ON THE EXISTENCE OF (γ_p) k-SET CONTRACTIVE RETRACTIONS IN $L_p[0,1]$ SPACES, $1 \le p < \infty$

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Abstract. We prove that for any $\varepsilon > 0$ there exists a retraction of the closed unit ball in the space $L_p[0,1]$, $1 \le p < \infty$, onto the unit sphere being a (γ_p) $(2+\varepsilon)$-set contractive retraction.

1 Introduction. Let X be an infinite-dimensional Banach space with the closed unit ball B and the unit sphere S. A continuous mapping $R : B \to S$ with $Rx = x$ for any $x \in S$ is a retraction of the ball onto the sphere. Since the works of Nowak [5] and Benyamini and Sternfeld [2] it is known that, for any infinite-dimensional Banach space X, there exists a k-lipschitzian retraction $R : B \to S$ (i.e. a retraction satisfying the Lipschitz condition $\|Rx - Ry\| \le k\,\|x - y\|$, for all $x, y \in B$). Let ψ be a measure of noncompactness defined on X (see Section 2). A mapping $T : D(T) \subset X \to X$ is said to be a (ψ) k-set contraction if there exists a constant $k \ge 0$ such that

$$\psi(TA) \le k\psi(A), \quad \text{for all bounded sets } A \subset D(T).$$

We set

$$k_0(X) := \inf\{k \ge 1 : \text{there is a } k\text{-lipschitzian retraction } R : B \to S\},$$

$$k_\psi(X) := \inf\{k \ge 1 : \text{there is a } (\psi)\ k\text{-set contractive retraction } R : B \to S\}.$$

In [3] it is proved that $k_0(X) \ge 3$. Recall that the Hausdorff measure of noncompactness γ on a Banach space X is defined by

$$\gamma(A) := \inf\{r > 0 : A \text{ can be covered by a finite number of balls centered in } X\},$$

for all bounded sets $A \subset X$. If R is a k-lipschitzian retraction it is also (γ) k-set contractive. So that $k_\gamma(X) \le k_0(X)$ for any infinite-dimensional Banach space X. See the book of Toledano, Benavides and Acedo [7] and the references therein for more details concerning measures of noncompactness and (ψ) k-set contractions. In [9], the author proved that $k_\gamma(C[0,1]) = 1$ and that, for any infinite-dimensional Banach space X, there is no retraction $R : B \to S$ being both, k-lipschitzian for some constant k and (γ) 1-set contractive.

Further he posed the problem to estimate $k_\gamma(X)$ for particular classical Banach spaces and to establish for which spaces is $k_\gamma(X) < k_0(X)$. For $1 \le p < \infty$, let γ_p be the Hausdorff measure of noncompacness on $L_p[0,1]$. In the present note we prove that $k_{\gamma_p}(L_p[0,1]) \le 2$, $1 \le p < \infty$. Moreover, we observe that, for any infinite-dimensional Banach space X and for any measure of noncompacness ψ defined on X, there is no (γ) 1-set contractive retraction $R : B \to S$ being k-lipschitzian for some constant k.

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2 Notations and definitions. Let X be a Banach space and $\mathcal{B}$ the family of all bounded subsets of X. A mapping $\psi : \mathcal{B} \to [0, +\infty[$ is called a measure of noncompactness on X if it satisfies the following properties:

1) $\psi(A) = 0$ if and only if A is precompact ;
2) $\psi(\overline{co}A) = \psi(A)$, where $\overline{co}A$ denotes the closed convex hull of A ;
3) $\psi(A \cup B) = \max\{\psi(A), \psi(B)\}$;
4) $\psi(A + B) \leq \psi(A) + \psi(B)$;
5) $\psi(\lambda A) = |\lambda| \psi(A), \ \lambda \in \mathbb{R}$.

Let $L_p := L_p[0,1]$, $1 \leq p < \infty$, be the classical Lebesgue spaces with the usual norm denoted by $\|\cdot\|_p$. In the following we will assume $1 \leq p < \infty$ and we will always denote by S_p and B_p the unit sphere and the unit closed ball of L_p, respectively. Moreover, every function $f \in L_p$ will be extended outside $[0,1]$ by 0. Then for $f \in L_p$ and $h > 0$ consider the Steklow function

$$f_h(t) = \frac{1}{2h} \int_{[t-h,t+h]} f(s)ds,$$

for each $t \in [0,1]$. For any bounded set $A \subset L_p[0,1]$, we set

$$\omega_p(A) := \lim_{\delta \to 0} \sup_{f \in A} \max_{0<h\leq\delta} \|f - f_h\|_p .$$

It can be shown that ω_p is a measure of non compactness on L_p. Moreover, as a straightforward consequence of the Kolmogorov compactness criterion in the spaces L_p (see, e. g., [4]) we get the following

Theorem 1 *Let A be a bounded subset of L_p. Then*

$$\frac{1}{2}\omega_p(A) \leq \gamma_p(A) \leq \omega_p(A).$$

Remark 2 *In[8], Väth notes that the precise formula for the Hausdorff measure of noncompactness in L_p*

$$\gamma_p(A) = \frac{1}{2}\omega_p(A),$$

given in [1] *is false; see also* [16].

3 Results. Define a mapping $Q_p : B_p \to B_p$ by

$$(Q_p f)(t) = \begin{cases} \left(\frac{2}{1+\|f\|_p}\right)^{\frac{1}{p}} f\left(\frac{2}{1+\|f\|_p}t\right), & \text{for } t \in \left[0, \frac{1+\|f\|_p}{2}\right], \\ 0, & \text{for } t \in \left]\frac{1+\|f\|_p}{2}, 1\right]. \end{cases}$$

It is easy to see that $\|Q_p f\|_p = \|f\|_p$ for all $f \in B_p$ and $Q_p f = f$ for all $f \in S_p$.

Proposition 3 *The mapping Q_p is continuous.*

Proof. Let $\{f_n\}$ be a sequence of elements of B_p such that $f_n \to f(n \to \infty)$ with respect to the norm $\|\cdot\|_p$. Set

$$A_n := \left[0, \frac{1+\|f_n\|_p}{2}\right] \cap \left[0, \frac{1+\|f\|_p}{2}\right],$$

$$B_n := \left[0, \frac{1+\|f_n\|_p}{2}\right] \vartriangle \left[0, \frac{1+\|f\|_p}{2}\right],$$

for all $n \in \mathbb{N}$, where $\vartriangle$ denotes the symmetric difference. Let $\varepsilon > 0$. Since the family $\{f, f_1, f_2, ...\}$ has uniformly continuous norms and $\|f_n - f\|_p \to 0(n \to \infty)$, we can find an $n_1 \in \mathbb{N}$ such that $\|f_n - f\|_p \leq \frac{\varepsilon}{5}$ and

$$\|Q_p f_n - Q_p f\|_p \leq \left\|(Q_p f_n - Q_p f)\chi_{A_n}\right\|_p + \left\|(Q_p f_n - Q_p f)\chi_{B_n}\right\|_p$$

$$\leq \left\|(Q_p f_n - Q_p f)\chi_{A_n}\right\|_p + \frac{\varepsilon}{5}, \text{ for all } \ n \geq n_1.$$

Suppose $\|f_n\|_p \leq \|f\|_p$. Then, by the change of variables

$$s := \frac{2}{1+\|f_n\|_p} t \left(t \in \left[0, \frac{1+\|f_n\|_p}{2}\right]\right), \text{ it follows that}$$

$$\left\|(Q_p f_n - Q_p f)\chi_{A_n}\right\|_p$$

$$= \left[\int_{\left[0, \frac{1+\|f_n\|_p}{2}\right]} \left| \left(\frac{2}{1+\|f_n\|_p}\right)^{\frac{1}{p}} f_n\left(\frac{2}{1+\|f_n\|_p} t\right) - \left(\frac{2}{1+\|f\|_p}\right)^{\frac{1}{p}} f\left(\frac{2}{1+\|f\|_p} t\right)\right|^p dt\right]^{\frac{1}{p}}$$

$$= \left[\int_{[0,1]} \left| f_n(s) - \left(\frac{1+\|f_n\|_p}{1+\|f\|_p}\right)^{\frac{1}{p}} f\left(\frac{1+\|f_n\|_p}{1+\|f\|_p} s\right)\right|^p ds\right]^{\frac{1}{p}}$$

$$\leq \|f_n - f\|_p + \left[\int_{[0,1]} \left| f(s) - \left(\frac{1+\|f_n\|_p}{1+\|f\|_p}\right)^{\frac{1}{p}} f\left(\frac{1+\|f_n\|_p}{1+\|f\|_p} s\right)\right|^p ds\right]^{\frac{1}{p}}.$$

Now, suppose $\|f_n\|_p > \|f\|_p$. Then, by the change of variables

$$s := \frac{2}{1+\|f\|_p} t \left(t \in \left[0, \frac{1+\|f\|_p}{2}\right]\right), \text{ it follows that}$$

$$\left\|(Q_p f_n - Q_p f)\chi_{A_n}\right\|_p$$

$$= \left[\int_{\left[0, \frac{1+\|f\|_p}{2}\right]} \left| \left(\frac{2}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f_n \left(\frac{2}{1+\|f_n\|_p} t \right) - \left(\frac{2}{1+\|f\|_p} \right)^{\frac{1}{p}} f \left(\frac{2}{1+\|f\|_p} t \right) \right|^p dt \right]^{\frac{1}{p}}$$

$$= \left[\int_{[0,1]} \left| \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f_n \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} s \right) - f(s) \right|^p ds \right]^{\frac{1}{p}}$$

$$\leq \left[\int_{[0,1]} \left| \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f_n \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} s \right) - \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} s \right) \right|^p ds \right]^{\frac{1}{p}}$$

$$+ \left[\int_{[0,1]} \left| \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} s \right) - f(s) \right|^p ds \right]^{\frac{1}{p}} .$$

Moreover, by the change of variables $u := \frac{1+\|f\|_p}{1+\|f_n\|_p} s \; (s \in [0,1])$, we have

$$\left[\int_{[0,1]} \left| \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f_n \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} s \right) - \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} s \right) \right|^p ds \right]^{\frac{1}{p}}$$

$$= \left[\int_{\left[0, \frac{1+\|f\|_p}{1+\|f_n\|_p}\right]} |f_n(u) - f(u)|^p \, du \right]^{\frac{1}{p}} \leq \|f_n - f\|_p .$$

Then

$$\left\| (Q_p f_n - Q_p f) \chi_{A_n} \right\|_p$$

$$\leq \|f_n - f\|_p + \left[\int_{[0,1]} \left| \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} s \right) - f(s) \right|^p ds \right]^{\frac{1}{p}} .$$

We set

$$h_n(t) := \begin{cases} \left(\frac{1+\|f_n\|_p}{1+\|f\|_p} \right)^{\frac{1}{p}} f \left(\frac{1+\|f_n\|_p}{1+\|f\|_p} t \right), & \text{if } \|f_n\|_p \leq \|f\|_p , \\ \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} \right)^{\frac{1}{p}} f \left(\frac{1+\|f\|_p}{1+\|f_n\|_p} t \right), & \text{if } \|f_n\|_p > \|f\|_p , \end{cases} \qquad (t \in [0,1]),$$

for any $n \in \mathbb{N}$.

Choose a continuous function $g:[0,1]\to\mathbb{R}$ such that $\|g-f\|_p\le\frac{\varepsilon}{5}$. We put

$$g_n(t):=\begin{cases}\left(\frac{1+\|f_n\|_p}{1+\|f\|_p}\right)^{\frac{1}{p}}g\left(\frac{1+\|f_n\|_p}{1+\|f\|_p}t\right), & \text{if } \|f_n\|_p\le\|f\|_p\,,\\ \left(\frac{1+\|f\|_p}{1+\|f_n\|_p}\right)^{\frac{1}{p}}g\left(\frac{1+\|f\|_p}{1+\|f_n\|_p}t\right), & \text{if } \|f_n\|_p>\|f\|_p\,,\end{cases}\qquad (t\in[0,1])\,,$$

for any $n\in\mathbb{N}$. Suppose $\|f_n\|_p\le\|f\|_p$. By the change of variables $s:=\frac{1+\|f_n\|_p}{1+\|f\|_p}t\,(t\in[0,1])$, we obtain that

$$\|g_n-h_n\|_p$$

$$=\left[\int_{[0,1]}\left|\left(\frac{1+\|f_n\|_p}{1+\|f\|_p}\right)^{\frac{1}{p}}\left[g\left(\frac{1+\|f_n\|_p}{1+\|f\|_p}t\right)-f\left(\frac{1+\|f_n\|_p}{1+\|f\|_p}t\right)\right]\right|^p dt\right]^{\frac{1}{p}}$$

$$=\left[\int_{\left[0,\frac{1+\|f_n\|_p}{1+\|f\|_p}\right]}|g\,(s)-f(s)|^p\,ds\right]^{\frac{1}{p}}\le\|g-f\|_p\le\frac{\varepsilon}{5}.$$

If $\|f_n\|_p>\|f\|_p$, by the change of variables $s:=\frac{1+\|f\|_p}{1+\|f_n\|_p}t\,(t\in[0,1])$, it follows again $\|g_n-h_n\|_p\le\|g-f\|_p\le\frac{\varepsilon}{5}$. Since g is continuous, $\frac{1+\|f_n\|_p}{1+\|f\|_p}\to 1(n\to\infty)$ and $\frac{1+\|f\|_p}{1+\|f_n\|_p}\to 1(n\to\infty)$ we have that

$$|g_n\,(t)-g(t)|\to 0(n\to\infty),$$

for each $t\in[0,1]$. Then $g_n(t)\to g(t)(n\to\infty)$, for each $t\in[0,1]$. On the other hand

$$\|g-g_n\|_p=\begin{cases}\left[\int_{[0,1]}\left|g\,(t)-\left(\frac{1+\|f_n\|_p}{1+\|f\|_p}\right)^{\frac{1}{p}}g\left(\frac{1+\|f_n\|_p}{1+\|f\|_p}t\right)\right|^p dt\right]^{\frac{1}{p}}, & \text{if } \|f_n\|_p\le\|f\|_p\,,\\ \left[\int_{[0,1]}\left|g\,(t)-\left(\frac{1+\|f\|_p}{1+\|f_n\|_p}\right)^{\frac{1}{p}}g\left(\frac{1+\|f\|_p}{1+\|f_n\|_p}t\right)\right|^p dt\right]^{\frac{1}{p}}, & \text{if } \|f_n\|_p>\|f\|_p\,.\end{cases}$$

Moreover, we have that

$$\|g_n\|_p=\begin{cases}\left[\int_{\left[0,\frac{1+\|f_n\|_p}{1+\|f\|_p}\right]}|g\,(s)|^p\,ds\right]^{\frac{1}{p}}, & \text{if } \|f_n\|_p\le\|f\|_p\,,\\ \left[\int_{\left[0,\frac{1+\|f\|_p}{1+\|f_n\|_p}\right]}|g\,(s)|^p\,ds\right]^{\frac{1}{p}}, & \text{if } \|f_n\|_p>\|f\|_p\,.\end{cases}$$

Then

$$\lim_n\|g_n\|_p=\|g\|_p\,.$$

So that $\lim_n\|g_n-g\|_p=0$. Let $n_2\in\mathbb{N}$ such that $\|g_n-g\|_p\le\frac{\varepsilon}{5}$, for any $n\ge n_2$. Set $\nu:=\{n_1,n_2\}$, we have

$$\|Q_pf_n-Q_pf\|_p\le\left\|(Q_pf_n-Q_pf)\chi_{A_n}\right\|_p+\frac{\varepsilon}{5}$$

$$\leq \|f_n - f\|_p + \|f - h_n\|_p + \frac{\varepsilon}{5}$$

$$\leq \|f_n - f\|_p + \|f - g\|_p + \|g - g_n\|_p + \|g_n - h_n\|_p + \frac{\varepsilon}{5} \leq \varepsilon$$

for all $n \geq \nu$. ■

Proposition 4 *The mapping Q_p is an (γ_p) 2-set contraction.*

Proof. Let $f \in B_p$ and $0 < h \leq \frac{1}{4}$. Set $\alpha := \frac{1+\|f\|_p}{2}$. In this proof we consider the Steklov function $f_{\frac{h}{\alpha}}$ of f defined by

$$f_{\frac{h}{\alpha}}(t) = \frac{\alpha}{2h} \int_{\left[t-\frac{h}{\alpha}, t+\frac{h}{\alpha}\right]} f(s)ds,$$

for $t \in [0, 3/2]$ and equal to 0 elsewhere. Moreover, we still denote by $\|\cdot\|_p$ the usual norm on $L_p[0, 3/2]$. We start to prove that

$$\left\|f - f_{\frac{h}{\alpha}}\right\|_p = \|Q_p f - (Q_p f)_h\|_p .$$

Infact

$$\left\|f - f_{\frac{h}{\alpha}}\right\|_p^p = \int_{\left[0, \frac{3}{2}\right]} \left|f(\tau) - f_{\frac{h}{\alpha}}(\tau)\right|^p d\tau$$

$$= \int_{\left[0, \frac{h}{\alpha}\right]} \left|f(\tau) - f_{\frac{h}{\alpha}}(\tau)\right|^p d\tau + \int_{\left[\frac{h}{\alpha}, 1-\frac{h}{\alpha}\right]} \left|f(\tau) - f_{\frac{h}{\alpha}}(\tau)\right|^p d\tau$$

$$+ \int_{\left[1-\frac{h}{\alpha}, 1\right]} \left|f(\tau) - f_{\frac{h}{\alpha}}(\tau)\right|^p d\tau + \int_{\left[1, 1+\frac{h}{\alpha}\right]} \left|f(\tau) - f_{\frac{h}{\alpha}}(\tau)\right|^p d\tau$$

$$= \int_{\left[0, \frac{h}{\alpha}\right]} \left|f(\tau) - \frac{\alpha}{2h} \int_{\left[0, \tau+\frac{h}{\alpha}\right]} f(s)ds\right|^p d\tau + \int_{\left[\frac{h}{\alpha}, 1-\frac{h}{\alpha}\right]} \left|f(\tau) - \frac{\alpha}{2h} \int_{\left[\tau-\frac{h}{\alpha}, \tau+\frac{h}{\alpha}\right]} f(s)ds\right|^p d\tau$$

$$+ \int_{\left[1-\frac{h}{\alpha}, 1\right]} \left|f(\tau) - \frac{\alpha}{2h} \int_{\left[\tau-\frac{h}{\alpha}, 1\right]} f(s)ds\right|^p d\tau + \int_{\left[1, 1+\frac{h}{\alpha}\right]} \left|-\frac{\alpha}{2h} \int_{\left[\tau-\frac{h}{\alpha}, 1\right]} f(s)ds\right|^p d\tau.$$

By the change of variables $\tau = \frac{t}{\alpha}$ and $s = \frac{x}{\alpha}$, we obtain that

$$\left\|f - f_{\frac{h}{\alpha}}\right\|_p^p = \int_{[0,h]} \left|\frac{1}{\alpha^{\frac{1}{p}}} f(\frac{t}{\alpha}) - \frac{1}{2h} \int_{[0,t+h]} \frac{1}{\alpha^{\frac{1}{p}}} f(\frac{x}{\alpha})dx\right|^p dt$$

$$+ \int_{[h,\alpha-h]} \left|\frac{1}{\alpha^{\frac{1}{p}}} f(\frac{t}{\alpha}) - \frac{1}{2h} \int_{[t-h,t+h]} \frac{1}{\alpha^{\frac{1}{p}}} f(\frac{x}{\alpha})dx\right|^p dt$$

$$+\int_{[\alpha-h,\alpha]}\left|\frac{1}{\alpha^{\frac{1}{p}}}f(\frac{t}{\alpha})-\frac{1}{2h}\int_{[t-h,\alpha]}\frac{1}{\alpha^{\frac{1}{p}}}f(\frac{x}{\alpha})dx\right|^p dt$$

$$+\int_{[\alpha,\alpha+h]}\left|-\frac{1}{2h}\int_{[t-h,\alpha]}\frac{1}{\alpha^{\frac{1}{p}}}f(\frac{x}{\alpha})dx\right|^p dt.$$

Hence

$$\left\|f-f_{\frac{h}{\alpha}}\right\|_p^p=\int_{[0,h]}|Q_pf(t)-(Q_pf)_h(t)|^p\,dt+\int_{[h,\alpha-h]}|Q_pf(t)-(Q_pf)_h(t)|^p\,dt$$

$$+\int_{[\alpha-h,\alpha]}|Q_pf(t)-(Q_pf)_h(t)|^p\,dt+\int_{[\alpha,\alpha+h]}|Q_pf(t)-(Q_pf)_h(t)|^p\,dt$$

$$=\|Q_pf-(Q_pf)_h\|_p^p.$$

Then, for any set $A\subset B_p$, we have that

$$\omega'_p(Q_pA)=\lim_{\delta\to 0}\sup_{f\in A}\max_{0<h\le\delta}\|Q_pf-(Q_pf)_h\|_p\le\lim_{\delta\to 0}\sup_{f\in A}\max_{0<h\le 2\delta}\|f-f_h\|_p=\omega'_p(A),$$

where ω'_p is defined on $L_p[0,3/2]$ as ω_p. Let γ'_p be the Hausdorff measure of noncompactness on $L_p[0,3/2]$. Then, since $\gamma_p(C)=\gamma'_p(C)$ for all sets $C\subset B_p$ and an analogous of Theorem 1 holds in $L_p[0,3/2]$, we have that

$$\gamma_p(Q_pA)\le\omega'_p(Q_pA)\le\omega'_p(A)\le 2\gamma_p(A),$$

for all sets $A\subset B_p$. ■

For any $u>0$,we define the mapping $P_{p,u}:B_p\to L_p$ putting

$$(P_{p,u}f)(t):=\max\left\{0,\frac{u}{2}\left(2t-\|f\|_p-1\right)\right\},(t\in[0,1]).$$

Observe that, for any $u>0$ and for all $f\in B_p$, we have $(P_{p,u}f)(t)=0$ for any $t\in\left[0,\frac{1+\|f\|_p}{2}\right]$. Moreover, it easy to see that, for any $u>0$, the mapping $P_{p,u}$ is continuous and compact. We set, for any $u>0$,

$$F_{p,u}(\lambda):=\lambda^p+\frac{1}{2(p+1)}\left(\frac{u}{2}\right)^p(1-\lambda)^{p+1},(\lambda\in[0,1]).$$

Then, it is simple to verify that $F_{p,u}$ attains its minimum for a unique $\lambda_u\in\,]0,1[$. Moreover, $0<F_{p,u}(\lambda_u)<1$ and

$$\lim_{u\to\infty}\lambda_u=1.$$

For any $u>0$, consider the mapping $T_{p,u}:B_p\to L_p$ defined by

$$T_{p,u}f=Q_pf+P_{p,u}f.$$

Clearly, the mapping $T_{p,u}$ is an (γ_p) 2-set contraction, and $T_{p,u}f = f$ for any $f \in B_p$. Further, for any $u > 0$ and for all $f \in B_p$, we have that

$$\|T_{p,u}f\|_p^p = \int_{\left[0,\frac{1+\|f\|_p}{2}\right]} |Q_p f(t)|^p \, dt + \int_{\left[\frac{1+\|f\|_p}{2},1\right]} |Q_p f(t) + P_{p,u} f(t)|^p \, dt$$

$$= \|f\|_p^p + \int_{\left[\frac{1+\|f\|_p}{2},1\right]} \left|\frac{u}{2}\left(2t - \|f\|_p - 1\right)\right|^p dt$$

$$= \|f\|_p^p + \frac{1}{2(p+1)}\left(\frac{u}{2}\right)^p \left(1 - \|f\|_p\right)^{p+1} = F_{p,u}(\|f\|_p^p).$$

So that, $\|T_{p,u}f\|_p^p \geq F_{p,u}(\lambda_u)$ for any $f \in B_p$. Now, we define

$$R_{p,u}f = \frac{1}{\|T_{p,u}f\|_p} T_{p,u}f.$$

Then, for any set $A \subset B_p$, we have that

$$\gamma_p(R_{p,u}A) \leq \left(\frac{1}{F_{p,u}(\lambda_u)}\right)^{\frac{1}{p}} 2\gamma_p(A).$$

Therefore $R_{p,u} : B_p \to S_p$ is a (γ_p) $2\left(\frac{1}{F_{p,u}(\lambda_u)}\right)^{\frac{1}{p}}$-set contractive retraction. Since $\lim_{u\to\infty} F_{p,u}(\lambda_u) = 1$, for any $\varepsilon > 0$ there exists $u > 0$ such that the mapping $R_{p,u} : B_p \to S_p$ is a (γ_p) $(2+\varepsilon)$-set contractive retraction. Thus, the following result holds.

Theorem 5 $k_{\gamma_p}(L_p) \leq 2$.

In the context above described the following question remains open.

Problem 6 *Let X be an infinite-dimensional Banach space and let ψ be a measure of noncompactness on X. Does there exist a (ψ) 1-set contractive retraction $R : B \to S$?*

However, we have that

Theorem 7 *Let X be an infinite-dimensional Banach space and let ψ be a measure of noncompactness on X. If $R : B \to S$ is a (ψ) 1-set contractive retraction, then it is not $k-$lipschitzian for any constant k.*

The proof of the above theorem is carried out analogously to the proof of Theorem II in [9].

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