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A MOVING TARGET SEARCH PROBLEM WITH NESTED CONSTRAINTS OF SEARCH EFFORT

Ryusuke Hohzaki

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Abstract. This paper deals with an optimal distribution of search effort for a target moving on two-dimensional search space, say a geographical space with time flow. In most researches published so far on this subject, the amount of available search effort has an upper limit only at each time so that the algorithm of repeatedly solving a kind of resource allocation problem at each time worked well to obtain optimal solutions. However, on the two-dimensional search space, we have to consider the nested constraints of search effort, which mean constraints on the total amount of available effort on the whole space as well as at each time and make the problem difficult to be solved. In this paper, we derive necessary and sufficient conditions for optimality and propose two algorithms for an optimal solution, which perform better than some well-known nonlinear programming methods in terms of computational time.

1 Introduction This paper deals with an optimal distribution of search effort for a target moving on two-dimensional search space, say a geographical space with time flow. Since Koopman[9] studied first the search problem where continuous search effort is distributed for a stationary target on one-dimensional geographical space, many researchers have been dealing with the problem. de Guenin[2] generalized the Koopman's work and derived some optimality conditions by a variational method. In the case of discrete search effort, Kadane[7] found an optimal distribution for a stationary target hiding in one of some cells. When we consider a moving target, we need another space other than the geographical space to express the movement of the target, that is time space. Earlier studies on the moving target problem owe to Brown[1], Washburn[15], Iida[6] and other researchers who clarified the similarity and the difference between the stationary target problem and the moving target one, and devised methods of giving optimal solutions. In their problems, they set limits on the amount of the available effort only at each time but not during the whole time. It enabled them to construct methods for optimal solutions by repeatedly solving a stationary target problem at each time. Stromquist and Stone[14] mathematically generalized those results and Stone[13] compiled a wide variety of optimal distribution problems of search effort in a book. Among those researchers, only Stone had been going to deal with the multi-layered constraints, which mean constraints on the total amount of available search effort during the whole time as well as at each time. He developed his theory with the concept of deterministic target motion which means that a certain parameter at initial time determines the whole motion of the target ever since. Furthermore, he assumed the separability of time t and geographical coordinate x for the function to be dealt with, which he called factorability. Those characteristics made calculations on the process be executed separably with respect to t and x so that he avoided the difficulties involved in the two-dimensional problem. We[3] confronted a problem on one-dimensional space, where an

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objective is non-separable for variables but the constraint is given only on the total amount of variables, and proposed a new method for optimal solution. However, it is not strong enough to solve the general search problem for the moving target. It has to be extended to the problem with the nested constraints of variables on a two dimensional search space.

In this paper, we formulate the search problem with the nested constraints of search effort as a concave maximization problem and propose two methods of giving an optimal solution. The problem is more difficult than so-called resource allocation problem[5], which is usually defined as a concave maximization problem with a constraint only on the total amount of resources. However it is simpler than nonlinear problem or global optimization problem[8, 10, 11, 4]. The problem will be modeled as a search problem. But the methods proposed in this paper could be applied to many practical problems as long as they are formulated as a concave maximization with doubly layered constraints. In this paper, it will be clarified how better the proposed methods perform than well-known nonlinear programming methods in terms of computational efficiency.

We describe a search problem and formulate a concave maximization problem in the next section. In Section 3, we derive necessary and sufficient conditions for an optimal solution by introducing Lagrange multipliers. In Section 4, two methods are proposed for an optimal solution with the proof of these algorithmic validities. We compare the computational efficiency of the proposed methods with some methods known well in the field of nonlinear programming.

2 Modeling and Formulation of Problem Let us consider a search problem, where a searcher is searching a target with the limited amount of available search effort.

(1) A searcher wants to detect a target by distributing search effort in the search space of a discrete cell space and a discrete time space. The cell space and the time space are denoted by $\boldsymbol{K} = \{1, \cdots, K\}$ and $\boldsymbol{T} = \{1, \cdots, T\}$, respectively.

(2) The search effort can be continuously divided in the search space. Let $\varphi(i,t)$ be the effort distributed at a point $(i,t) \in \boldsymbol{K} \times \boldsymbol{T}$. The available search effort has some constraints on its amount. Its local constraint is given by $0 \le \varphi(i,t) \le m_{it}$. The total amount of effort at each time t and during the whole time periods must not be beyond $\Phi(t)$ and M, respectively, that is, $\sum_i \varphi(i,t) \le \Phi(t)$ for $t \in \boldsymbol{T}$ and $\sum_t \sum_i \varphi(i,t) \le M$.

(3) The target has a set of several possible paths, denoted by Ω. The target selects a path $\omega \in \Omega$ with probability $\pi(\omega)$. The path ω is represented by a sequence of cells according to time flow, $\{\omega(t), t \in \boldsymbol{T}\}$ where $\omega(t)$ is the target's position in the cell space at time t. Such path information is given to the searcher in advance. It is assumed that $\sum_{\omega \in \Omega} \pi(\omega) = 1$.

(4) By search effort $\varphi(i,t)$ distributed at a point $(i,t) \in \boldsymbol{K} \times \boldsymbol{T}$, the target is detected at time t with probability $1 - \exp(-\alpha_i \varphi(i,t))$ given that he exists there. The positive real number α_i indicates the detectability of cell i. Events of the detection at each time are assumed to occur independently one another.

(5) The searcher wants to know an optimal distribution of search effort so as to maximize the detection probability of the target.

First of all, let us derive the detection probability, which is an objective function. Using a search plan $\varphi = \{\varphi(i,t), (i,t) \in \boldsymbol{K} \times \boldsymbol{T}\}$, the detection probability of target on path ω is given by $1 - \exp(-\sum_t \alpha_{\omega(t)}\varphi(\omega(t), t))$ and therefore the entire detection probability $P(\varphi)$ has the following expression.

$$P(\varphi) = 1 - \sum_{\omega \in \Omega} \pi(\omega) \exp\left(-\sum_{t=1}^{T} \alpha_{\omega(t)}\varphi(\omega(t), t)\right) . \tag{1}$$

The function is finite, increasing and strictly concave for variable $\{\varphi(i,t),\ (i,t) \in S\}$ where $S \equiv \{(\omega(t), t),\ t = 1, \cdots, T,\ \omega \in \Omega\} \subseteq \boldsymbol{K} \times \boldsymbol{T}$. Now we can formulate the problem as a concave maximization problem as follows.

$$\max_{\varphi} P(\varphi) \tag{2}$$

$$s.t. \quad 0 \le \varphi(i,t) \le m_{it} \ , \quad (i,t) \in \boldsymbol{K} \times \boldsymbol{T} \tag{3}$$

$$\sum_{i} \varphi(i,t) \le \Phi(t) \ , \quad t \in \boldsymbol{T} \tag{4}$$

$$\sum_{t}\sum_{i} \varphi(i,t) \le M. \tag{5}$$

In the case of $\sum_{i=1}^{K} m_{it} \le \Phi(t)$, the limit $\Phi(t)$ is not necessary for the problem and similarly in the case of $\sum_{t=1}^{T} \Phi(t) \le M$, so is M. Therefore, we assume the following inequalities without loss of generality.

$$\sum_{i=1}^{K} m_{it} > \Phi(t),\ t \in \boldsymbol{T} \tag{6}$$

$$\sum_{t=1}^{T} \Phi(t) > M \ . \tag{7}$$

If $\sum_{i,t} \varphi(i,t) < M$, we can increase $P(\varphi)$ by additionally distributing residual effort $M - \sum_{i,t} \varphi(i,t)$ on any point (i,t). Therefore, we can replace an inequality sign with an equality sign in constraint (5). In result, the last formulation of our problem is as follows.

$$P_M \ : \quad \max_{\varphi} P(\varphi) \tag{8}$$

$$s.t. \quad 0 \le \varphi(i,t) \le m_{it} \ , \quad (i,t) \in \boldsymbol{K} \times \boldsymbol{T} \tag{9}$$

$$\sum_{i} \varphi(i,t) \le \Phi(t) \ , \quad t \in \boldsymbol{T} \tag{10}$$

$$\sum_{t}\sum_{i} \varphi(i,t) = M \ . \tag{11}$$

In the field of search theory, many researcher[6, 1, 15] have so far concentrated their effort on a special case of this problem, that is the problem with only the constraint of (10). Since constraints are given each time point in those studies, they can obtain an optimal solution as a convergence point by repeating the polynomial-time algorithm proposed for the typical resource allocation problem and improving current solution on the process at each time t. An additional constraint (11) of searching resources makes problem P_M harder to be solved because the balance of searching effort has to be taken account of on time flow.

3 Optimality Conditions Let Ψ be a feasible region satisfying constraints (9)-(11), which is a closed convex set. Since problem P_M is to maximize a strictly concave objective function on a closed convex feasible region, it has a unique optimal solution. For some (i,t), we denote a set of paths coming in cell i at time t by Ω_{it}, that is $\Omega_{it} = \{\omega \in \Omega | \omega(t) = i\}$. Here we elucidate necessary and sufficient conditions for optimality by introducing Lagrange multipliers and the relation between the constraints of the problem and the multipliers.

3.1 Necessary and sufficient conditions for optimality

Theorem 1 (Necessary and Sufficient Conditions) *A solution $\varphi \in \Psi$ is optimal if and only if there exit a positive multiplier λ and nonnegative multipliers $\nu_t,\ t = 1, \cdots, T$, which satisfy, for every $(i,t) \in \boldsymbol{K} \times \boldsymbol{T}$,*

$$if \quad \varphi(i,t) = 0 \ , \quad \alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(-\sum_{\tau=1}^{T} \alpha_{\omega(\tau)} \varphi(\omega(\tau), \tau)\right) \leq \lambda + \nu_t \tag{12}$$

$$if \quad 0 < \varphi(i,t) < m_{it} \ , \quad \alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(-\sum_{\tau=1}^{T} \alpha_{\omega(\tau)} \varphi(\omega(\tau), \tau)\right) = \lambda + \nu_t \tag{13}$$

$$if \quad \varphi(i,t) = m_{it} \ , \quad \alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(-\sum_{\tau=1}^{T} \alpha_{\omega(\tau)} \varphi(\omega(\tau), \tau)\right) \geq \lambda + \nu_t \tag{14}$$

and for every $t \in \boldsymbol{T}$,

$$if \quad \nu_t > 0 \ , \quad \sum_{i=1}^{K} \varphi(i,t) = \Phi(t) \ . \tag{15}$$

Proof: Except for the positivity of λ, the conditions above in the theorem are given as Kuhn-Tucker conditions[12] by considering the following Lagrangean function

$$\begin{aligned} L(\varphi; \lambda, \nu_t, \eta^1_{it}, \eta^2_{it}) \quad &= \quad P(\varphi) + \lambda\left(M - \sum_{t=1}^{T}\sum_{i=1}^{K} \varphi(i,t)\right) + \sum_{t=1}^{T} \nu_t \left(\Phi(t) - \sum_{i=1}^{K} \varphi(i,t)\right) \\ &\quad + \sum_{t=1}^{T}\sum_{i=1}^{K} \eta^1_{it} \varphi(i,t) + \sum_{t=1}^{T}\sum_{i=1}^{K} \eta^2_{it} (m_{it} - \varphi(i,t)) \ , \end{aligned} \tag{16}$$

where $\lambda,\ \nu_t, \eta^1_{it}$ and $\eta^2_{it},\ i = 1, \cdots, K,\ t = 1, \cdots, T$ are nonnegative multipliers and by noting that

$$\frac{\partial P}{\partial \varphi(i,t)} = \alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(-\sum_{\tau=1}^{T} \alpha_{\omega(\tau)} \varphi(\omega(\tau), \tau)\right) \ .$$

We can prove $\lambda > 0$ as follows. $\sum_{i=1}^{K} \varphi(i,t) = \Phi(t)$ can not be true for all $t \in \boldsymbol{T}$ because of condition (7). It follows that $\nu_t = 0$ for some t. For the t, condition (14) can not always hold for $i \in \boldsymbol{K}$ because of condition (6). It needs $\lambda > 0$ to hold either of conditions (12) and (13) for some $i \in \boldsymbol{K}$. **Q.E.D.**

The solution satisfying conditions (12)-(14) can be expressed as follows, using optimal multipliers $\lambda > 0,\ \nu_t \geq 0$.

$$\varphi(i,t) = [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}} \tag{17}$$

where

$$\gamma_{it}(\lambda + \nu_t; \varphi) \equiv \frac{1}{\alpha_i} \log \frac{\alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(-\sum_{\tau=1, \tau \neq t}^{T} \alpha_{\omega(\tau)} \varphi(\omega(\tau), \tau)\right)}{\lambda + \nu_t}$$

and symbol $[\cdot]_a^b$ indicates $[x]_a^b = \{b \ if \ b \le x; \ x \ if \ a < x < b; \ a \ if \ x \le a\}$. Using this notation, we can rewrite conditions (15) and (11) as follows.

$$If \quad \nu_t > 0 \ , \quad \sum_{i=1}^{K} [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}} = \Phi(t) \tag{18}$$

$$\sum_t \sum_i [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}} = M \ . \tag{19}$$

3.2 Relation between upper limits and Lagrange multipliers Consider two problems of P_{M_1} and P_{M_2}. The problem P_{M_1} has limits $\{M_1, \Phi_1(t)\}$ and the problem P_{M_2} has limits $\{M_2, \Phi_2(t)\}$ while other local upper limits $\{m_{it}\}$ are the same for both problems. Assume that P_{M_1} and P_{M_2} have φ_1 and φ_2 as their optimal solutions, respectively, and $\{\lambda_1, \nu_{1t}\}$ and $\{\lambda_2, \nu_{2t}\}$ as their optimal Lagrange multipliers. For φ_1 and φ_2, $\varphi_1 \neq \varphi_2$, we have the following inequality from the strict concavity of objective function $P(\cdot)$.

$$P(\varphi_2) < P(\varphi_1) + \sum_{i,t} \frac{\partial P}{\partial \varphi_1(i,t)} (\varphi_2(i,t) - \varphi_1(i,t)) \ . \tag{20}$$

Let $I_0 \equiv \{(i,t) | \varphi_1(i,t) = 0\}$, $I_1 \equiv \{(i,t) | 0 < \varphi_1(i,t) < m_{it}\}$ and $I_2 \equiv \{(i,t) | \varphi_1(i,t) = m_{it}\}$. From the optimality of φ_1 satisfying conditions (12)-(14), it follows that $\partial P / \partial \varphi_1(i,t) \le \lambda_1 + \nu_{1t}$ for $(i,t) \in I_0$, $\partial P / \partial \varphi_1(i,t) = \lambda_1 + \nu_{1t}$ for $(i,t) \in I_1$ and $\partial P / \partial \varphi_1(i,t) \ge \lambda_1 + \nu_{1t}$ for $(i,t) \in I_2$. Now we proceed further the transformation of (20).

$$\begin{aligned}
P(\varphi_2) &< P(\varphi_1) + \sum_{(i,t) \in I_0} (\lambda_1 + \nu_{1t}) \varphi_2(i,t) + \sum_{(i,t) \in I_1} (\lambda_1 + \nu_{1t})(\varphi_2(i,t) - \varphi_1(i,t)) \\
&\quad + \sum_{(i,t) \in I_2} (\lambda_1 + \nu_{1t})(\varphi_2(i,t) - m_{it}) \\
&= P(\varphi_1) + \sum_{i,t} (\lambda_1 + \nu_{1t})(\varphi_2(i,t) - \varphi_1(i,t)) \\
&= P(\varphi_1) + \lambda_1 \left(\sum_{i,t} \varphi_2(i,t) - \sum_{i,t} \varphi_1(i,t) \right) + \sum_t \nu_{1t} \left(\sum_i \varphi_2(i,t) - \sum_i \varphi_1(i,t) \right) \\
&= P(\varphi_1) + \lambda_1 (M_2 - M_1) + \sum_{\{t | \nu_{1t} > 0\}} \nu_{1t} \left(\sum_i \varphi_2(i,t) - \Phi_1(t) \right) \\
&\le P(\varphi_1) + \lambda_1 (M_2 - M_1) + \sum_t \nu_{1t} (\Phi_2(t) - \Phi_1(t)) \ .
\end{aligned}$$

Our final result is as follows.

$$P(\varphi_2) < P(\varphi_1) + \lambda_1 (M_2 - M_1) + \sum_t \nu_{1t} (\Phi_2(t) - \Phi_1(t)) \ . \tag{21}$$

Lemma 1 *Between total limit M of search effort and Lagrange multiplier λ, there exist the following relations.*

- (i) *When two problems P_{M_1}, P_{M_2} with different total limits M_1, M_2 and the same subtotal limits $\Phi(t)$ have optimal multipliers λ_1 and λ_2, respectively, it follows that $\lambda_1 < \lambda_2$ if $M_1 > M_2$ and vice versa.*
- (ii) *When λ approaches $\lambda_{max} \equiv \max_{i,t} \alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega)$, an optimal solution becomes $\varphi = \{0, \cdots, 0\}$. That is, $\lambda \to \lambda_{max}$ corresponds to the limit of $M = 0$.*

Proof: Now let us prove part (i). Suppose that problems P_{M_1}, P_{M_2} have optimal solutions $\varphi_1,\ \varphi_2$ and optimal multipliers $\lambda_1,\ \lambda_2$, respectively, where $\varphi_1 \neq \varphi_2$ of course. By applying $\Phi_1(t) = \Phi_2(t)$ to inequality (21), we obtain $P(\varphi_2) < P(\varphi_1) + \lambda_1(M_2 - M_1)$. Similarly we have $P(\varphi_1) < P(\varphi_2) + \lambda_2(M_1 - M_2)$ and consequently the following relation holds.

$$\lambda_1(M_1 - M_2) < P(\varphi_1) - P(\varphi_2) < \lambda_2(M_1 - M_2)\ .$$

It tells us that $\lambda_1 < \lambda_2$ if $M_1 > M_2$ and $\lambda_1 > \lambda_2$ if $M_1 < M_2$.

From (i), as M becomes smaller, the corresponding multiplier λ becomes larger. As M approaches 0, all search effort are shrinking to 0 and then all ν_t are going to be 0 because of the complementary slackness (15). Applying these to the expression of optimal solution (17), it follows that λ is supposed to be λ_{max} as $M \to 0$. Now the assertion of (ii) has be done. **Q.E.D.**

Similarly we can obtain the relation between $\Phi(t)$ and ν_t.

Lemma 2 *There are the following relations between subtotal limit of search effort $\Phi(t)$ and Lagrange multiplier ν_t at time point $t \in \boldsymbol{T}$.*

(i) Two problems, which have different subtotal limits $\Phi_1(t)$ and $\Phi_2(t)$ only at time t but the same subtotal limits at any other time and the same total limit M, have optimal multipliers $\nu_{1t} > 0$ and $\nu_{2t} > 0$, respectively. Then it follows that $\nu_{1t} < \nu_{2t}$ if $\Phi_1(t) > \Phi_2(t)$ and vice versa.

(ii) Let φ be an optimal solution of the following relaxed problem which is made by removing limit Φ_t at a certain time t.

$$RPL_t\ :\qquad \max_{\varphi}\ P(\varphi) \tag{22}$$

s.t.

$$0 \leq \varphi(i,\tau) \leq m_{i\tau},\quad i \in \boldsymbol{K},\quad \tau \in \boldsymbol{T} \tag{23}$$

$$\sum_{i=1}^{K} \varphi(i,\tau) \leq \Phi(\tau),\ t \neq \tau \in \boldsymbol{T} \tag{24}$$

$$\sum_{\tau=1}^{T}\sum_{i=1}^{K} \varphi(i,\tau) \leq M\ . \tag{25}$$

Let us denote the subtotal amount of the solution at time t by $\Phi_{max}(t) = \sum_i \varphi(i,t)$. For any problem with subtotal limit $\Phi(t) > \Phi_{max}(t)$, an optimal solution remains unchanged and an optimal multiplier ν_t is zero.

Proof: Let us denote optimal solutions of relevant two problems by φ_1 and φ_2. Noting that $\varphi_1 \neq \varphi_2$ from $\sum_i \varphi_1(i,t) = \Phi_1(t) \neq \Phi_2(t) = \sum_i \varphi_2(i,t)$, we may follow the same way used in Lemma 1 to verify part (i) in this Lemma.

Let φ and φ_Φ be optimal solutions for the relaxed problem RPL_t and the original one with limit $\Phi(t)$ at time t, respectively. Since $\Phi_{max}(t) = \sum_i \varphi(i,t)$, we have $\varphi = \varphi_{\Phi_{max}}$ and hence $P(\varphi) = P(\varphi_{\Phi_{max}})$. For arbitrary $\Phi(t)$ ($> \Phi_{max}(t)$), $P(\varphi_\Phi) \geq P(\varphi)$ holds, of course. At the same time, $P(\varphi_\Phi) \leq P(\varphi)$ because φ is an optimal solution for the relaxed problem. Now we have obtained $P(\varphi_\Phi) = P(\varphi)$ or $\varphi_\Phi = \varphi$ from the uniqueness of optimal solution. In this case, we have $\sum_i \varphi_\Phi(i,t) = \Phi_{max}(t) < \Phi(t)$ which indicates $\nu_t = 0$ from (15). **Q.E.D.**

In Lemma 1, we have clarified that a total upper limit of search effort M, $0 < M < \sum_t \Phi(t)$ corresponds to a unique optimal multiplier λ, $\lambda_{max} > \lambda > 0$. There is a similar correspondence between subtotal limit $\Phi(t)$ and multiplier ν_t with the exception of the case of $\nu_t = 0$, as we see in Lemma 2. These properties implicitly point out that Lagrange multipliers could be pointers to be easily manipulated for finding an optimal solution.

4 Methods for Optimal Solution Here we propose two methods for optimal solution of P_M.

4.1 Gradient-completion method This method repeats making up feasible solutions such that Eqs. (18) and (19) are kept satisfied until Eqs. (12)-(14) are fulfilled. The equations (12)-(14) are optimal conditions associated with the gradient of the Lagrangean function. That is why we call the algorithm the gradient-completion method. Before describing the algorithm, let us discuss about how to compute optimal Lagrange multipliers by using an optimal solution from now.

Let $\varphi^* = \{\varphi^*(i,t), i \in \boldsymbol{K}\ t \in \boldsymbol{T}\}$ be an optimal solution for problem P_M. Here we consider a procedure to derive optimal Lagrange multipliers λ^*, $\{\nu_t^*\}$ from φ^*. We define a function of $y \geq 0$ and $\varphi \in \Psi$ which indicates the subtotal amount of search effort at time t using the expression of optimal solution (17).

$$S_t(y,\varphi) \equiv \sum_{i=1}^{K} [\gamma_{it}(y;\varphi)]_0^{m_{it}} \quad . \tag{26}$$

$S_t(y,\varphi)$ is a monotone continuous decreasing function for y in the range of values of $(0, \sum_i m_{it})$ and there exists a unique root $\xi(t,\varphi)$ in $(0, \max_i\{\alpha_i \sum_{\omega\in\Omega_{it}} \pi(\omega)\})$ for the equation $S_t(\xi(t,\varphi),\varphi) = \Phi(t)$, where $0 < \Phi(t) < \sum_i m_{it}$. Now we sort $\{\xi(t,\varphi^*),\ t \in \boldsymbol{T}\}$ in the order of values, such as $\xi(t_1,\varphi^*) \leq \xi(t_2,\varphi^*) \leq \cdots \leq \xi(t_T,\varphi^*)$. Suppose $\lambda^* < \xi(t,\varphi^*)$. Then since $S_t(\lambda^*,\varphi^*) > S_t(\xi(t,\varphi^*),\varphi^*) = \Phi(t)$, condition $S_t(\lambda^*+\nu_t^*,\varphi^*) \leq \Phi(t)$ requires $\nu_t^* > 0$ and we have $\lambda^* + \nu_t^* = \xi(t,\varphi^*)$ from condition (15). Suppose $\lambda^* \geq \xi(t,\varphi^*)$. If $\nu_t^* > 0$, $S_t(\lambda^*+\nu_t^*,\varphi^*) < S_t(\lambda^*,\varphi^*) \leq S_t(\xi(t,\varphi^*),\varphi^*) = \Phi(t)$ holds which is inconsistent with (18). Therefore $\nu_t^* = 0$. Summarizing the above discussion, we have the following relation among λ^*, ν_t^* and $\xi(t,\varphi^*)$.

$$If\ \lambda^* < \xi(t,\varphi^*),\ \lambda^* + \nu_t^* = \xi(t,\varphi^*)\ and\ \sum_i \varphi^*(i,t) = \Phi(t)\ , \tag{27}$$

$$if\ \lambda^* \geq \xi(t,\varphi^*),\ \nu_t^* = 0\ . \tag{28}$$

The multiplier λ^* must be equal to or greater than $\xi(t_1,\varphi^*)$. Otherwise the supposition of $\lambda^* < \xi(t_1,\varphi^*)$ implies that equality holds in condition (10) at all $t \in \boldsymbol{T}$ from (27) and contradicts the basic assumption (7). Now there is a certain integer I such that $\xi(t_I,\varphi^*) \leq \lambda^* < \xi(t_{I+1},\varphi^*)$. From (27) and (28), we obtain $\nu_{t_1}^* = \cdots = \nu_{t_I}^* = 0$ and $\lambda^* + \nu_{t_k}^* = \xi(t_k,\varphi^*)$, $k = I+1,\cdots,T$, and the optimal solution as follows.

$$\varphi^*(i,t_k) = \begin{cases} [\gamma_{it_k}(\lambda^*;\varphi^*)]_0^{m_{it_k}} \ , & k = 1,\cdots,I\ , \\ [\gamma_{it_k}(\xi(t_k,\varphi^*);\varphi^*)]_0^{m_{it_k}}\ , & k = I+1,\cdots,T\ . \end{cases} \tag{29}$$

Now we can define the total amount of distributed search effort by the above expression.

$$\begin{aligned} Q(\lambda,\varphi) &\equiv \sum_{k=1}^{\mu_\lambda(\varphi)} \sum_{i=1}^{K} [\gamma_{it_k}(\lambda;\varphi)]_0^{m_{it_k}} + \sum_{k=\mu_\lambda(\varphi)+1}^{T} \sum_{i=1}^{K} [\gamma_{it_k}(\xi(t_k,\varphi);\varphi)]_0^{m_{it_k}} \\ &= \sum_{k=1}^{\mu_\lambda(\varphi)} \sum_{i=1}^{K} [\gamma_{it_k}(\lambda;\varphi)]_0^{m_{it_k}} + \sum_{k=\mu_\lambda(\varphi)+1}^{T} \Phi(t_k) \end{aligned} \tag{30}$$

where

$$\mu_\lambda(\varphi) \equiv \max\{k \mid \xi(t_k, \varphi) \le \lambda\} \ . \tag{31}$$

By the definition that the second term of (30) ought to be omitted in the case of $\mu_\lambda(\varphi) = T$, $Q(\lambda, \varphi)$ is available in the case of $\xi(t_T, \varphi^*) \le \lambda^*$ as well. We can derive the optimal multiplier λ^* by solving the equation $Q(\lambda^*, \varphi^*) = M$. Other optimal multipliers $\{\nu_t^*\}$ are given by $\nu_{t_k}^* = 0$ for $k = 1, \cdots, \mu_{\lambda^*}(\varphi^*)$ and $\nu_{t_k}^* = \xi(t_k, \varphi^*) - \lambda^*$ for $k = \mu_{\lambda^*}(\varphi^*) + 1, \cdots, T$. Then the optimal solution φ^* is represented by

$$\varphi^*(i, t_k) = \begin{cases} [\gamma_{it_k}(\lambda^*; \varphi^*)]_0^{m_{it_k}} \ , & k = 1, \cdots, \mu_{\lambda^*}(\varphi^*) \\ [\gamma_{it_k}(\xi(t_k, \varphi^*); \varphi^*)]_0^{m_{it_k}} \ , & k = \mu_{\lambda^*}(\varphi^*) + 1, \cdots, T \end{cases} \tag{32}$$

where T time points are sorted into $t_1, t_2, \cdots, t_T$ according to values of $\{\xi(t, \varphi^*),\ t = 1, \cdots, T\}$, such that $\xi(t_1, \varphi^*) \le \xi(t_2, \varphi^*) \le \cdots \le \xi(t_T, \varphi^*)$.

We are ready to explain the algorithm of the gradient-completion method.

Algorithm GC

(G1) Set $j = 0$ and make an initial feasible solution φ^j, e.g. $\varphi(i, t) = M\Phi(t)/(K \sum_t \Phi(t))$.

(G2) For $t \in \boldsymbol{T}$, calculate $\{\xi(t, \varphi^j) \mid t \in \boldsymbol{T}\}$ satisfying $S_t(\xi(t, \varphi^j), \varphi^j) = \Phi(t)$ and assign each element of $\boldsymbol{T}$ numbers $t_1, \cdots, t_T$ in the order of $0 < \xi(t_1, \varphi^j) \le \cdots \le \xi(t_T, \varphi^j)$. Obtain λ satisfying $Q(\lambda, \varphi^j) = M$.

Generate a new feasible solution $\widehat{\varphi}^j$ using $\xi(t_k, \varphi^j)$ and λ as follows.

$$\widehat{\varphi}^j(i, t_k) = \begin{cases} \left[\gamma_{it_k}(\lambda; \varphi^j)\right]_0^{m_{it_k}} \ , & k = 1, \cdots, \mu_\lambda(\varphi^j) \\ \left[\gamma_{it_k}(\xi(t_k, \varphi^j); \varphi^j)\right]_0^{m_{it_k}} \ , & k = \mu_\lambda(\varphi^j) + 1, \cdots, T. \end{cases} \tag{33}$$

(G3) If $\widehat{\varphi}^j = \varphi^j$, terminate. The current φ^j is the optimal solution. Otherwise, execute the following line search and generate the next feasible solution $\varphi^{j+1} = \varphi^j + \theta^*(\widehat{\varphi}^j - \varphi^j)$.

$$P(\varphi^j + \theta^*(\widehat{\varphi}^j - \varphi^j)) = \max_{0 < \theta \le \overline{\theta}} P(\varphi^j + \theta(\widehat{\varphi}^j - \varphi^j)) \ , \tag{34}$$

where $\overline{\theta} = \min\{\theta_1, \theta_2, \theta_3\}$ and $\theta_1, \theta_2, \theta_3$ are given by four estimations.

$$\theta_1 \equiv \begin{cases} \infty \ , \ \textit{if there is no } (i,t) \textit{ satisfying } \widehat{\varphi}^j(i,t) - \varphi^j(i,t) < 0 \\ \min_{i,t}\{ -\varphi^j(i,t)/(\widehat{\varphi}^j(i,t) - \varphi^j(i,t)) \ \big| \ \widehat{\varphi}^j(i,t) - \varphi^j(i,t) < 0\}, \ \textit{otherwise} \end{cases} \tag{35}$$

$$\theta_2 \equiv \begin{cases} \infty \ , \ \textit{if there is no } (i,t) \textit{ satisfying } \widehat{\varphi}^j(i,t) - \varphi^j(i,t) > 0 \\ \min_{i,t}\{ (m_{it} - \varphi^j(i,t))/(\widehat{\varphi}^j(i,t) - \varphi^j(i,t)) \ \big| \ \widehat{\varphi}^j(i,t) - \varphi^j(i,t) > 0\}, \textit{otherwise} \end{cases} \tag{36}$$

$$\theta_3 \equiv \begin{cases} \infty \ , \ \textit{if there is no } t \textit{ satisfying } \sum_i(\widehat{\varphi}^j(i,t) - \varphi^j(i,t)) > 0 \\ \min_t\{ (\Phi(t) - \sum_i \varphi^j(i,t))/(\sum_i(\widehat{\varphi}^j(i,t) - \varphi^j(i,t))) \big| \sum_i(\widehat{\varphi}^j(i,t) - \varphi^j(i,t)) > 0\}, \\ \qquad \textit{otherwise} \end{cases} \tag{37}$$

Increase j by one, $j = j + 1$, and go to (G2).

The range of θ in the line search (34) is estimated by considering the feasibility of $\varphi'(i,t) = \varphi^j(i,t) + \theta(\widehat{\varphi}^j(i,t) - \varphi^j(i,t))$, that is, $0 \le \varphi'(i,t) \le m_{it}$ and $\sum_i \varphi'(i,t) \le \Phi(t)$. From definitions (35)-(37), all of $\theta_1, \theta_2, \theta_3$ are equal to or more than 1 and at least one of them is finite and hence $1 \le \overline{\theta} < \infty$.

The representation (33) in Step (G2) are the same as (29) assuming that φ^j is optimal. By (33), $\widehat{\varphi}^j$ is generated so as to satisfy conditions (18) and (19). If $\widehat{\varphi}^j = \varphi^j$, the conditions (12)-(14) become valid at the same time, which indicates that current solution is optimal.

In the case of $\widehat{\varphi}^j \neq \varphi^j$ in (G2), the new solution φ^{j+1} is generated. Then if we can clarify that $P(\varphi^j) < P(\varphi^{j+1})$, the proof that $\lim_{j\to\infty} P(\varphi^j)$ converges to an optimal value will be completed from the boundedness of the feasible region Ψ and the finiteness of $P(\cdot)$. Since φ^{j+1} is given by the line search in the direction of $\widehat{\varphi}^j - \varphi^j$, if the direction always becomes an ascent direction, that is, $\nabla P(\varphi^j)(\widehat{\varphi}^j - \varphi^j) > 0$, we have $P(\varphi^j) < P(\varphi^{j+1})$. The assertion of $\nabla P(\varphi^j)(\widehat{\varphi}^j - \varphi^j) > 0$ is seen from the strict concavity of $P(\cdot)$ as follows.

$$\nabla P(\varphi^j)(\widehat{\varphi}^j - \varphi^j) = \sum_{i,t} \alpha_i \sum_{\omega\in\Omega_{it}} \pi(\omega) \exp\left(-\sum_{\tau=1}^{T} \alpha_{\omega(\tau)} \varphi^j(\omega(\tau),\tau)\right)(\widehat{\varphi}^j(i,t) - \varphi^j(i,t))$$

$$= \sum_{i,t} \alpha_i \sum_{\omega\in\Omega_{it}} \pi(\omega) \exp\left(-\sum_{\tau=1,\tau\neq t}^{T} \alpha_{\omega(\tau)} \varphi^j(\omega(\tau),\tau) - \alpha_i \widehat{\varphi}^j(i,t)\right)$$
$$\times \exp\left(-\alpha_i(\varphi^j(i,t) - \widehat{\varphi}(i,t))\right)(\widehat{\varphi}^j(i,t) - \varphi^j(i,t)) \ .$$

We subdivide $\boldsymbol{K} \times \boldsymbol{T}$ into $I_0 \equiv \{(i,t) \mid \widehat{\varphi}^j(i,t) = 0\}$, $I_1 \equiv \{(i,t) \mid 0 < \widehat{\varphi}^j(i,t) < m_{it}\}$ and $I_2 \equiv \{(i,t) \mid \widehat{\varphi}^j(i,t) = m_{it}\}$, and then we can transform the above expression as follows.

$$\geq \sum_{(i,t)\in I_0} (\lambda + \nu_t) \exp\left(-\alpha_i \varphi^j(i,t)\right)(-\varphi^j(i,t))$$
$$+ \sum_{(i,t)\in I_1} (\lambda + \nu_t) \exp\left(-\alpha_i(\varphi^j(i,t) - \widehat{\varphi}(i,t))\right)(\widehat{\varphi}^j(i,t) - \varphi^j(i,t))$$
$$+ \sum_{(i,t)\in I_2} (\lambda + \nu_t) \exp\left(-\alpha_i(\varphi^j(i,t) - m_{it})\right)(m_{it} - \varphi^j(i,t))$$
$$= \sum_{i,t} (\lambda + \nu_t) \exp\left(-\alpha_i(\varphi^j(i,t) - \widehat{\varphi}(i,t))\right)(\widehat{\varphi}^j(i,t) - \varphi^j(i,t))$$
$$= \sum_{i,t} (\lambda + \nu_t) \left\{\exp\left(\alpha_i(\widehat{\varphi}(i,t) - \varphi^j(i,t))\right) - 1\right\}(\widehat{\varphi}^j(i,t) - \varphi^j(i,t))$$
$$+ \sum_{i,t} (\lambda + \nu_t)(\widehat{\varphi}^j(i,t) - \varphi^j(i,t))$$
$$> \sum_{i,t} (\lambda + \nu_t)(\widehat{\varphi}^j(i,t) - \varphi^j(i,t))$$
$$= \lambda(M - M) + \sum_{\{t\mid\nu_t>0\}} \nu_t \sum_i (\widehat{\varphi}^j(i,t) - \varphi^j(i,t))$$
$$= \sum_{\{t\mid\nu_t>0\}} \nu_t \left(\Phi(t) - \sum_i \varphi^j(i,t)\right) \geq 0 \ .$$

We can see the validity of these transformation from the facts that $(\exp \beta x - 1)x$ is positive-valued for a coefficient $\beta > 0$ with the exception of $x = 0$ and $\sum_i \widehat{\varphi}^j(i,t) = \Phi(t)$ for $\nu_t > 0$. Now the proof is completed. The gradient-completion method generates a sequence of feasible solutions $\varphi^0, \varphi^1, \varphi^2, \cdots$ with the increasing values of $P(\varphi^0) < P(\varphi^1) < P(\varphi^2) < \cdots$ and terminates to give an optimal solution.

4.2 Total amount-completion method We know that there is a simple relation between the optimal multiplier λ and the total amount of search effort as stated in Lemma

1. We calculate an optimal solution $\varphi^{\lambda *}$ and the marginal limit M_λ corresponding to λ. If $M_\lambda > M$, the algorithm adjusts λ in an adequate direction which is pointed out in Lemma 1 and finds a solution satisfying $\sum_{i,t} \varphi^{\lambda *}(i,t) = M$ at last. The varying of λ means the indirect adjustment of the total amount of search effort. That is why we call the algorithm the total amount-completion method. Let φ^* be the optimal solution of the problem. The outline of the algorithm is as follows.

<u>Algorithm TAC</u>

(T1) Set $\underline{\lambda} = 0$ and $\overline{\lambda} = \lambda_{max}$ which is defined in Lemma 1.

(T2) Update λ by $\lambda = (\underline{\lambda} + \overline{\lambda})/2$ and calculate $\varphi^{\lambda *}$ by subprocedure AL_λ.

- (i) If $\sum_{i,t} \varphi^{\lambda *}(i,t) = M$, terminate. The current solution $\varphi^{\lambda *}$ is optimal.
- (ii) If $\sum_{i,t} \varphi^{\lambda *}(i,t) > M$, set $\underline{\lambda} = \lambda$ and repeat (T2).
- (iii) If $\sum_{i,t} \varphi^{\lambda *}(i,t) < M$, set $\overline{\lambda} = \lambda$ and repeat (T2).

AL_λ is the subprocedure to give the optimal solution $\varphi^{\lambda *}$ corresponding to the given multiplier λ.

<u>Algorithm AL_λ</u>

(A1) As a tentative solution, take φ delivered from the procedure TAC. Let us substitute $\{\tilde{\varphi}(i,t), i \in \boldsymbol{K}\}$, which is derived after the execution of steps (A2)-(A4) at time $t \in \boldsymbol{T}$, for a part $\{\varphi(i,t),\ i \in \boldsymbol{K}\}$ of φ. We denote this operation by operator Λ_t. Repeat the operation of $\varphi = \Lambda_t \varphi$ for all $t \in \boldsymbol{T}$ and make φ converge to a vector which is $\varphi^{\lambda *}$.

(A2) At a fixed time point t, initialize $\nu_t = 0$ and obtain $\{\varphi(i,t),\ i \in \boldsymbol{K}\}$ satisfying $\varphi(i,t) = [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}}$ by a subprocedure $AL_\lambda(\nu_t)$ which is described later. If $\sum_i \varphi(i,t) \le \Phi(t)$, terminate. Otherwise estimate ν_t and φ satisfying $\sum_i [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}} = \Phi(t)$ by the following steps (A3) and (A4).

(A3) Set $\underline{\nu} = 0$ and $\overline{\nu}$ being large enough.

(A4) Set $\nu_t = (\underline{\nu} + \overline{\nu})/2$ and obtain $\{\varphi(i,t), i \in \boldsymbol{K}\}$ satisfying $\varphi(i,t) = [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}}$ by subprocedure $AL_\lambda(\nu_t)$.

- (i) If $\sum_i \varphi(i,t) = \Phi(t)$, terminate.
- (ii) If $\sum_i \varphi(i,t) > \Phi(t)$, set $\underline{\nu} = \nu_t$ and repeat (A4).
- (iii) If $\sum_i \varphi(i,t) < \Phi(t)$, set $\overline{\nu} = \nu_t$ and repeat (A4).

<u>Algorithm $AL_\lambda(\nu_t)$</u>

Substitute $[\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}}$ for $\varphi(i,t)$ for $i = 1, 2, \cdots, K$.

The total amount-completion method terminates and then gives an optimal solution φ^* as the gradient-completion method does. Algorithm $AL_\lambda(\nu_t)$ produces a solution satisfying $\varphi(i,t) = [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}}$ at time point t.

In steps (A3) and (A4) of Algorithm AL_λ, while fixing φ at any other time point except t, $\{\varphi(i,t),\ i \in \boldsymbol{K}\}$ satisfying the following conditions at t is obtained.

$$\varphi(i,t) = [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}} \ , \ i \in \boldsymbol{K}$$
$$if \ \ \nu_t > 0 \ , \quad \sum_i \varphi(i,t) = \Phi(t) \ .$$

These conditions are equivalent to the following conditions as seen by analogy to Theorem 1.

$$If \ \ \varphi(i,t) = 0 \ , \ \ \alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(- \sum_{\tau=1}^{T} \alpha_{\omega(\tau)} \varphi(\omega(\tau), \tau) \right) \le \lambda + \nu_t$$
$$if \ \ 0 < \varphi(i,t) < m_{it} \ , \ \ \alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(- \sum_{\tau=1}^{T} \alpha_{\omega(\tau)} \varphi(\omega(\tau), \tau) \right) = \lambda + \nu_t$$
$$if \ \ \varphi(i,t) = m_{it} \ , \ \ \alpha_i \sum_{\omega \in \Omega_{it}} \pi(\omega) \exp\left(- \sum_{\tau=1}^{T} \alpha_{\omega(\tau)} \varphi(\omega(\tau), \tau) \right) \ge \lambda + \nu_t$$

for $i \in \boldsymbol{K}$ and

$$if \ \ \nu_t > 0 \ , \quad \sum_i \varphi(i,t) = \Phi(t) \ .$$

These conditions are necessary and sufficient for an optimal solution of the following convex programming problem $LP_t(\lambda)$ with a fixed t.

$$LP_t(\lambda) \ : \qquad \max_{\varphi} \ \left\{ P(\varphi) - \lambda \sum_{i,\tau} \varphi(i,\tau) \right\}$$

s.t.

$$0 \le \varphi(i,t) \le m_{it}, \quad i \in \boldsymbol{K}$$
$$\sum_i \varphi(i,t) \le \Phi(t)$$
$$\{\varphi(i,\tau),\ i \in \boldsymbol{K},\ t \ne \tau \in \boldsymbol{T}\} \ \ are \ \ given.$$

Since the objective function $\widehat{P}(\varphi) = P(\varphi) - \lambda \sum_{i,\tau} \varphi(i,\tau)$ is strictly concave and the feasible region is a closed convex set, the problem has a unique optimal solution. By the operation of $\Lambda_t\varphi$, $\{\varphi(i,t),\ i \in \boldsymbol{K}\}$ is changed to $\{\widetilde{\varphi}(i,t), i \in \boldsymbol{K}\}$ which is the optimal solution of problem $LP_t(\lambda)$ and hence $\widehat{P}(\varphi) \le \widehat{P}(\Lambda_t\varphi)$ where equality holds only if $\varphi = \Lambda_t\varphi$. By the repetition of Λ_t for $t = 1, \cdots, T$, the solution converges to a solution which is just the optimal solution of the following problem $LP(\lambda)$.

$$LP(\lambda) \ : \qquad \max_{\varphi} \ \left\{ P(\varphi) - \lambda \sum_{i,t} \varphi(i,t) \right\}$$
$$s.t. \quad 0 \le \varphi(i,t) \le m_{it}, \quad i \in \boldsymbol{K},\ t \in \boldsymbol{T}$$
$$\sum_i \varphi(i,t) \le \Phi(t), \quad t \in \boldsymbol{T} \ .$$

Denoting the total amount of the solution of AL_λ by $M_\lambda = \sum_{i,t} \varphi(i,t)$, it follows that $\varphi(i,t) = [\gamma_{it}(\lambda + \nu_t; \varphi)]_0^{m_{it}}$ for all (i,t) and $\sum_i \varphi(i,t) = \Phi(t)$ if $\nu_t > 0$ for $t \in \boldsymbol{T}$ and moreover $M_\lambda = \sum_{i,t} \varphi(i,t)$. This means that we already have obtained the optimal solution $\varphi^{\lambda *}$ with its total amount M_λ corresponding to the multiplier λ. Therefore, just when $\sum_{i,t} \varphi^{\lambda *}(i,t) = M$ occurs by adjusting multiplier λ in (T2), we have obtained an ultimately optimal solution. Now we have proved the validity of the total amount-completion method.

The revising method of $\lambda = (\underline{\lambda} + \overline{\lambda})/2$ in (T2) or $\nu_t = (\underline{\nu} + \overline{\nu})/2$ in (A4) is simply binary search. There could be other ideas for revising the multipliers. We can exploit the monotonicity of the relation between the multipliers and the upper limits of searching effort as explained in Lemma 1 and 2. That is, assuming that there is an inversely proportional relation between them, we obtain a new revising method for λ in (T2) or ν_t in (A4).

Let $\overline{M}$ and $\underline{M}$ ($\overline{M} > \underline{M}$) be the total limits corresponding to $\underline{\lambda}$ and $\overline{\lambda}$, respectively. The new revising of λ is as follows.

$$\lambda = \frac{\overline{M} - M}{\overline{M} - \underline{M}}\, \overline{\lambda} + \frac{M - \underline{M}}{\overline{M} - \underline{M}}\, \underline{\lambda}\ . \tag{38}$$

Similarly, letting $\overline{\Phi_t}$ and $\underline{\Phi_t}$ ($\overline{\Phi_t} > \underline{\Phi_t}$) be the subtotal limits corresponding to $\underline{\nu}$ and $\overline{\nu}$ at time $t \in \boldsymbol{T}$, respectively, we may revise ν_t in (A4) by the following estimation.

$$\nu_t = \frac{\overline{\Phi_t} - \Phi(t)}{\overline{\Phi_t} - \underline{\Phi_t}}\, \overline{\nu} + \frac{\Phi(t) - \underline{\Phi_t}}{\overline{\Phi_t} - \underline{\Phi_t}}\, \underline{\nu}\ . \tag{39}$$

5 Computational Efficiency of Proposed Methods We could find some intriguing examples concerning with the optimal distribution of search effort. However we concentrate our effort on investigating the computational efficiency of the proposed methods. It will be done by comparing them with some well-known nonlinear programming methods; the gradient projection method and the multiplier method.

Problems are randomly generated as follows. First we decide the number of cells K, the number of time points T and the number of target paths $|\Omega|$. A target path is constructed in such a way that a cell is randomly selected from K cells at each of T time points. The one-path-construction is repeated until the procedure generates $|\Omega|$ paths in all, and we set $\pi(\omega) = 1/|\Omega|$. For the detectability parameter α_i of cell i, a real number is randomly chosen in the interval $[\underline{\alpha}, \overline{\alpha}]$. Finally, we set the limits of search effort m_{it}, $\Phi(t)$, M and then finish the generation of a problem. Through all computer experiments here, we set $|\Omega| = 10$, $\underline{\alpha} = 0.1$, $\overline{\alpha} = 0.5$, $M = 5$, $\Phi(t) = 1$ and $m_{it} = 6$ which means that the local limit m_{it} gives no practical constraint on the amount of search effort. We change K and T by $K = 5, 20(5)$, $T = 5, 20(5)$ to measure CPU-times of solving many sizes of problems. Using a HITACHI S3600/120A mainframe computer and programming language FORTRAN 77, we solve each problem by four methods: the gradient-completion method, the total amount-completion method, the gradient projection method and the multiplier method which are abbreviated to the GC method, the TAC method, the GP method and the M method for short, respectively. For each size, 50 problems are generated and solved by each of four methods. CPU-times are averaged for 50 problems and shown in Table 1. A symbol *** indicates the case that the algorithm did not terminate in 300 seconds. Approximately, problems larger than $K = 10$ and $T = 10$ could not be solved by the M method, or larger than $K = 20$ and $T = 20$ by the GP method.

The computational time of the M method is approximately $100 \sim 1000$ times as much as the proposed methods. The GP method always expends more CPU-time than the proposed methods, especially for large K and T. For small problems, its computational time is $3 \sim 10$ times as much as the proposed methods and $10 \sim 100$ times as much for large problems. Among two of the proposed methods, superiority varies depending on the size of problem.

On the whole, the GC method is superior to the TAC method. We have verified the superiority of the proposed methods over the well-known nonlinear programming methods.

Table 1 CPU-times(sec) of four methods.

$K \setminus T$		5	10	15	20
5	GC	5.3×10^{-2}	2.7×10^{-1}	4.7×10^{-1}	7.1×10^{-1}
	TAC	4.0×10^{-2}	4.1×10^{-1}	8.3×10^{-1}	1.3×10^{0}
	GP	1.9×10^{-1}	9.2×10^{-1}	2.6×10^{0}	5.9×10^{0}
	M	7.9×10^{0}	$2.6 \times 10^{+2}$	$***$	$***$
10	GC	8.9×10^{-2}	4.4×10^{-1}	7.8×10^{-1}	1.2×10^{0}
	TAC	5.8×10^{-2}	7.6×10^{-1}	1.4×10^{0}	2.3×10^{0}
	GP	1.1×10^{0}	6.2×10^{0}	$2.3 \times 10^{+1}$	$5.5 \times 10^{+1}$
	M	$6.0 \times 10^{+1}$	$***$	$***$	$***$
15	GC	1.4×10^{-1}	6.5×10^{-1}	1.1×10^{0}	1.8×10^{0}
	TAC	7.6×10^{-2}	1.1×10^{0}	1.9×10^{0}	3.7×10^{0}
	GP	4.0×10^{0}	$2.6 \times 10^{+1}$	$7.7 \times 10^{+1}$	$***$
	M	$***$	$***$	$***$	$***$
20	GC	1.8×10^{-1}	6.6×10^{-1}	1.3×10^{0}	2.1×10^{0}
	TAC	1.1×10^{-1}	1.3×10^{0}	2.5×10^{0}	4.3×10^{0}
	GP	$1.0 \times 10^{+1}$	$6.1 \times 10^{+1}$	$***$	$***$
	M	$***$	$***$	$***$	$***$

6 Conclusions This paper deals with a moving target search problem with nested constraints of search effort, which have been left untouched so far. We derive necessary and sufficient conditions for optimality and propose two methods to give an optimal solution. Since this problem can be formulated as a concave maximization problem, other nonlinear programming methods can be applied to it. However, by numerical examination, it is clarified that the proposed methods are 10 to 1000 times as fast as some of well-known nonlinear programming methods for large size of problems.

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Department of Computer Science, National Defense Academy,
1-10-20 Hashirimizu, Yokosuka, 239-8686, Japan.
Phone: 0468-41-3810 Ext.3621
Fax: 0468-44-5912
E-mail address: hozaki@cc.nda.ac.jp