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NOTE ON THE NUMBER OF SEMISTAR-OPERATIONS, IV

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Abstract. We study a certain kind of integral domains D with dimension three, and construct star-operations on D of new type. Furthermore, we prove a proposition in [MS] whose proof was wrong.

This note is a continuation of our [M1], [M2] and [M3] on the number of semistar-operations on a domain. Let D be a three-dimensional Prüfer domain with exactly two maximal ideals M and N, and assume that there exist prime ideals P_1 and P_2 of D such that $M \cap N \supsetneqq P_2 \supsetneqq P_1 \supsetneqq (0)$, and assume that there exist elements π_1, π_2, p and q of D such that $P_1 D_{P_1} = \pi_1 D_{P_1}, P_2 D_{P_2} = \pi_2 D_{P_2}, M = (p)$ and $N = (q)$. We will study such kind of the domains D, and will construct star-operations on D of new type. On the other hand, [MS] showed the following two facts:

1. Let D be an integrally closed quasi-local domain with dimension n. Then D is a valuation domain if and only if $n+1 \leq |\Sigma'(D)| \leq 2n+1$, where $\Sigma'(D)$ denotes the set of semistar-operations on D.

2. Let D be an integrally closed domain with dimension $n \leq 3$. If $n+1 \leq |\Sigma'(D)| \leq 2n+1$, then D is a valuation domain.

We show that the proof in [MS] of the above fact 2 is wrong, and we give its correct proof. The above fact 1 and its proof in [MS] are right.

Let D be an integral domain, and let $\mathrm{F}(D)$ be the set of non-zero fractional ideals of D. A mapping $I \longmapsto I^*$ of $\mathrm{F}(D)$ into itself is called a star-operation on D if it satisfies the following conditions:

(1) $(a)^* = (a)$ for each non-zero element of K, where K is the quotient field of D.

(2) $(aI)^* = aI^*$ for each non-zero element a of K and for each element $I \in \mathrm{F}(D)$.

(3) $I \subset I^*$ for each element $I \in \mathrm{F}(D)$.

(4) $I \subset J$ implies $I^* \subset J^*$ for all elements I and J in $\mathrm{F}(D)$.

(5) $(I^*)^* = I^*$ for each element $I \in \mathrm{F}(D)$.

Let $\mathrm{F}'(D)$ be the set of non-zero D-submodules of K. A mapping $I \longmapsto I^*$ of $\mathrm{F}'(D)$ into itself is called a semistar-operation on D if it satisfies the following conditions:

(1) $(aI)^* = aI^*$ for each non-zero element a of K and for each element $I \in \mathrm{F}'(D)$.

(2) $I \subset I^*$ for each element $I \in \mathrm{F}'(D)$.

(3) $I \subset J$ implies $I^* \subset J^*$ for all elements I and J in $\mathrm{F}'(D)$.

(4) $(I^*)^* = I^*$ for each element $I \in \mathrm{F}'(D)$.

The set of star-operations (resp. semistar-operations) on D is denoted by $\Sigma(D)$ (resp. $\Sigma'(D)$). The identity mapping d on $\mathrm{F}(D)$ is a star-operation, and is called the d-operation on D. The mapping $I \longmapsto I^v = (I^{-1})^{-1}$ of $\mathrm{F}(D)$ is a star-operation, and is called the v-operation on D. The identity mapping d$'$ on $\mathrm{F}'(D)$ is a semistar-operation on D, and is called the d$'$-operation on D. We set $I^{v'} = I^v$ for each element $I \in \mathrm{F}(D)$, and set $I^{v'} = K$ for each element $I \in \mathrm{F}'(D) - \mathrm{F}(D)$, where K is the quotient field of D. Then v$'$ is a

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semistar-operation on D, and is called the v$'$-operation on D. Let $*$ be a star-operation on D, and let $*'$ be a semistar-operation on D. If the restriction of $*'$ to $\mathrm{F}(D)$ coincides with $*$, then $*'$ is called an extension of $*$ to a semistar-operation. Let R be a domain, let D be a subdomain of R, and let $*$ be a semistar-operation on D. If we set $I^{\alpha(*)} = I^*$ for each $I \in \mathrm{F}'(R)$, then $\alpha(*)$ is a semistar-operation on R, and is called the ascent of $*$ to R. Let $*$ be a semistar-operation on R. If we set $I^{\delta(*)} = (IR)^*$, then $\delta(*)$ is a semistar-operation on D, and is called the descent of $*$ to D.

In this note, D denotes a domain, K denotes the quotient field of D, n denotes a positive integer, and the descent of the d$'$-operation d'_R on R is also denoted by $*_R$.

Proposition 1. Let D be a three-dimensional Prüfer domain with exactly two maximal ideals M and N. Assume that there exist prime ideals P_1 and P_2 of D such that $M \cap N \supsetneqq P_2 \supsetneqq P_1 \supsetneqq (0)$, and that there exist elements π_1, π_2, p and q of D such that $P_1 D_{P_1} = \pi_1 D_{P_1}, P_2 D_{P_2} = \pi_2 D_{P_2}$, $M = (p)$ and $N = (q)$. Then

(1) Each non-zero element x of K can be expressed as $\pi_1^{l_1}\pi_2^{l_2}p^{l_3}q^{l_4}$ up to a unit of D with the integers l_i. This expression is unique up to a unit of D.

(2) Each finitely generated ideal of D is principal.

(3) Define the fractional ideals $A_2 = (1/\pi_2, 1/\pi_2^2, \cdots)$, $A = (1/p, 1/p^2, \cdots)$, $B = (1/q, 1/q^2, \cdots)$ and $C = (1/(pq), 1/(pq)^2, \cdots)$ of D. Then each non-finitely generated ideal I of D is of the form dA_2 or dA or dB or dC with $d \in D$.

(4) We have $P_1 = P_1^v = \pi_1 A_2, P_2 = P_2^v = \pi_2 C = \pi_2 A^v = \pi_2 B^v, C = C^v$, $A_2 = A_2^v$ and $A \neq A^v, B \neq B^v$.

(5) For a fractionl ideal I of D, set $I^{*_1} = I$ if I is of the form xA, and set $I^{*_1} = I^v$ otherwise. Then $*_1$ is a star-operation on D. Set $I^{*_2} = I$ if I is of the form xB, and set $I^{*_2} = I^v$ otherwise. Then $*_2$ is a star-operation on D.

Proof. (1) and (2) are straightforward.

(3) We may assume that there exist principal ideals $I_n = (x_n)$ of D such that $I_1 \subsetneqq I_2 \subsetneqq I_3 \subsetneqq \cdots$ and $I = \cup_1^\infty I_n$. We may assume that each x_i is of the form $\pi_1^{a_i}\pi_2^{b_i}p^{c_i}q^{d_i}$ with integers a_i, b_i, c_i and d_i. Next, we may assume that $x_i = \pi_2^{b_i}p^{c_i}q^{d_i}$ with integers b_i, c_i and d_i for each i. If $\inf\,(b_i) = -\infty$, then $I = dA_2$. If $\inf\,(b_i) > -\infty$, then we may assume that $x_i = p^{c_i}q^{d_i}$ for each i. If $\inf\,(c_i) > -\infty$, then $I = dB$. If $\inf\,(d_i) > -\infty$, then $I = dA$. If $\inf\,(c_i) = \inf\,(d_i) = -\infty$, then $A = dC$.

(4) We have $P_1 = \cap_1^\infty(\pi_2^n)$, and hence $P_1 = P_1^v$. Next, $P_2 = \cap_1^\infty(pq)^n$, and hence $P_2 = P_2^v$. Next, $\pi_2 C = (\pi_2/(pq), \pi_2/(pq)^2, \cdots) = P_2$, and hence $C = C^v$. Next, $\pi_1 A_2 = (\pi_1/\pi_2, \pi_1/\pi_2^2, \cdots) = P_1$, and hence $A_2 = A_2^v$. Assume that $\pi_2 A \subset (\alpha)$ for an element $\alpha \in K$. It follows that $P_2 \subset (\alpha)$. Hence $\pi_2 A^v = P_2$. Similarly, $\pi_2 B^v = P_2$. Clearly, $A \neq A^v$ and $B \neq B^v$.

(5) Let I and J be non-zero fractional ideals of D such that $I \subset J$. We must show that $I^{*_1} \subset J^{*_1}$. We may assume that I is not of the form xA, and J is of the form xA. Next, we may assume that $I = B$ and $J = xA$ for an element $x \in K$. x is expressed as $\pi_1^a\pi_2^b p^c q^d$ up to a unit of D with integers a, b, c and d. Then we see that either $a < 0$ or $a = 0 > b$. Hence $I^{*_1} = P_2/\pi_2 \subset \pi_1^a\pi_2^b p^c q^d A = J$. Similarly, $*_2$ is a star-operation on D.

In the proof of [MS, Proposition 8], we asserted that: For the domain D in Proposition 1, there exists an ideal I of D such that $M \not\supset I \neq I^v$. But this is clearly impossible. We state [MS, Proposition 8] again, and prove it.

Proposition 2. Let D be an integrally closed domain with dimension $n \leq 3$. If $n + 1 \leq\, \mid \Sigma'(D) \mid\, \leq 2n + 1$, then D is a valuation domain.

Proof. Suppose the contrary. We may assume that D is as in Proposition 1. Then, by Proposition 1(4), the semistar-operations e,$*_{U_1}, *_{U_2}, *_V, *_W$,d$'$ and v$'$ are distinct each other. Let $*_1$ and $*_2$ be star-operations on D constructed in Proposition 1(5), then they induce semistar-operations $*_1'$ and $*_2'$ on D. There does not exist an element $x \in K$ such that $B = xA$. Therefore the star-operations $*_1, *_2$, d and v are distinct each other. It follows that the semistar-operations e,$*_{U_1}, *_{U_2}, *_V, *_W$,d$'$, v$'$, $*_1$ and $*_2$ are distinct each other. Hence $\mid \Sigma'(D) \mid \geq 9$; a contradiction.

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