^k = (x * a^k) * y = (y * a^k) * y = 0 * a^k \in SP_k(X)$. From this it follows that $x * y \in SP_k(X)$ since $SP_k(X)$ is a k-ideal of X, and $a = 0 * (0 * a)^k = (0 * (x * y)^k) * (0 * a^k) = (0 * (0 * a^k)) * (x * y)^k = a * (x * y)^k$

In particular, $0 * (x * y)^k = 0$, and thus $x * y = 0 * (0 * (x * y^k) = 0$.

Similarly, we have $y * x = 0$, Therefore $x = y$ Hence a_r^k is injective. On the other hard, for any

$x \in X, ((x * a^k) * (0 * a)) * x = ((x * a^k) * x) * (0 * a) = (0 * a^k) * (0 * a) = 0 * a^{k-1} = a * a^k = (0 * (0 * a^k)) * a^k = 0$ and $(0 * a)_r^k a_r^k (x * ((x * a^k) * (0 * a))) = ((x * ((x * a^k) * (0 * a))) * a^k) * (0 * a)^k = ((x * ((x * a^k) * (0 * a))) * a^k) * (0 * a)^k = ((x * a^k) * (0 * a)^k) * (0 * a))) * a^k) * (0 * a)^k = ((x * a^k) * (0 * a)^k) * ((x * a^k) * (0 * a)) = 0 * (0 * a)^{k-1} = (0 * a) * (0 * a)^k = (0 * (0 * a)^k) * a = a * a = 0 = (0 * a^k) * (0 * a^k) = (0 * (0 * a^k)) * a^k = (0 * a)_r^k a_r^k$

Since $(0 * a)_r^k$ and a_r^k are injective, we have

$$x * ((x * a^k) * (0 * a)) = 0$$

Hence $x = (x * a^k) * (0 * a) = (x * (0 * a)) * a^k = a_r^k(x * (0 * a))$

Therefore a_r^k is surjective.

Conversely if a_r^k is bijective for any $a \in SP_k(X)$, then $SP_k(X)$ is a k-ideal of X by Theorem 2.1.

From the proof of Theorem 2.1, it's easy to see that $(0*a)_r^k$ is the inverse of a_r^k.

Theorem 2.3. Let X be a BCI-algebra. If $SP_k(X)$ is a k-ideal of X, then $a_r^k b_r^k = (a*(0*b^k))_r^k$ for any $a, b \in SP_k(X)$.

Proof. For any $x \in X$. $(a*(0*b^k))_r^k(((x*b^k)*a^k)*(x*(a*(0*b^k))^k) = (((x*b^k)*a^k)*(x*(a*(0*b^k))^k))*(a*(0*b^k))^k = (0*b^k)*a^k = 0*(a*(0*b^k))^k = (a*(0*b^k))_r^k(0)$ and $a_r^k b_r^k((x*(a*(0*b^k)*((x*b^k)*a^k)) = (((x*(a*(0*b^k))^k)*((x*b^k))*a^k))*a^k = 0*(a*(0*b^k))^k = (0*a^k)*(0*(0*b^k))^k = (0*a^k)*b^k = a_r^k b_r^k(0)$

Since $(a*(0*b^k)_r^k$ and $a_r^k b_r^k$ are injective, we have $((x*b^k)*a^k)*(x*(a*(0*b^k))^k) = 0$ and $(x*(a*(0*b^k))k)*((x*b^k)*a^k) = 0$. Hence $x*(a*(0*b^k))^k) = (x*b^k)*a^k$ and that $a_r^k b_r^k(x) = (a*(0*b^k))_r^k(x)$ for any $x \in X$. Hence $a_r^k b_r^k = (a*(0*b^k))_r^k$.

References

[1] Zhan Jianming & Tan Zhisong, *On the BCI-KG part of BCI-algebras*, Math. Japon **55** (2002), 149–152.

[2] Huang Wenping, *On the P-semisimple part in BCI-algebras*, Math. Japon **37** (1992), 159-161.

[3] L. Tiande and X. Changchang, *P-radical in BCI-algebras*, Math. Japon **30** (1985), 511-517.

[4] Y. B. Jun and Eoh. E. W., *On the BCI-G part of BCI-algebras*, Math. Japon **38** (1993), 697-702.

Department of Mathematics, Hubei Institute for Nationalities, Enshi, Hubei Province, 445000, China
E-mail:zhanjianming@hotmail.com