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THE NUMBER OF EXTREMAL DISKS EMBEDDED IN COMPACT RIEMANN SURFACES OF GENUS TWO

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Dedicated to Professor Masayuki Itô in honour of his sixtieth birthday

Abstract. A compact Riemann surface is said to be extremal if it admits extremal disks. In the present paper we shall find every extremal disk embedded in extremal surfaces of genus two. As a consequence, we shall show that an extremal surface of genus two admits at most four extremal disks.

1 Introduction Let S be a compact Riemann surface of genus $g \geq 2$. Then S is equipped with a metric induced by the hyperbolic metric of the hyperbolic plane. We shall use the unit disk $\Delta = \{z \in \mathbf{C}\ ;\ |z| < 1\}$ as a model of the hyperbolic plane with the metric derived from the differential

$$ds = \frac{2|dz|}{1-|z|^2}.$$

Let $D(r)$ be a disk of radius $r > 0$ isometrically embedded in S. It is known that radius r is bounded by R_g, which is determined by the genus g as follows [1]:

$$\cosh R_g = \frac{1}{2\sin\frac{\pi}{12g-6}}.$$

A disk $D(R_g)$ is said to be extremal, and a surface S admitting extremal disks is also said to be extremal. There exists an extremal surface that can admit more than one extremal disk if $g = 2$ [1] or $g = 3$ [3]. But every extremal surface admits a unique extremal disk if $g \geq 4$ [3]. In the present paper we shall find every extremal disk embedded in the extremal surface of genus two. As a consequence, we shall show that an extremal surface of genus two admits at most four extremal disks.

2 A regular polygon Let S be an extremal surface of genus $g \geq 2$ and $\pi : \Delta \to S$ the universal covering map. Then S has a regular $(12g-6)$-gon $F_g \subset \Delta$ with every angle $2\pi/3$ as a closed fundamental region. One of the extremal disks embedded in S is the image by π of the disk inscribed in the boundary ∂F_g with center at the origin [1].

We shall consider a regular 18-gon $F = F_2$ in the case of genus two. We label each side of the polygon as C_n (Figure 1), where we take n modulo 18. Put $R = R_2$ and $\beta = \pi/18$. Each side C_n is an arc of the Euclidean circle

$$\left| z - \frac{e^{2in\beta}}{\tanh R} \right| = \frac{1}{\sinh R}, \tag{1}$$

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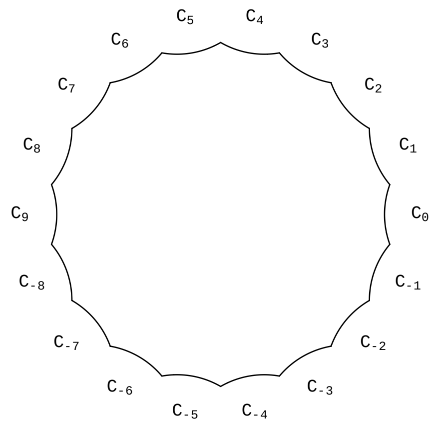

Figure 1: An 18-gon

so that

$$F = \bigcap_{n=0}^{17} \{z \in \Delta\ ;\ |z - \frac{e^{2in\beta}}{\tanh R}| \geq \frac{1}{\sinh R}\}. \tag{2}$$

We note that F is inscribed in the Euclidean circle

$$\{z \in \Delta\ ;\ |z| = \frac{2\sin 2\beta}{\tanh R}\}. \tag{3}$$

Let $A_{n,m}$ be the side pairing transformation of F from C_n onto C_m, that is, the Möbius transformation of Δ onto itself which maps C_n onto C_m. Then $A_{n,m}$ is of the form

$$A_{n,m}(z) = \frac{i\cosh R\, e^{i(m-n)\beta} z - i\sinh R\, e^{i(m+n)\beta}}{i\sinh R\, e^{-i(m+n)\beta} z - i\cosh R\, e^{-i(m-n)\beta}}. \tag{4}$$

In particular, $A_{9,m} : C_9 \to C_m$ is of the form

$$A_{9,m}(z) = \frac{\cosh R\, e^{im\beta} z + \sinh R\, e^{im\beta}}{\sinh R\, e^{-im\beta} z + \cosh R\, e^{-im\beta}}. \tag{5}$$

By using the rotation $r_\theta(z) = e^{i\theta} z$ around the origin, we see that every side pairing transformation $A_{n,m}$ is conjugate to $A_{9,l}$ for some l, precisely, $A_{n,m} = r_\theta^{-1} A_{9,l}\, r_\theta$, where $l \equiv 9+m-n \pmod{18}$ and $\theta \equiv 2(9-n)\beta \pmod{2\pi}$.

Lemma 2.1 *If $m \equiv n \pm 3, n \pm 4, \ldots, n \pm 8, n+9 \pmod{18}$, then $A_{n,m}$ is hyperbolic; if $m \equiv n \pm 1, n \pm 2 \pmod{18}$, then $A_{n,m}$ is elliptic. Hence $A_{n,n\pm1}$ and $A_{n,n\pm2}$ do not appear in the side pairing transformations of F.*

Proof. By considering the rotations around the origin, it is sufficient to show in the case $n = 9$ and $m = 0, \pm1, \ldots, \pm8$. Since $\cosh R = 1/(2\sin\beta)$, the trace of $A_{9,m}$ is

$$\begin{aligned} \mathrm{trace}(A_{9,m}) &= |\cosh R\, e^{im\beta} + \cosh R\, e^{-im\beta}| \\ &= |2\cosh R \cos m\beta| = \frac{\cos m\beta}{\sin\beta}. \end{aligned}$$

If $m = 0, \pm 1, \ldots, \pm 6$, then trace$(A_{9,m}) > 2$, namely, $A_{9,m}$ is hyperbolic. If $m = \pm 7, \pm 8$, then trace$(A_{9,m}) < 2$, namely, $A_{9,m}$ is elliptic. □

The axis of hyperbolic transformation $A_{9,m}$ is the hyperbolic line through the two fixed points of $A_{9,m}$. The following lemma gives the form of the axis of $A_{9,m}$.

Lemma 2.2 *The axis of $A_{9,m}$ $(m = \pm 1, \ldots, \pm 6)$ is an arc of the Euclidean circle*

$$\left| z - \frac{\tanh R\, e^{i(9+m)\beta}}{\sin m\beta} \right| = \frac{\sqrt{\tanh^2 R - \sin^2 m\beta}}{|\sin m\beta|}.$$

The axis of $A_{9,0}$ is the Euclidean segment on the real axis.

We have an equation for the hyperbolic distance with respect to the axis (cf. [2] p.163).

Lemma 2.3 *Let $z_0 \in \Delta$. Let $L = \{z \in \Delta\ ;\ |z - z_0| = r\}$ be a hyperbolic line of Δ, that is, $r^2 = |z_0|^2 - 1$. Then the hyperbolic distance $\rho(w, L)$ between $w \in \Delta$ and L satisfies the equation*

$$\sinh \rho(w, L) = \frac{|1 + |w|^2 - 2\Re(\bar{z}_0 w)|}{r(1 - |w|^2)}.$$

3 The number of extremal disks Let S be an extremal surface and $p \in S$ the center of an extremal disk. Let $\{\rho_k\}_{k=1}^{\infty}$ be the strictly increasing sequence of the hyperbolic distances of two distinct points in $\pi^{-1}(p) \subset \Delta$. Then, as mentioned in [1] (p.197), $\rho_1 = 2\sinh^{-1}(\sinh 2R \sin \beta) = 2R \approx 3.438$, $\rho_2 = 2\sinh^{-1}(\sinh 2R \sin 2\beta) \approx 4.746$, $\rho_3 = 2\sinh^{-1}(\sinh 2R \sin 3\beta) \approx 5.496$.

We define the closed subset K_n as

$$K_n = \{z \in F\ ;\ \rho(z, C_n) \le R\}\ (n = 0, \pm 1, \ldots, \pm 8, 9).$$

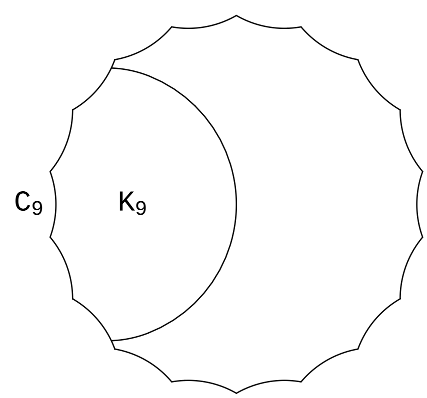

Figure 2: K_9

Lemma 3.1 *If $z \in K_n$, then it follows that*

$$\left|z - \frac{e^{2in\beta}}{2\tanh R}\right| \leq \frac{1}{2\tanh R}. \tag{6}$$

In particular, if $n = 9$, then

$$x \leq -|z|^2 \tanh R. \tag{7}$$

Proof. By Lemma 2.3 and (1), the inequality $\rho(z, C_n) \leq R$ implies

$$\sinh R \frac{|1 + |z|^2 - 2\Re(\frac{1}{\tanh R} e^{-2in\beta} z)|}{1 - |z|^2} \leq \sinh R,$$

that is,

$$\left|1 + |z|^2 - \frac{2}{\tanh R}(x \cos 2n\beta + y \sin 2n\beta)\right| \leq 1 - |z|^2. \tag{8}$$

From (2) it follows that

$$\left|z - \frac{e^{2in\beta}}{\tanh R}\right| \geq \frac{1}{\sinh R},$$

that is,

$$1 + |z|^2 - \frac{2}{\tanh R}(x \cos 2n\beta + y \sin 2n\beta) \geq 0.$$

Therefore (8) becomes

$$|z|^2 - \frac{1}{\tanh R}(x \cos 2n\beta + y \sin 2n\beta) \leq 0.$$

Hence (6) holds. □

Remark. More precisely, we note that K_n is a proper subset of

$$F \cap \{z \in \Delta\,;\ |z - \frac{e^{2in\beta}}{2\tanh R}| \leq \frac{1}{2\tanh R}\}.$$

Lemma 3.2 *For $z \in F$, the number of K_n containing z is 4, 5, 6, 7, 8, 9, or 18 according to the location of z (Figure 3), where if z is on the curve in Figure 3, then the number is the greatest one; if z is at the origin, then the number is 18.*

Theorem 3.3 *For a side pairing transformation $A_{n,m} : C_n \to C_m$ and for $z \in K_n$, it follows that $\rho(z, A_{n,m}(z)) < \rho_3$. Consequently $\rho(z, A_{n,m}(z)) = \rho_k$ $(k = 1, 2)$ because $\pi(z) = \pi(A_{n,m}(z))$.*

Proof. It is sufficient to show our theorem when $n = 9$. Hence suppose that $z \in K_9$. From (5) we have

$$|A_{9,m}(z)|^2 = \left|\frac{z \cosh R + \sinh R}{z \sinh R + \cosh R}\right|^2 = \frac{|z|^2 + \tanh^2 R + 2x \tanh R}{|z|^2 \tanh^2 R + 1 + 2x \tanh R}.$$

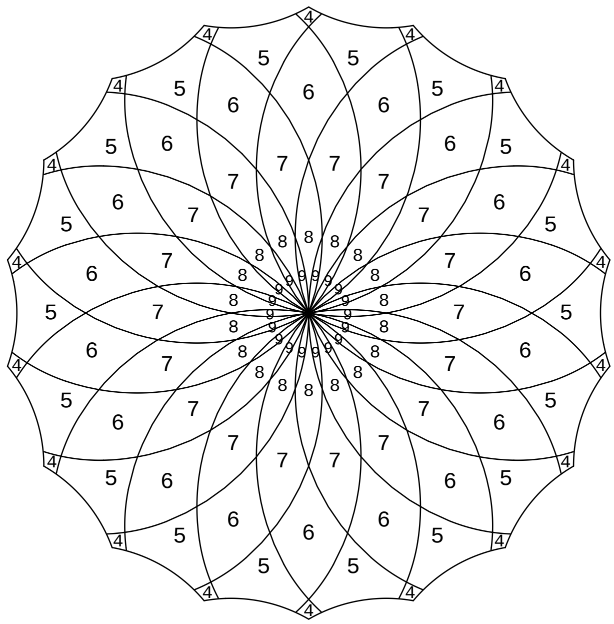

Figure 3: The number of the intersection of $\{K_n\}$

From [2] (p.131) and (7) we have

$$\begin{aligned}
\sinh^2 \frac{1}{2}\rho(z, A_{9,m}(z)) &= \frac{|z - A_{9,m}(z)|^2}{(1-|z|^2)(1-|A_{9,m}(z)|^2)} \\
&= \frac{|z - A_{9,m}(z)|^2}{(1-|z|^2)^2}(|z|^2 \sinh^2 R + \cosh^2 R + 2x \sinh R \cosh R) \\
&\leq \frac{(|z|+1)^2}{(1-|z|^2)^2}(|z|^2 \sinh^2 R + \cosh^2 R - 2|z|^2 \tanh R \sinh R \cosh R) \\
&= \frac{\cosh^2 R - |z|^2 \sinh^2 R}{(1-|z|)^2}.
\end{aligned}$$

Since $f(r) = (\cosh^2 R - r^2 \sinh^2 R)/(1-r)^2$ is the increasing function of r, $0 \leq r < 1$, $(\cosh^2 R - |z|^2 \sinh^2 R)/(1-|z|)^2$ has its maximum at $z \in K_9$ with the maximal absolute value $|z|$. As noted in (3), $K_n \subset \{z \in \Delta\ ;\ |z| \leq (2\sin 2\beta)/\tanh R\}$. Hence we obtain

$$\sinh^2 \frac{1}{2}\rho(z, A_{9,m}(z)) \leq f\left(\frac{2\sin 2\beta}{\tanh R}\right) \approx 60.27 < \sinh^2 \frac{1}{2}\rho_3 \approx 60.45.$$

Consequently $\rho(z, A_{9,m}(z)) < \rho_3$. □

Theorem 3.4 *The set* $E_{n,m}(\rho_k) = \{z \in \Delta\ ;\ \rho(z, A_{n,m}(z)) = \rho_k\}$ $(k = 1, 2)$ *is on the set* $L_{n,m}(\rho_k) \cup M_{n,m}(\rho_k)$ *described as follows:*

if $k=1$, *then*

$$L_{n,m}(\rho_1) \quad : \quad \left|z-\frac{\tanh R}{2\cos(n-m)\beta}e^{i(n+m)\beta}\right| = \frac{\tanh R}{2|\cos(n-m)\beta|} \ (m \not\equiv n+9 \pmod{18}),$$

$$M_{n,m}(\rho_1) \quad : \quad z=\frac{e^{2in\beta}}{\tanh R}-te^{i(n+m+9)\beta} \ (t\in \boldsymbol{R}),$$

if $k=2$, *then*

$$L_{n,m}(\rho_2) \quad : \quad \left|z-\frac{\tanh R\ e^{i(n+m)\beta}}{M+\cos(n-m)\beta}\right| = \frac{2\sinh R\sin 2\beta}{M+\cos(n-m)\beta},$$

$$M_{n,m}(\rho_2) \quad : \quad \left|z+\frac{\tanh R\ e^{i(n+m)\beta}}{M-\cos(n-m)\beta}\right| = \frac{2\sinh R\sin 2\beta}{M-\cos(n-m)\beta},$$

where $M=\sqrt{\tanh^2 R(4\cos^2\beta-1)+\cos^2(n-m)\beta}$.

Remark. In the case of $m\equiv n+9 \pmod{18}$, we can regard $L_{n,m}(\rho_1)$ as the Euclidean line $M_{n,m}(\rho_1)$.

Proof of Theorem 3.4. By considering the rotations around the origin, it is sufficient to consider the case $n=9$. Furthermore, since the set $E_{9,m}(\rho_k)$ derived from $A_{9,m}$ are complex conjugate to that from $A_{9,-m}$, it is sufficient to consider $A_{9,m}$ ($m=0,1,\ldots,6$).

Since $A_{9,m}$ is a hyperbolic transformation, we have the equation [2](p.174)

$$\sinh\frac{1}{2}\rho(z,A_{9,m}(z)) = \cosh\rho(z,\mathrm{ax}(A_{9,m}))\sinh\frac{1}{2}T(A_{9,m}), \tag{9}$$

where $\mathrm{ax}(A_{9,m})$ denotes the axis of $A_{9,m}$, and $T(A_{9,m})$ denotes the translation length of $A_{9,m}$. Since the translation length $T(A_{9,m})$ is determined by

$$\cosh\frac{1}{2}T(A_{9,m}) = \frac{1}{2}|\mathrm{trace}(A_{9,m})| = \cosh R\ \cos m\beta, \tag{10}$$

we have

$$\sinh\frac{1}{2}T(A_{9,m}) = \sqrt{\cosh^2 R\ \cos^2 m\beta-1}. \tag{11}$$

Also, since $\rho_k = 2\sinh^{-1}(\sinh 2R\sin k\beta)$ $(k=1,2)$, the equation $\rho(z,A_{9,m}(z))=\rho_k$ implies

$$\sinh\frac{1}{2}\rho(z,A_{9,m}(z)) = \sinh 2R\sin k\beta. \tag{12}$$

Substitute (11) and (12) for (9), and we have

$$\cosh\rho(z,\mathrm{ax}(A_{9,m})) = \frac{\sinh 2R\ \sin k\beta}{\sqrt{\cosh^2 R\ \cos^2 m\beta-1}}. \tag{13}$$

Hence

$$\begin{aligned} \sinh\rho(z,\mathrm{ax}(A_{9,m})) &= \sqrt{\frac{\sinh^2 2R\ \sin^2 k\beta-\cosh^2 R\ \cos^2 m\beta+1}{\cosh^2 R\ \cos^2 m\beta-1}} \\ &= \sqrt{\frac{4\sinh^2 R\ \sin^2 k\beta-\tanh^2 R+\sin^2 m\beta}{\tanh^2 R-\sin^2 m\beta}}. \end{aligned} \tag{14}$$

On the other hand, by Lemma 2.2 and 2.3,

$$\sinh \rho(z, \mathrm{ax}(A_{9,m})) = \frac{|\sin m\beta\,(|z|^2+1) - 2\tanh R\,(y\cos m\beta - x\sin m\beta)|}{\sqrt{\tanh^2 R - \sin^2 m\beta}(1-|z|^2)}. \tag{15}$$

Therefore, by (14) and (15), we obtain

$$\begin{aligned} &\frac{|\sin m\beta\,(|z|^2+1) - 2\tanh R\,(y\cos m\beta - x\sin m\beta)|}{1-|z|^2} \\ &= \sqrt{4\sinh^2 R\,\sin^2 k\beta - \tanh^2 R + \sin^2 m\beta}. \end{aligned} \tag{16}$$

If $k=1$, $m \neq 0$, and $\sin m\beta\,(|z|^2+1) - 2\tanh R\,(y\cos m\beta - x\sin m\beta) \geq 0$, then (16) becomes

$$\left(x + \frac{\tanh R}{2}\right)^2 + \left(y - \frac{\tanh R\,\cos m\beta}{2\sin m\beta}\right)^2 = \frac{\tanh^2 R}{4\sin^2 m\beta},$$

that is, z is on the curve

$$L_{9,m}(\rho_1):\ \left|z - \frac{\tanh R}{2\sin m\beta} e^{i(9+m)\beta}\right| = \frac{\tanh R}{2\sin m\beta}.$$

If $k=1$ and $\sin m\beta\,(|z|^2+1) - 2\tanh R\,(y\cos m\beta - x\sin m\beta) < 0$, then (16) becomes

$$x\sin m\beta - y\cos m\beta + \frac{\sin m\beta}{\tanh R} = 0,$$

that is, z is on the Euclidean line

$$M_{9,m}(\rho_1):\ z = te^{im\beta} - \frac{1}{\tanh R}\ (t \in \mathbf{R}).$$

If $k=2$ and $\sin m\beta\,(|z|^2+1) - 2\tanh R\,(y\cos m\beta - x\sin m\beta) \geq 0$, then (16) becomes

$$\left(x + \frac{\tanh R\,\sin m\beta}{M+\sin m\beta}\right)^2 + \left(y - \frac{\tanh R\,\cos m\beta}{M+\sin m\beta}\right)^2 = \frac{4\sinh^2 R\sin^2 2\beta}{(M+\sin m\beta)^2},$$

that is, z is on the curve

$$L_{9,m}(\rho_2):\ \left|z - \frac{\tanh R\ e^{i(9+m)\beta}}{M+\sin m\beta}\right| = \frac{2\sinh R\,\sin 2\beta}{M+\sin m\beta}.$$

If $k=2$ and $\sin m\beta\,(|z|^2+1) - 2\tanh R\,(y\cos m\beta - x\sin m\beta) < 0$, then (16) becomes

$$\left(x - \frac{\tanh R\,\sin m\beta}{M-\sin m\beta}\right)^2 + \left(y + \frac{\tanh R\,\cos m\beta}{M-\sin m\beta}\right)^2 = \frac{4\sinh^2 R\sin^2 2\beta}{(M-\sin m\beta)^2},$$

that is, z is on the curve

$$M_{9,m}(\rho_2):\ \left|z + \frac{\tanh R\ e^{i(9+m)\beta)}}{M-\sin m\beta}\right| = \frac{2\sinh R\,\sin 2\beta}{M-\sin m\beta}.$$

Hence we showed our theorem. □

The possible side pairings of the fundamental region F of 18 sides are completely obtained, and there are essentially 8 cases [4].

Proposition 3.5 *The side pairings of F are listed below, where (n, m) denotes the pair of C_n and C_m.*

Case 1. $(0,9),(1,4),(2,6),(3,7),(5,8),(-8,-5),(-7,-3),(-6,-2),(-4,-1)$

Case 2. $(0,9),(1,4),(2,-3),(3,-2),(5,8),(6,-7),(7,-6),(-8,-5),(-4,-1)$

Case 3. $(0,3),(1,7),(2,8),(4,-6),(5,-3),(6,9),(-8,-4),(-7,-2),(-5,-1)$

Case 4. $(0,9),(1,-5),(2,6),(3,-2),(4,-8),(5,-1),(7,-6),(8,-4),(-7,-3)$

Case 5. $(0,9),(1,-7),(2,5),(3,-4),(4,-3),(6,-8),(7,-1),(8,-6),(-5,-2)$

Case 6. $(0,5),(1,8),(2,-5),(3,-2),(4,9),(6,-7),(7,-4),(-8,-3),(-6,-1)$

Case 7. $(0,9),(1,-6),(2,-4),(3,7),(4,-2),(5,-8),(6,-1),(8,-5),(-7,-3)$

Case 8. $(0,9),(1,4),(2,-7),(3,-6),(5,8),(6,-3),(7,-2),(-8,-5),(-4,-1)$

We shall give our main theorem, which shows that an extremal surface of genus two admits at most four extremal disks.

Theorem 3.6 *Let S be an extremal surface of genus two. Then the centers of extremal disks embedded in S are the following, where $\pi = \pi|_F : F \to S$:*

Case 1. $\pi(0),\ \pi(-\tanh\frac{R}{2})$;

Case 2. $\pi(0),\ \pi(-\tanh\frac{R}{2})$;

Case 3. $\pi(0),\ \pi(\frac{i\cosh R}{\tanh R\sqrt{4\cosh^2 R-1}})$;

Case 4. $\pi(0),\ \pi(-\tanh\frac{R}{2})$;

Case 5. $\pi(0),\ \pi(\frac{1}{\sinh R})$;

Case 6. $\pi(0),\ \pi(\frac{\sqrt{4\cosh^2 R-1}}{\sinh R\cosh R}e^{3i\beta}),\ \pi(\frac{\sqrt{4\cosh^2 R-1}}{\sinh R\cosh R}e^{7i\beta}),\ \pi(\frac{\sqrt{4\cosh^2 R-1}}{\sinh R\cosh R}e^{-i\beta})$;

Case 7. $\pi(0),\ \pi(\frac{1}{\tanh R\,(1+\tanh^2 R)})$;

Case 8. $\pi(0)$.

Proof. Let $z \in F$ and suppose that the image of z by $\pi|_F : F \to S$ is the center of an extremal disk embedded in S. If $z \in K_n$ for some n, then, by Theorem 3.3, the side pairing transformation $A_{n,m}$ from the side C_n to another side C_m satisfies $\rho(z, A_{n,m}(z)) = \rho_k$ $(k = 1, 2)$. Hence z must be on the set $K_n \cap E_{n,m}(\rho_k)$. Consequently z is on $K_n \cap E_{n,m}(\rho_k)$ for every n satisfying $K_n \ni z$. So that a necessary condition for $\pi(z) \in S$ to be the center of an extremal disk is that z is the intersection point of such sets $K_n \cap E_{n,m}(\rho_k)$ with the intersection number N, where N is determined by the location of z in Figure 3. For each case $1, 2, \ldots, 8$, Figure 4 shows points $\{z_i\}$ satisfying the necessary condition. By simple calculations, we have $\{z_i\}$ as follows:

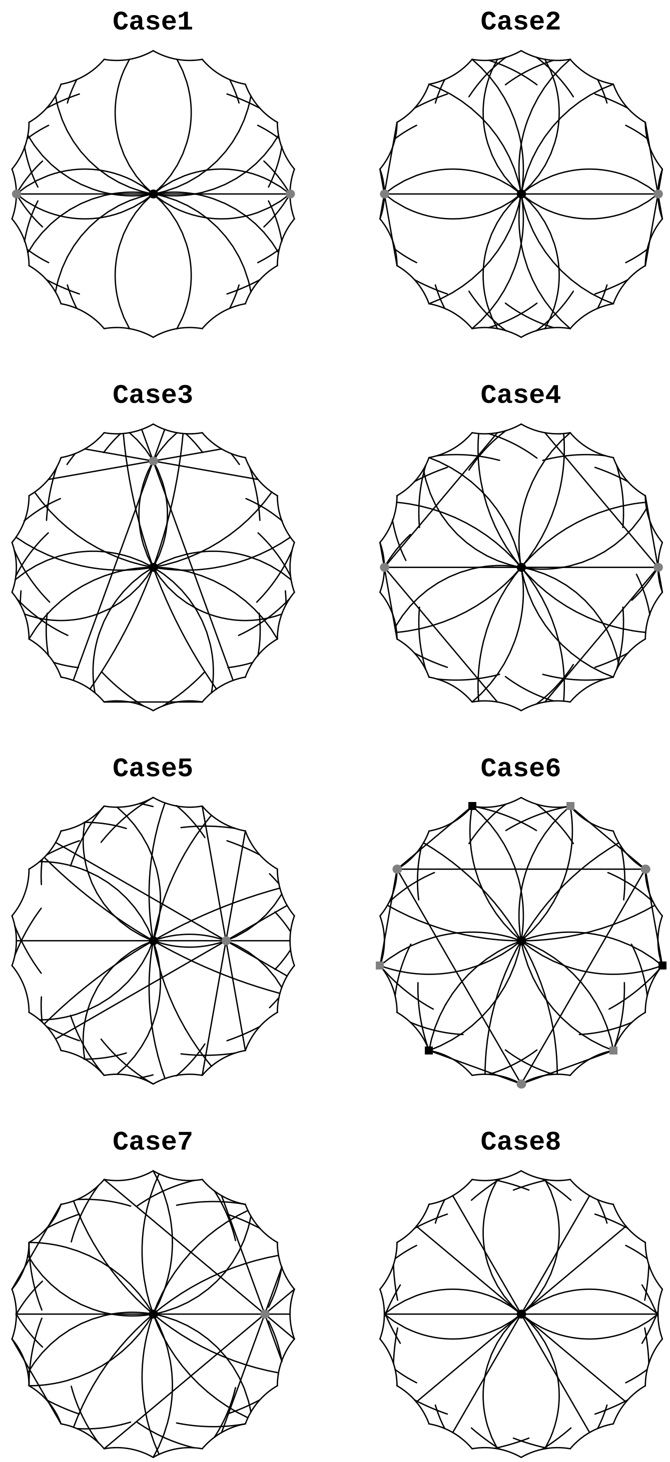

Figure 4: Intersection points

Case 1. $z_0 = 0$, $z_1 = -\tanh(R/2)$, $z_2 = \tanh(R/2)$. Note that $\pi(z_1) = \pi(z_2)$;

Case 2. $z_0 = 0$, $z_1 = -\tanh(R/2)$, $z_2 = \tanh(R/2)$. Note that $\pi(z_1) = \pi(z_2)$;

Case 3. $z_0 = 0$, $z_1 = i\cosh R/(\tanh R\sqrt{4\cosh^2 R - 1})$;

Case 4. $z_0 = 0$, $z_1 = -\tanh(R/2)$, $z_2 = \tanh(R/2)$. Note that $\pi(z_1) = \pi(z_2)$;

Case 5. $z_0 = 0$, $z_1 = 1/\sinh R$;

Case 6. $z_0 = 0$, $z_1 = re^{3i\beta}$, $z_2 = re^{15i\beta}$, $z_3 = re^{27i\beta}$, $z_4 = re^{7i\beta}$, $z_5 = re^{19i\beta}$, $z_6 = re^{31i\beta}$, $z_7 = re^{11i\beta}$, $z_8 = re^{23i\beta}$, $z_9 = re^{35i\beta}$, where $r = \sqrt{4\cosh^2 R - 1}/(\sinh R\cosh R)$. Note that $\pi(z_1) = \pi(z_2) = \pi(z_3)$, $\pi(z_4) = \pi(z_5) = \pi(z_6)$, and $\pi(z_7) = \pi(z_8) = \pi(z_9)$;

Case 7. $z_0 = 0$, $z_1 = 1/(\tanh R\,(1 + \tanh^2 R))$;

Case 8. $z_0 = 0$.

Since $\pi(0)$ is the center of an extremal disk [1], a sufficient condition for $\pi(z_i)$ $(i \neq 0)$ to be the center of an extremal disk is that there exists an automorphism $f : S \to S$ such that $f(\pi(0)) = \pi(z_i)$. Let Γ be a Fuchsian group representing S, namely, $\Delta/\Gamma = S$. Then every automorphism f is represented by an Möbius transformation $\gamma_f : \Delta \to \Delta$ such that $\Gamma\gamma_f = \gamma_f\Gamma$. Since Γ is finitely generated by the side pairing transformations $\{A_{n,m}\}$, it is sufficient to find $\gamma \in \Gamma$ satisfying that $\gamma A_{n,m}\gamma^{-1} \in \Gamma$ and that $\gamma(0) = z_i$ for the existence of f such that $f(\pi(0)) = \pi(z_i)$. For every z_i $(i \neq 0)$, we shall give γ, hence we see that $\pi(z_i)$ is the center of an extremal disk embedded in S.

Case 1. $z_1 = -\tanh(R/2)$, $\gamma(z) = (z + z_1)/(z_1 z + 1)$.

$$\begin{array}{lll}
\gamma A_{0,9} = A_{0,9}\gamma & \gamma A_{1,4} = A_{5,8}\gamma & \gamma A_{2,6} = A_{5,8}A_{7,3}\gamma \\
\gamma A_{3,7} = A_{0,9}A_{7,3}\gamma & \gamma A_{5,8} = A_{0,9}A_{8,5}\gamma & \gamma A_{-8,-5} = A_{-5,-8}A_{9,0}\gamma \\
\gamma A_{-7,-3} = A_{-3,-7}A_{9,0}\gamma & \gamma A_{-6,-2} = A_{-3,-7}A_{-8,-5}\gamma & \gamma A_{-4,-1} = A_{-8,-5}\gamma
\end{array}$$

Case 2. $z_1 = -\tanh(R/2)$, $\gamma(z) = (z + z_1)/(z_1 z + 1)$.

$$\begin{array}{lll}
\gamma A_{0,9} = A_{0,9}\gamma & \gamma A_{1,4} = A_{5,8}\gamma & \gamma A_{2,-3} = A_{6,-7}\gamma \\
\gamma A_{3,-2} = A_{7,-6}\gamma & \gamma A_{5,8} = A_{0,9}A_{8,5}\gamma & \gamma A_{6,-7} = A_{0,9}A_{2,-3}A_{9,0}\gamma \\
\gamma A_{7,-6} = A_{0,9}A_{3,-2}A_{9,0}\gamma & \gamma A_{-8,-5} = A_{-5,-8}A_{9,0}\gamma & \gamma A_{-4,-1} = A_{-8,-5}\gamma
\end{array}$$

Case 3. $z_1 = i\cosh R/(\tanh R\sqrt{4\cosh^2 R - 1})$, $\gamma(z) = (-z + z_1)/(z_1 z + 1)$.

$$\begin{array}{lll}
\gamma A_{0,3} = A_{6,9}\gamma & \gamma A_{1,7} = A_{7,1}\gamma & \gamma A_{2,8} = A_{8,2}\gamma \\
\gamma A_{4,-6} = A_{-6,4}\gamma & \gamma A_{5,-3} = A_{-3,5}\gamma & \gamma A_{6,9} = A_{0,3}\gamma \\
\gamma A_{-8,-4} = A_{-3,5}A_{3,0}\gamma & \gamma A_{-7,-2} = A_{-3,5}A_{4,-6}\gamma & \gamma A_{-5,-1} = A_{9,6}A_{4,-6}\gamma
\end{array}$$

Case 4. $z_1 = -\tanh(R/2)$, $\gamma(z) = (z + z_1)/(z_1 z + 1)$.

$$\begin{array}{lll}
\gamma A_{0,9} = A_{0,9}\gamma & \gamma A_{1,-5} = A_{4,-8}\gamma & \gamma A_{2,6} = A_{0,9}A_{6,2}\gamma \\
\gamma A_{3,-2} = A_{7,-6}\gamma & \gamma A_{4,-8} = A_{0,9}A_{8,-4}\gamma & \gamma A_{5,-1} = A_{8,-4}\gamma \\
\gamma A_{7,-6} = A_{4,-8}A_{2,6}A_{9,0}\gamma & \gamma A_{8,-4} = A_{4,-8}A_{9,0}\gamma & \gamma A_{-7,-3} = A_{7,-6}A_{-4,8}A_{9,0}\gamma
\end{array}$$

Case 5. $z_1 = 1/\sinh R$, $\gamma(z) = (-z + z_1)/(-z_1 z + 1)$.

$$\begin{array}{lll} \gamma A_{0,9} = A_{9,0}\gamma & \gamma A_{1,-7} = A_{-7,1}\gamma & \gamma A_{2,5} = A_{-5,-2}\gamma \\ \gamma A_{3,-4} = A_{-4,3}\gamma & \gamma A_{4,-3} = A_{-3,4}\gamma & \gamma A_{6,-8} = A_{9,0}A_{-1,7}\gamma \\ \gamma A_{7,-1} = A_{-1,7}\gamma & \gamma A_{8,-6} = A_{-7,1}A_{0,9}\gamma & \gamma A_{-5,-2} = A_{2,5}\gamma \end{array}$$

Case 6.1. $z_1 = re^{3i\beta}$, $\gamma(z) = (-z + z_1)/(-\bar{z_1} z + 1)$.

$$\begin{array}{lll} \gamma A_{0,5} = A_{5,0}\gamma & \gamma A_{1,8} = A_{8,1}\gamma & \gamma A_{2,-5} = A_{-5,2}\gamma \\ \gamma A_{3,-2} = A_{-2,3}\gamma & \gamma A_{4,9} = A_{8,1}A_{0,5}\gamma & \gamma A_{6,-7} = A_{-5,2}A_{3,-2}A_{0,5}\gamma \\ \gamma A_{7,-4} = A_{-5,2}A_{1,8}\gamma & \gamma A_{-8,-3} = A_{-5,2}A_{-3,-8}A_{2,-5}\gamma & \gamma A_{-6,-1} = A_{-2,3}A_{2,-5}\gamma \end{array}$$

Case 6.2. $z_4 = re^{7i\beta}$, $\gamma(z) = (-z + z_4)/(-\bar{z_4} z + 1)$.

$$\begin{array}{lll} \gamma A_{0,5} = A_{5,0}\gamma & \gamma A_{1,8} = A_{-2,3}A_{5,0}\gamma & \gamma A_{2,-5} = A_{-2,3}A_{7,-4}\gamma \\ \gamma A_{3,-2} = A_{9,4}\gamma & \gamma A_{4,9} = A_{-2,3}\gamma & \gamma A_{6,-7} = A_{-2,3}A_{2,-5}\gamma \\ \gamma A_{7,-4} = A_{9,4}A_{2,-5}\gamma & \gamma A_{-8,-3} = A_{9,4}A_{3,-2}\gamma & \gamma A_{-6,-1} = A_{9,4}A_{-4,7}A_{3,-2}\gamma \end{array}$$

Case 6.3. $z_9 = re^{35i\beta}$, $\gamma(z) = (-z + z_9)/(-\bar{z_9} z + 1)$.

$$\begin{array}{lll} \gamma A_{0,5} = A_{-6,-1}\gamma & \gamma A_{1,8} = A_{-6,-1}A_{-3,-8}\gamma & \gamma A_{2,-5} = A_{8,1}A_{-3,-8}\gamma \\ \gamma A_{3,-2} = A_{-2,3}\gamma & \gamma A_{4,9} = A_{-6,-1}A_{-3,-8}A_{-1,-6}\gamma & \gamma A_{6,-7} = A_{5,0}A_{-1,-6}\gamma \\ \gamma A_{7,-4} = A_{8,1}A_{-1,-6}\gamma & \gamma A_{-8,-3} = A_{8,1}A_{0,5}\gamma & \gamma A_{-6,-1} = A_{0,5}\gamma \end{array}$$

Case 7. $z_1 = 1/(\tanh R\,(1 + \tanh^2 R))$, $\gamma(z) = (-z + z_1)/(-z_1 z + 1)$.

$$\begin{array}{lll} \gamma A_{0,9} = A_{9,0}\gamma & \gamma A_{1,-6} = A_{-6,1}\gamma & \gamma A_{2,-4} = A_{-4,2}\gamma \\ \gamma A_{3,7} = A_{9,0}A_{-3,-7}\gamma & \gamma A_{4,-2} = A_{-2,4}\gamma & \gamma A_{5,-8} = A_{9,0}A_{-1,6}\gamma \\ \gamma A_{6,-1} = A_{-1,6}\gamma & \gamma A_{8,-5} = A_{-6,1}A_{0,9}\gamma & \gamma A_{-7,-3} = A_{7,3}A_{0,9}\gamma \end{array}$$

Finally we shall draw pictures of disks in Δ which are the inverse images of extremal disks by $\pi : \Delta \to S$ (Figure 5).

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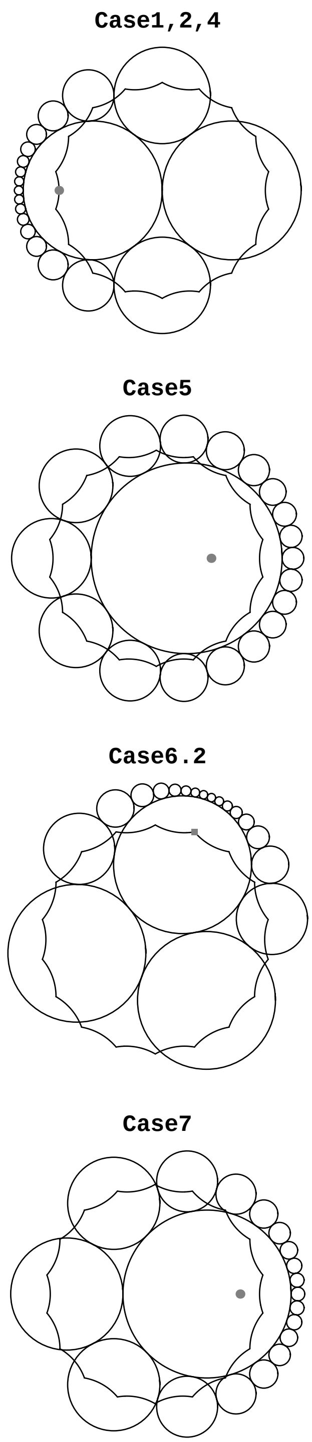

Figure 5: Inverse images of the extremal disks