ON COMPLETE SPACES AND REGULAR TOPOLOGICAL SPACES

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Abstract. This paper contains a generalization of fine uniform spaces and related properties. We consider some sequences which we call I-thick sequences in a regular topological space. We then obtain a generalization of the concept of Cauchy sequence in uniformisable spaces. The definitions and properties for complete regular topological space and uniformly continuous functions follow straighttforwards . We obtain a generalization of Baire's category theorem.

1 Introduction In section 2, we define the concepts of I-thick sequence, I-thick net and I-thick filter in a regular topological space that is, a separated topological space, in which each point has a basis of closed neighborhoods, in the sense of [SCHWA]. We conclude that, in each such space, we may consider one fixed index set I, for which there exists a surjection onto a basis of the neighborhoods of each point. This is the keystone for the definition of the I-thick sequences, as well for nets and filters, and I-complete spaces, and it makes possible the definition of $(I$-$J)$-uniformly continuous functions between two regular topological spaces.

In §3, we obtain that the concepts of Cauchy sequence, and Cauchy net and filter in a uniformisable space, are particular cases of our definitions. Hence each complete uniformisable space becomes an I-complete regular topological space in the sense that we have defined.

In §4, we obtain a generalization of Cantor's theorem to I-complete regular topological spaces that satisfy the first Axiom of countability. Next, we obtain that each regular I-complete topological space, which satisfies the first axiom of countability, is a Baire space, which is a generalization of Baire's category theorem. We obtain the unique extension of an $(I$-$J)$-uniformly continuous function, from a subset A of a regular topological space in a I-complete regular topological space, to the closure of A. We also obtain the I-completion of a regular topological space, which is the analogue for the metric spaces.

The concepts of I-thick sequence, net and filter, and the corresponding concepts of I-complete regular topological space, are not topological, since that they are not preserved under homeomorphisms. They are beyond the context of uniformisable topological spaces, since they are defined in regular topological spaces, which are not necessarily completely regular.

2 Equally distributed spaces, I-thick sequences and nets

Definition 1 *We say that the topological space* (X, T) *is* **equally distributed**, *if there exists a set I such that, for each* $a \in X$, *there exists a surjection from I onto a basis of neighborhoods* $\mathcal{B}_a$ *of the filter* $\mathcal{V}_a$ *of all neighborhoods of a, such that we may write* $\mathcal{B}_a = \{V_i(a) : i \in I\}$.

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Remark 1 *Every regular topological space is equally distributed. For if $a, b \in X$ with $a \neq b$, the mapping $\theta : \mathcal{V}_{a,b} \to \mathcal{U}_b$, where $\mathcal{V}_{a,b}$ is the class of the open sets V, such that $a \in V$, $b \notin V$ and, $\mathcal{U}_b$ is the class of the closed neighborhoods of b which do not contain a, defined through $\theta(V) = V^c$ clearly is an injection. Moreover, if W is a closed neighborhood of b that does not contain a, then W^c is in $\mathcal{V}_a$, since that there exists $U \in \mathcal{V}_a$ such that $U \subseteq W^c$, due of X being a regular space. Thus θ is a bijection. It follows that, since the class $\mathcal{V}_a$ of all open neighborhoods of a contains each such class $\mathcal{V}_{a,b}$ for each $b \in X$, there exists a surjection from $\mathcal{V}_a$ onto $\mathcal{U}_b$. Therefore we may take for I an index set with the cardinality of $\mathcal{V}_a$.*

Remark 2 *In what follows, we consider all topological spaces to be regular, which we state precisely sometimes, and we write $\mathcal{B}_p = \{(V_i(p)) : i \in I\}$ for each point p in the space.*

Definition 2 *We say that the sequence (x_n) in the topological space (X, T) is an I-thick sequence if it satisfies the condition*

$$(\text{th-}I) \qquad (\forall i \in I)(\exists a_i \in X)(\exists k)(\forall n)(n \geq k \Rightarrow x_n \in int(V_i(a_i))).$$

Here, as in the following, we consider I for the fixed index set, corresponding to the topological space, without mentioning it explicitly.

Clearly, (th-I) generalizes the concept of Cauchy sequence in a metric space: take $x_k = a$ in the definition and consider, for given k, that

$$n, m \geq k \Rightarrow x_n, x_m \in int(B(a, \frac{\varepsilon}{2})).$$

As for a topological vector space, clearly if V is a neighborhood of 0, then $x_n \in int(V) + x_k$ if and only if $x_n - x_k \in int(V)$. In what follows, $(A, \prec)$ is a directed system, whose elements are usually denoted by letters of the greek alphabet.

Definition 3 *(1) The topological space (X, T) is said to be I-sequentially complete, if each I-thick sequence in X has a limit in X.*

(2) If $(x_\alpha)_{\alpha \in A}$ is a net in the topological space (X, T), we say that it is an I-thick net if

$$(\text{thn-}I) \qquad (\forall i \in I)(\exists a_i \in X)(\exists \beta \in A)(\forall \alpha \in A)(\alpha \succ \beta \Rightarrow x_\alpha \in int(V_i(a_i))).$$

(3) We say that X is an I-complete topological space, if each I-thick net in X has a limit in X.

(4) Finally, let us call a filter $\{F_\alpha : \alpha \in A\}$ of subsets of X an I-thick filter whenever it satisfies the condition

$$(\text{thf-}I) \qquad (\forall i \in I)(\exists a_i \in X)(\exists \alpha \in A)(F_\alpha \subseteq int(V_i(a_i))).$$

Then clearly X is complete if and only if each I-thick filter of subsets of X is convergent.

Remark 3 *Sometimes we write complete instead of I-complete whenever it is clear what index set shall be used.*

3 Regular topological spaces, separated uniformi-sable spaces and I-thick sequences Recall that, if X is a set and $W, V \subseteq X \times X$, then $W^{-1} = \{(y, x) : (x, y) \in W\}$ is the inverse of W, and the composition of V and W is the relation

$$V \circ W = \{(x, z) : (\exists y \in X)((x, y) \in W \wedge (y, z) \in V)\}.$$

We also write $\Delta = \{(x, x) : x \in X\}$ for the diagonal of X.

Definition 4 *A filter M over $X \times X$ is said to define a* uniform structure *in X if it satisfies the conditions:*

(U1) $W \supseteq \Delta$ for each $W \in M$;

(U2) $W \in M \Rightarrow W^{-1} \in M$;

(U3) for each $W \in M$, there exists $V \in M$ with $V \circ V \subseteq W$.

The family of all subsets G of X such that, for each $x \in G$, there exists $W \in M$ with

$$\{y : (x, y) \in W\} \subseteq G$$

is a topology on X, which we call the uniform topology; the topological spaces that can be obtained this way, are called the uniformisable spaces.

According to [SCHWA], a topological space is separated and uniformisable iff it is completely regular, which is a condition that implies regularity.

If X is a separated uniformisable space, with its topology defined by means of the uniform structure M, and $a \in X$, we may consider the sets $V(N, a) = \{y \in X : (a, y) \in N\}$, where $N \in M$; these sets form the neighborhoods of a. If (u_n) is a Cauchy sequence in X, we have:

$$(\forall N \in M)(\exists k)(\forall n, m \geq k)((u_n, u_m) \in N)$$

Hence, for given $N \in M$, we have $(u_k, u_n) \in N$ and $(u_k, u_m) \in N$, so that u_n, $u_m \in V(N, a)$ with $a = u_k$, provided that n, $m \geq k$; this means that (u_n) is an I-thick sequence.

Conversely, if (x_n) is an I-thick sequence in X, and we take $L \in M$ such that $L \circ L \subseteq N$, then

$$(\forall N \in M)(\exists a \in X)(\exists k)(\forall m, n \geq k)(x_k, x_m \in V(L, a) \wedge x_k, x_n \in V(L, a)).$$

Therefore, if $n, m \geq k$, and since (x_n, x_k), $(x_k, x_m) \in L$, we see that $(x_n, x_m) \in N$, and therefore that (x_n) is a Cauchy sequence in the uniform space X.

Since the above clearly holds with nets, in place of sequences, and since the concept of convergence is the same, we see that a separated uniformisable space is sequentially complete (resp. complete) if and only if it is a sequentially I-complete (resp. I-complete) topological space, in the sense of definitions 4 (1) and (2).

4 On completeness and uniform (I-J)-continuity for regular topological spaces

Definition 5 *If $f : X \to Y$ is a function between two (regular) topological spaces, I and J are index sets for X and Y respectively, we say that f is $(I$-$J)$-uniformly continuous if it satisfies the condition*

$$\text{(u)} \qquad (\forall j \in J)(\forall a \in X)(\exists i \in I)(f(int(U_i(a))) \subseteq int(V_j(f(a))).$$

Sometimes we write uniformly continuous instead of $(I$-$J)$-uniformly continuous whenever it is clear what index sets I and J shall be used.

Clearly, each $(I$-$J)$-uniformly continuous function from X to Y is continuous (start with $(\forall a \in X)$ in (u)).

If X is a uniformisable space, with its topology defined by means of the fine uniform structure M_X, and if Y is a uniformisable space, relatively to the uniform structure M_Y, then a function $f : X \to Y$ is uniformly continuous in the sense of uniform structures (that is, for each $N \in M_Y$, there exists $M \in M_X$ satisfying $f \times f(M) \subseteq N$) if and only if it is continuous for the uniform topologies. Hence if f is $(I$-$J)$-uniformly continuous in the sense of def. 5, then for every $N \in M_Y$ there exists $M \in M_X$ such that for all $y \in V(M, x)$

we have $f(y) \in V(N, f(x))$, and it follows that f is uniformly continuous between the uniformisable spaces, due of

$$y \in V(M, x) \Leftrightarrow x, y \in M, \quad f(y) \in V(N, f(x)) \Leftrightarrow (f(x), f(y)) \in N.$$

For the converse, it suffices to consider index sets containing M_X, M_Y.

According to def. 3, if (x_α) is an I-thick net indexed by the directed set A in the topological space X, we have:

$$(\forall i \in I)(\exists a_i \in X)(\exists \alpha \in A)(\forall \beta \in A)(\beta \succ \alpha \Rightarrow x_\beta \in int(V_i(a_i))).$$

It follows that, if $f : X \to Y$ is $(I\text{-}J)$-uniformly continuous, then the net $(f(x_\alpha))$ is a J-thick net in Y. In fact, for given j in the index set J for Y, if we consider $i \in I$ as in (u), def. 5, then for each $V_j(f(a)) \in \mathcal{V}_{f(a)}$, it holds that each $V_i(a) \in \mathcal{V}_a$ sastisfies $f(int(V_i(a))) \subseteq int(V_j(f(a)))$; therefore $f(x_\beta) \in int(V_j(f(a)))$ if $\beta \succ \alpha$. Here as in the following, we write $\mathcal{V}_p$ for the class of all neighborhoods of p.

Conversely:

Claim 1 *If $f : X \to Y$ maps I-thick nets in X to J-thick nets in Y, then f is $(I\text{-}J)$-uniformly continuous.*

For suppose that f is not $(I\text{-}J)$-uniformly continuous. Then there exists $j \in J$, where J is as above, such that for some $a \in X$, we have: for each $i \in I$ as in def. 5, there exists some $V_j(f(a)$ such that $f(int(V_i(a)))$ is not a subset of $int(V_j(f(a)))$. Considering the partial ordering $\succ$ in I defined by $j \succ i$ iff $V_j(a) \subseteq V_i(a)$, we obtain a directed set, in which we may take a maximal chain; it follows that a net (a_k) chosen so that $f(a_k) \notin int(V_j(f(a)))$ and $a_k \in int(V_k(a)\ )$ will be an I-thick net in X, but the image net $(f(a_k))$ is not a J-thick net in Y.

It holds that if a net in the topological space X is an I-thick net, and it has a convergent partial net, with limit x, then clearly $x = a$ in def. 3 (2), and the net converges to x. Consequently, since any convergent net is an I- thick net, we have: If $f : X \to Y$ is continuous, where X, Y are topological spaces, with X compact, then f is $(I\text{-}J)$-uniformly continuous in the sense of def. 5. Clearly a continuous function between two regular spaces, maps convergent nets to convergent nets.

We now look for the analogue of Cantor's theorem in complete metric spaces. Consider a topological space X, with an index set I for the neighborhoods of its points, and let us convey that for $B \subseteq X$ and $\text{diam}(B) =$ diameter of B:

$\text{diam}(B) \leqq i_0$ if there exists some $a \in X$ such that $B \subseteq V_{i_0}(a)$;

$\text{diam}(B) = 0$ if $\text{diam}(B) \leqq i$ for each $i \in I$, that is,

$$(\forall i \in I)(\exists a_i \in X)(B \subseteq V_i(a));$$

for a given family $\{A_n : n \in \mathbb{N}\}$ of subsets of X, $\text{diam}(A_n) \to 0$ if the condition

$$(\forall i \in X)(\exists a_i \in X)(\exists n(i) \in \mathbb{N})(n \geq n(i) \Rightarrow A_n \subseteq (V_i(a_i))$$

holds.

Proposition 1 *Let $\{A_n : n \in \mathbb{N}\}$ be a family of nonempty closed sets in a complete topological space X, such that there exists a denumerable set of neighborhoods of each point, the A_n satisfying $A_{n+1} \subseteq A_n$ for each n, and $diam(A_n) \to 0$. Then $\bigcap\{A_n : n \in \mathbb{N}\}$ consists of a single point.*

Proof. Since each point has a denumerable family of neighborhoods, we may take N for a subset of the index set I, in such a way that for each $a \in X$, we have $V_{n+1}(a) \subseteq V_n(a)$ with $\{V_n(a) : n \in \mathbb{N}\}$ a basis of open neighborhoods of a.

If we consider $x_n \in A_n \subseteq V_n(a_n)$ where

$$(\forall m)(m \geq n \Rightarrow A_m \subseteq V_n(a_n))$$

(which is possible, according to the hypothesis), the sequence (x_n) is an I-thick sequence. Hence there exists $x = \lim x_n$ in X, and since the sets A_n are closed, we have $V \cap A_n \neq \emptyset$ for each n, whenever $V \in \mathcal{V}_x$, that is, $x \in \bigcap\{A_n : n \in \mathbb{N}\}$. If y is a point in this intersection, then with (x_n) as above, we may consider the I-thick sequence $(x_1, y, x_2, y, ...)$ which must converge to $x = y$, since the space is separated, and the theorem is proved. ∎

Lemma 1 *An I-complete topological space X is a Baire space if and only if every denumerable intersection of dense open sets is dense.*

Proof. Let I be the index set for X, and assume that X is a Baire space. Let $\{A_n : n \in \mathbb{N}\}$ be a denumerable family of dense open sets. We prove that, for W a nonempty open set, then $A \cap W \neq \emptyset$, where $A = \bigcap\{A_n : n \in \mathbb{N}\}$. If $A \cap W_0 = \emptyset$ for some open set $W_0 \neq \emptyset$ then $X = (A \cap W_0)^c = A^c \cup W_0^c$, so that

$$\begin{aligned} W_0 &= X \cap W_0 = A^c \cap W_0 = (\bigcap\{A_n : n \in \mathbb{N}\})^c \cap W_0 \\ &= \bigcup\{A_n^c : n \in \mathbb{N}\} \cap W_0 = \bigcup(A_n^c \cap W_0). \end{aligned}$$

This shows that W_0 is a set of the first category, since $\text{int}(\overline{B \cap W_0}) \subseteq \text{int}(B) = \emptyset$ for each $B = A_n^c$; (notice that $p \in V \subseteq A_n^c$ and V open implies V open, $V \neq \emptyset$, $V \cap A_n = \emptyset$, contradicting that A_n is dense). It follows that W_0 cannot be open, and we conclude by absurd that A is dense. Conversely, assume that every denumerable intersection of dense open sets is dense in X. If U is an open set of the first category in X, we prove that $U = \emptyset$, concluding the proof.

Let $\{A_n : n \in \mathbb{N}\}$ be a denumerable family of nowhere dense sets, such that $U = \bigcup\{A_n : n \in \mathbb{N}\}$. Then $\{(\overline{A_n})^c : n \in \mathbb{N}\}$ is a family of open sets, each of which is dense, and since $\text{int}(\overline{A_n}) = \emptyset$ implies that for each open set V, it follows that $V \cap (\overline{A_n})^c \neq \emptyset$. By hypothesis, $A = \bigcap\{(\overline{A_n})^c : n \in \mathbb{N}\}$ is dense in X. Therefore

$$U \subseteq \bigcup\{\overline{A_n} : n \in \mathbb{N}\} \Rightarrow \bigcap\{(\overline{A_n})^c : n \in \mathbb{N}\} = A \subseteq U^c,$$

which implies that $X \subseteq \overline{U^c} = U^c$, thence $U = \emptyset$, as wished. ∎

Theorem 1 *Let (X, T) be a regular, I-complete topological space, which satisfies the first axiom of countability. Then X is a Baire space.*

Proof. We may consider the indices for the neighborhoods, in such a way that $I \supseteq \mathbb{N}$, $V_{n+1}(a) \subseteq V_n(a)$ for each n and each $a \in X$. We have to prove that

$$A = \bigcap\{A_n : n \in \mathbb{N}\}$$

is dense in X, for each denumerable class of dense open sets A_n, according to the lemma. We prove that for $x \in X$, $n \in \mathbb{N}$ we have $V_n(x) \cap A \neq \emptyset$. Since A_1 is open and $\overline{A_1} = X$, and since we may suppose the neighborhoods $V_n(a)$ to be open sets, it follows that $x \in V_n(x) \cap A_1 \neq \emptyset$; since X is regular, there exists $n_1 \in \mathbb{N}$, with $x_1 \in \overline{V}_{n_1}(x_1) \subseteq V_n(x) \cap A_1$, for some $V_{n_1}(x_1) \in$

$V(x_1)$. It follows analogously, that there exists $x_2 \in X$ with $x_2 \in \overline{V}_{n_2}(x_2) \subseteq V_{n_1}(x_1) \cap A_2$, for some $n_2 > n_1$ since A_2 is dense and, $V_{n_1}(x_1) \cap A_2$ is open and nonempty.

Inductively, if x_1, ..., x_m, n_1, ..., n_m have been selected, we may choose x_{n+1}, n_{m+1} with

$$\overline{V}_{n_{m+1}}(x_{n+1}) \subseteq V_{n_m}(x_m) \cap A_{m+1},$$

(n_m) a strictly increasing sequence.

So, there exists a sequence (x_m) in X , and a strictly increasing sequence of natural numbers (n_m), with $\overline{V}_{n_{m+1}}(x_{m+1}) \subseteq V_{n_m}(x_m) \cap A_{m+1}$. We have: for each n_k given, there exists $a = x_k$ such that

$$(\forall m, n \geq k)(x_m, x_n \in V_{n_k}(x_k))$$

which shows that (x_m) is an I-thick sequence. Hence $x_m \to u$, for some $u \in X$. Therefore, for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $m \geq k$ implies $x_m \in V_n(u)$. For each m, it holds that $u \in \overline{V}_{n_m}(x_m) \subseteq A_m$, thence $u \in \bigcap\{A_m : m \in \mathbb{N}\} = A$.

Moreover, we have $u \in \overline{V}_{n_1}(x_1) \subseteq V_n(x)$, so that $u \in V_n(x) \cap A$, which proves the theorem.Therefore, for a suitable indexation by means of some index set I, if (X, T) is a regular I-complete topological space, satisfying the first axiom of countability, and if $X = \bigcup\{A_n : n \in \mathbb{N}\}$, then the interior of some $\overline{A_n}$ is nonempty. ■

If I is an index set for X, $\phi \neq A \subseteq X$ then $a \in \overline{A}$ if and only if there is a net in A indexed through the directed set $(I, \succ)$ where, $i \succ j$ if and only if $V_i(a) \subseteq V_j(a)$ such that, the net converges to a. In fact, suppose that $(\Omega, \geq)$ is a directed set and, $(x_\alpha) \in X^\Omega$ is a net such that $x_\alpha \to a$. It follows that the set $\Omega_i = \{\alpha(i) \in \Omega : \alpha \geq \alpha(i) \Rightarrow x_\alpha \in int(V_i(a))\}$ is nonempty for each $i \in I$. If we define $i \succ j$ if and only if $V_i(a) \subseteq V_j(a)$ clearly $(I, \succ)$ is a directed set. Denoting $\varphi : \{\Omega_i : i \in I\} \to \Omega$ the choice function, according to the Zermelo axiom of Choice and, $\varphi(\Omega_i) = \overline{\alpha}(i)$ for each i we have: for each given $k \in I$, it holds that $i \succ k$ implies $x_{\overline{\alpha}(i)} \in int(V_i(a)) \subseteq int(V_k(a))$. This means that the net $x_{\overline{\alpha}(i)} \to a$ and this net is idexed in $(I, \succ)$, if we let $\overline{\alpha}(i)$ constant on Ω_i.

Theorem 2 *Let X, Y be topological spaces, Y regular and complete, and consider $f : A \to Y$ with $A \subseteq X$ an $(I$-$J)$-uniformly continuous function. Then there exists a unique $(I$-$J)$-uniformly continuous extension of f to $\overline{A}$.*

Proof. We may consider J such that each $V_j(y)$ is closed, and such that $\{V_j(y) : j \in J\}$ constitutes a basis of neighborhoods of y, if $y \in Y$. Given $x \in \overline{A}$, we consider a net (x_i) in A indexed as above with $x_i \to x$. Since f is $(I$-$J)$-uniformly continuous, $(f(x_i))$ is a J-thick sequence in Y (see claim 1 above); therefore it converges in Y, to a point which we denote by $g(x)$. This definition makes sense, for if (u_k) is another net in A with limit x, indexed through the same directed set, then $(x_i, u_k) \to x$, where we consider $\overline{\beta}(k) = \varphi(\Omega'_k \backslash \{\overline{\alpha}(k)\})$, the Ω'_k standing for u_k analogously to the x_i above and, the same way for the $\overline{\beta}_k$. Therefore the net of the images is a J-thick net in Y, so converging to the unique limit $g(x)$.

We next prove that g is $(I$-$J)$-uniformly continuous. Let x, $z \in \overline{A}$. If (x_j), (z_l) are nets in A with $x = \lim x_j$, $z = \lim z_l$, k, $l \in J$, $a \in A$, there exists some $i \in I$ such that $f(V_i(a)) \subseteq V_j(f(a)) \cap V_l(f(a)$, for each a. Also there exist some $\overline{j}, \overline{l} \in I$ such that

$$(\forall j)(\forall l)(j \succ \overline{j} \wedge l \succ \overline{l} \Rightarrow (x_j, z_l) \in V_i(x) \times V_i(z)).$$

We then have

$$(f(x_j), f(z_l)) \in (V_j(f(x_j)) \cap V_l(f(x_j))) \times (V_j(f(z_l)) \cap V_l(f(z_l))).$$

It follows that

$$(g(x), g(z)) \in \overline{V(g(x), g(z))} = V_j(g(x)) \times V_l(g(z)).$$

We conclude that, for given $V_i(x)$ in $\overline{A}$, and for each $z \in V_i(x)$, then $g(z) \in V_j(g(x))$ (choose $z_l \to z$ above). This means precisely that $g(V_i(x)) \subseteq V_j(g(x))$ and the proof is complete. ■

Definition 6 *In what follows, if (X, T_X) is a topological space, with index set I, we say that the complete topological space (E, T) is an I-completion of X, if there exists an injective mapping $\theta : X \to E$, such that θ is $(I\text{-}J)$-uniformly continuous, θ^{-1} is $(J\text{-}I)$-uniformly continuous,and $\overline{\theta(X)} = E$. It follows from theorem 2 that if E, F are two I-completions of X, then there exists a bijection $v : E \to F$ such that both v and v^{-1} are uniformly continuous.*

Theorem 3 (Existence of the I-completion) *If E is a regular topological space, then there exists a regular topological space $\widehat{E}$ which is an I-completion of E.*

Proof. Recall that the filter $\mathcal{F}$ of subsets of E is I-thick if it satisfies the condition that there exists a set $\{a_i : i \in I\} \subseteq E$, where I is the index set for the neighborhoods of the points, such that for each $i \in I$ there exists $F \in \mathcal{F}$ such that $F \subseteq int(V_i(a_i))$. We put $(a_i) \in I(\mathcal{F})$ for such a set $\{a_i : i \in I\}$.

Consider the set C of all I-thick filters $\mathcal{F}$ on E, and the binary relation ρ defined by $\mathcal{F}\rho\mathcal{G}$ iff there exists $(a_i) \in I(\mathcal{F}) \cap I(\mathcal{G})$ on C that is, if we may consider

$$F \subseteq int(V_i(a_i)),\ G \subseteq int(V_i(a_i)),$$

with $F \in \mathcal{F}$ and $G \in \mathcal{G}$ in the above condition. Clearly ρ is an equivalence relation on C.

We denote by $\widetilde{\mathcal{F}}$ the equivalence class of $\mathcal{F}$, and $\widehat{E}$ will stand for the set of the $\widetilde{\mathcal{F}}$. In order to define a topology on $\widehat{E}$, we put for each index $k \in I$,

$$\widehat{U}_{\widetilde{\mathcal{F}},k} = \{\widetilde{\mathcal{G}} \in \widehat{E} : (\exists (a_i) \in I(\widetilde{\mathcal{F}}))(\exists (g_i) \in I(\widetilde{\mathcal{G}}))(V_k(a_k) = V_k(g_k) \wedge a_k \ = g_k)\}$$

where the notation $I(\widetilde{\mathcal{F}})$ instead of $I(\mathcal{F})$ is understood from above. Next, let

$$\overline{U}_{\widetilde{\mathcal{F}},k} = \{\widetilde{\mathcal{G}} \in \widehat{E} : (\forall j \in I)(\ \widehat{U}_{\widetilde{\mathcal{G}},j} \cap \widehat{U}_{\widetilde{\mathcal{F}},k} \neq \phi\}$$

for each $k \in I$.The class of sets $\{\bigcap\{\overline{U}_{\widetilde{\mathcal{F}},i} : i \in L\} : L \subseteq I \wedge L \text{ finite}\}$ is the basis of a filter $\widehat{U}_{\widetilde{\mathcal{F}}}$ on $\widehat{E}$ such that $\widetilde{\mathcal{F}}$ belongs to each of the sets. Also the intersection of each class reduces to $\{\widetilde{\mathcal{F}}\}$.

Considering $\widetilde{\mathcal{G}} \in \widehat{U}_{\widetilde{\mathcal{F}},k}$, and if $\widetilde{\mathcal{H}} \in \widehat{U}_{\widetilde{\mathcal{G}},k}$, then there exist $(g_i) \in I(\widetilde{\mathcal{G}})$, $(h_i) \in I(\widetilde{\mathcal{H}})$ such that $g_k = h_k$, $V_k(g_k) = V_k(h_k)$. Since we also may consider $(a_i) \in I(\widetilde{\mathcal{F}})$ with $a_k = h_k$, $V_k(a_k) = V_k(g_k)$, it follows that $a_k = h_k$, thence $\widetilde{\mathcal{H}} \in \widehat{U}_{\widetilde{\mathcal{F}},k}$. Thus it holds that, for each $\widetilde{\mathcal{G}} \in \widehat{U}_{\widetilde{\mathcal{F}},k}$ it follows that $\widehat{U}_{\widetilde{\mathcal{G}},k} \subseteq \widehat{U}_{\widetilde{\mathcal{F}},k}$. Consequently, we find that for each $\widetilde{\mathcal{G}} \in \overline{U}_{\widetilde{\mathcal{F}},k}$, also $\overline{U}_{\widetilde{\mathcal{G}},k} \subseteq \overline{U}_{\widetilde{\mathcal{F}},k}$, so that the classes $\{\overline{U}_{\widetilde{\mathcal{F}},k} : k \in I\}$ are basis of filters of neighborhoods of each $\widetilde{\mathcal{F}} \in \widehat{E}$, for a topology $\widehat{T}(\widehat{E})$ on $\widehat{E}$, for which $\widehat{E}$ is a (T_1) space. $(\widehat{E}, T(\widehat{E}))$ is a regular space, since each $\widehat{U}_{\widetilde{\mathcal{F}},i}$ is closed, as we prove in the following. It holds that $\overline{U}_{\widetilde{\mathcal{F}},k} = \widehat{U}_{\widetilde{\mathcal{F}},k}$. In fact, if $\widetilde{\mathcal{G}} \in \overline{U}_{\widetilde{\mathcal{F}},k} \backslash \widehat{U}_{\widetilde{\mathcal{F}},k}$, it follows that for each $j \in I$, $\widehat{U}_{\widetilde{\mathcal{G}},j} \cap \widehat{U}_{\widetilde{\mathcal{F}},k} \neq \phi$ and, for each

$(a_i) \in I(\widetilde{\mathcal{F}})$ and each $(g_i) \in I(\widetilde{\mathcal{G}})$ we have $V_k(a_k) \neq V_k(g_k)$ or $a_k \neq g_k$. In particular with $j = k$, it would hold that there exists some $\widetilde{\mathcal{H}} \in \widehat{U}_{\widetilde{\mathcal{G}},k} \cap \widehat{U}_{\widetilde{\mathcal{F}},k}$ that is: there exist $(h_i) \in I(\widetilde{\mathcal{H}})$, $(g_i) \in I(\widetilde{\mathcal{G}})$, $(a_i) \in I(\widetilde{\mathcal{F}})$ such that $V_k(h_k) = V_k(a_k)$ and $h_k = a_k$, $V_k\ (g_k) = V_k(a_k)$, whence $V_k(h_k) = V_k(g_k)$ and $h_k = g_k$, contradicting the above. Thus $\overline{U}_{\widetilde{\mathcal{F}},k} \subseteq \widehat{U}_{\widetilde{\mathcal{F}},k}$ and it follows that $\overline{U}_{\widetilde{\mathcal{F}},k} = \widehat{U}_{\widetilde{\mathcal{F}},k}$ for each k. We conclude next that each $\overline{U}_{\widetilde{\mathcal{F}},k}$ is closed in $(\widehat{E}, \widehat{T}(\widehat{E}\ ))$, so that this is a regular topological apace. For if $\widetilde{\mathcal{G}} \notin \overline{U}_{\widetilde{\mathcal{F}},k}$, there exists $j \in I$ such that $\overline{U}_{\widetilde{\mathcal{G}},j} = \widehat{U}_{\widetilde{\mathcal{G}},j} \subseteq \widehat{U}^c_{\widetilde{\mathcal{F}},k} = \overline{U}^c_{\widetilde{\mathcal{F}},k}$. Since that the cardinality of $\widehat{U}_{\widetilde{\mathcal{F}}} = \{\widehat{U}_{\widetilde{\mathcal{F}},k} : k \in I\}$, it follows that we may consider the same index set I, for the indexation of the neighbrhoods of each $\widetilde{\mathcal{F}} \in \widehat{E}$.

We now prove that the map $Q : a \mapsto \widetilde{\mathcal{V}_a}$, where $\mathcal{V}_a$ is the filter of neighborhoods of a, from E to $\widehat{E}$ (clearly $\mathcal{V}_a$ is an I-thick filter: take $a_i = a$ for $(a_i) \in I(\widetilde{\mathcal{V}_a})$), is $(I\text{-}I)$-uniformly continuous. Given $\mathcal{V}_a$, $\mathcal{F} \in \widehat{U}_{\widetilde{\mathcal{V}_a},k}$ iff there exists $(a_i) \in I(\widetilde{\mathcal{F}})$ with $V_k(a_k) = V_k(a)$, $a_k = a$ for all k in a cofinal subset of I, provided we consider the partial ordering in I defined through

$i \succeq j$ iff $V_i(a) \subseteq V_j(a)$ iff $V_i(a_k) \subseteq V_j(a_k)$ (but possibly for one of the a_k, see the remark following def. 1); for $b \in int(V_k(a))$, there exists $j \in I$ such that $V_j(b) \subseteq V_k(a)$. It follows that considering $\mathcal{V}_b$, $b \in int(V_k(a))$,we may take $b = b_k = a$, $V_k(b)$ for $V_k(a)$ in what concerns the index k, since $V_k(a) \in \mathcal{V}_b$; we conclude that $\widetilde{\mathcal{V}_b} \in \widehat{U}_{\widetilde{\mathcal{V}_a},k}$, therefore $Q(V_k(a)) \subseteq \widehat{U}_{\widetilde{\mathcal{V}_a},k}$. This shows that Q is $(I\text{-}I)$-uniformly continuous. Also Q clearly is an injection, and the inverse map from $Q(E)$ to E is $(I\text{-}I)$-uniformly continuous. In fact, given $\mathcal{V}_a$ with $a \in E$, consider $V_j(a)$. If $\widetilde{\mathcal{F}} \in \widehat{U}_{\widetilde{\mathcal{V}_a},k}$ it follows that, there is $(a_i) \in I(\widetilde{\mathcal{F}})$ such that $V_k(a_k) = V_k(a)$, $a_k = a$ for some $k \succeq j$, as we saw. This shows that if $\widetilde{\mathcal{V}_b} \in \widehat{U}_{\widetilde{\mathcal{V}_a},k}$ for such a k, then $b \in V_k(b_k) = V_k(a)$ so that $b \in V_k(a)$. Therefore the inverse of Q is $(I\text{-}I)$-uniformly continuous.

It holds that $Q(E)$ is dense in $\widehat{E}$, since for each $\widetilde{\mathcal{F}} \in \widehat{E}$ and each $i \in I$ we have with $(a_i) \in I(\widetilde{\mathcal{F}})$ that $\widetilde{\mathcal{V}}_{a_i} \in \widehat{U}_{\widetilde{\mathcal{F}},i}$, for $V_i(a_i)$ belongs to both filters $\mathcal{V}_{a_i}$, $\mathcal{F}$. Finally, we show that $\widehat{E}$ is I-complete, by proving that each I-thick filter of subsets of $\widehat{E}$ is convergent. Let $\widehat{\mathcal{M}}$ be a thick filter of subsets of $\widehat{E}$ constituted by sets $\widehat{M}$. This means that there exists a set $\{\widetilde{\mathcal{A}}_i : i \in I\} \subseteq \widehat{E}$ such that for each i, there exists some $\widehat{M}_i \in \widehat{\mathcal{M}}$, $\widehat{M}_i \subseteq \widehat{U}_{\widetilde{\mathcal{A}}_i,i}$. We have: the class

$$\{V_i(\widehat{M}) : \widehat{M} \in \widehat{\mathcal{M}}, i \in I\}$$

where

$$V_i(\widehat{M}) = \bigcup\{\widehat{U}_{\widetilde{\mathcal{F}},i} : \widetilde{\mathcal{F}} \in \widehat{M}\}$$

is the basis of a filter of subsets of $\widehat{E}$ and, the filter generated by this basis is an I- thick filter on $\widehat{E}$. In fact, $V_i(\widehat{M}) \subseteq V_j(\widehat{M})$ provided that we suppose $i \succeq j$ iff $\widehat{U}_{\widetilde{\mathcal{F}},i} \subseteq \widehat{U}_{\widetilde{\mathcal{F}},j}$ for each $\widetilde{\mathcal{F}} \in \widehat{M}$; also

$$V_i(\widehat{M}) \cap V_i(\widehat{N}) \supseteq V_i(\widehat{M} \cap \widehat{N}) \neq \phi$$

for each $\widehat{M}, \widehat{N} \in \widehat{\mathcal{M}}$. The filter generated by the class of the $V_i(\widehat{M})$ is an I-thick filter, due of $\widehat{M}_i \subseteq \widehat{U}_{\widetilde{\mathcal{A}}_i,i}$ implying $V_i(\widehat{M}_i) \subseteq \widehat{U}_{\widetilde{\mathcal{A}}_i,i}$. In fact, it holds that, under the hypothesis $\widehat{M}_i \subseteq \widehat{U}_{\widetilde{\mathcal{A}}_i,i}$,

$$\widetilde{\mathcal{W}} \in V_i(\widehat{M}_i) \Rightarrow \widetilde{\mathcal{W}} \in \widehat{U}_{\widetilde{\mathcal{F}},i}, \widetilde{\mathcal{F}} \in \widehat{M}_i$$

which implies that, in turn

$$\widetilde{W} \in \widehat{U}_{\widetilde{\mathcal{W}},i} \subseteq \widehat{U}_{\widetilde{\mathcal{F}},i} \subseteq \widehat{U}_{\widetilde{\mathcal{A}}_i,i}$$

Moreover, each of the sets $V_i(\widehat{M}) \cap Q(E)$ $(\widehat{M} \in \widehat{\mathcal{M}},\ i \in I)$ is non empty, due of $Q(E)$ being dense in $\widehat{E}$, since that the $V_i(\widehat{M})$ are open sets. Also each intersection $V_i(\widehat{M}) \cap V_j(\widehat{N}) \cap Q(E)$ is non empty, and the class

$$\bigcap \{V_i(\widehat{M}_j) : i \in L,\ j \in L', \widehat{M}_j \in \widehat{\mathcal{M}} \wedge L, L'\ finite\}$$

is the basis of an I- filter $\widehat{\mathcal{F}}$ on $Q(E)$ equipped with the induced topology, as follows from above. Let then $\mathcal{G} = Q^{-1}(\widehat{\mathcal{F}})$. Q^{-1} being $(I\text{-}I)$-uniformly continuous, it follows that $\mathcal{G}$ is an I-thick filter of subsets of E. We prove that $\widehat{\mathcal{M}} \to \widetilde{\mathcal{G}}$. Since $\mathcal{G}$ is an I-thick filter on E, there exists a set $\{a_i : i \in I\} \subseteq E$ such that for each i, there is some $G \in \mathcal{G}$ with $G \subseteq V_i(a_i)$. This implies that for each $i \in I$, there exists some $j(i) \in I$ such that,

$$Q^{-1}(V_{j(i)}(\widehat{M}_{j(i)}) \cap Q(E)) \subseteq V_i(a_i)$$

Therefore, also for each $i \in I$, there exists $k(i) \in I$ such that

$$\widehat{M}_{k(i)} \cap Q(E) \subseteq V_{k(i)}(\widehat{M}_{k(i)}) \cap Q(E) \subseteq \widehat{U}_{\widetilde{\mathcal{V}}_c,i}$$

where we write $c = a_i$, so that

$$\widehat{M}_{k(i)} \subseteq \widehat{U}_{\widetilde{\mathcal{V}}_c,i}$$

due of the $\widehat{U}_{\widetilde{\mathcal{F}},i}$ being clopen. Now $V_i(a_i) \in \mathcal{G}$, $(a_i) \in I(\mathcal{G})$ and $V_i(a_i) \subseteq V_i(a_i)$, $V_i(a_i) \in \mathcal{V}_{a_i}$, imply that the filter $\mathcal{V}_{a_i}$ which is generated by the basis $\{V_k(a_i) : k \in I\}$ is such that $\widetilde{\mathcal{V}}_{a_i} \in \widehat{U}_{\widetilde{\mathcal{G}},i}$ thence $\widehat{U}_{\widetilde{\mathcal{V}}c,i} \subseteq \widehat{U}_{\widetilde{\mathcal{G}},i}$. This implies that $\widehat{M} = \widehat{M}_{k(i)}$ above satisfies that $\widehat{M} \subseteq \widehat{U}_{\widetilde{\mathcal{G}},i}$ and the proof is
complete. ■

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