A SUBSET OF THE CLASS S

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Abstract. We investigate some properties of the class of univalent functions $f(z) = z + a_3z^3 + a_4z^4$, analytic in the unit disc and satisfying $\left|\left(\frac{z}{f(z)}\right)''\right| \leq \alpha \leq 2$.

Introduction

Let A denote the class of functions analytic in $U = \{z : |z| < 1\}$ and have the Taylor series

$$f(z) = z + a_2z^2 + a_3z^3 + \dots \tag{1}$$

and let S denote the well-known subclass of A consisting of univalent functions. A function $f(z) \in S$ is said to be starlike in U if and only if it satisfies

$$Re\, z\frac{f'(z)}{f(z)} > 0, \qquad z \in U. \tag{2}$$

In [1] the following theorem was established.

Theorem 1. *Let $f(z) \in A$ with $f(z) \neq 0$ for $0 < |z| < 1$ and let*

$$\left|\left(\frac{z}{f(z)}\right)''\right| < 1, \qquad z \in U. \tag{3}$$

Then, $f \in S$.

For $0 < \alpha \leq 2$, let $S(\alpha)$ denote the class of functions $f(z) \in A$ that satisfy

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq \alpha, \qquad z \in U, \quad f(z) \neq 0, \quad 0 < |z| < 1. \tag{4}$$

In [3] Theorem 1 was extended to the class $S(\alpha)$ and some results for the class $S(\alpha)$ were obtained. Here we prove the following theorem.

Theorem 2. *Let $f(z) \in S(\alpha)$ such that $f''(0) = 0$, then*

(i) *$Re\, \frac{f(z)}{z} \geq \frac{2}{2+\alpha}$, $z \in U$,*

(ii) *f is starlike in $|z| \leq \frac{\sqrt[4]{2}}{\sqrt{\alpha}}$, ($\sqrt{2} < \alpha < 2$). In particular, if $0 < \alpha \leq \sqrt{2}$, then $f(z)$ is starshaped in U.*

(iii) *$Re\, f'(z) > 0$ in $|z| \leq \frac{1}{\sqrt{\alpha}}$.*

Items (i), (ii) and (iii) are improvements of results in [3] and in fact (i) and (iii) are sharp as shown by the function

$$f(z) = \frac{z}{1 - \frac{\alpha}{2}z^2}. \tag{5}$$

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In (ii) the starlikeness of $f(z)$ for $0 < \alpha \le \sqrt{2}$ is in keeping with a result in [2] for a subclass of S that is related to $S(\alpha)$.

We need the notion of subordination. Let $f(z)$ and $g(z)$ be analytic functions in U with $f(0) = g(0)$. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in U$ such that $f(z) = g(\omega(z))$, $z \in U$.

Proof of the Theorem

Let

$$p(z) = \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' = z^2 \frac{f'(z)}{f^2(z)}. \tag{6}$$

In view of the expansion (1) it is easily checked that

$$\left(\frac{1}{z} - \frac{1}{f(z)}\right)\Big|_{z=0} = a_2 \tag{7}$$

and

$$z\left(\frac{z^2 f'(z)}{f^2(z)}\right)' = 2z^2(a_3 - a_2^2) + \dots \tag{8}$$

From (6) we obtain

$$p'(z) = -z\left(\frac{z}{f(z)}\right)'' = \left(\frac{z^2 f'(z)}{f^2(z)}\right)'. \tag{9}$$

Thus

$$\left|\left(\frac{z}{f(z)}\right)''\right| \le \alpha \iff z^2\left(\frac{z}{f(z)}\right)'' \prec \alpha z \tag{10}$$

and equivalently

$$z\left(\frac{z^2 f'(z)}{f^2(z)}\right)' \prec \alpha z. \tag{11}$$

On account of (8), (11) can be written in the form

$$z\left(\frac{z^2 f'(z)}{f^2(z)}\right)' = \alpha\omega(z), \qquad \omega(0) = \omega'(0) = 0, \quad |\omega(z)| \le |z|^2. \tag{12}$$

Therefore

$$\frac{z^2 f'(z)}{f^2(z)} = 1 + \alpha \int_0^1 \frac{\omega(tz)}{t} dt. \tag{13}$$

Further, because of the identity $z^2\left(\frac{1}{z} - \frac{1}{f(z)}\right)' = \frac{z^2 f'(z)}{f^2(z)} - 1$, and using (7) with the hypothesis $a_2 = 0$, we obtain in view of (13)

$$\frac{z}{f(z)} = 1 - \alpha \int_0^1 \frac{\omega(tz)(1-t)}{t^2} dt. \tag{14}$$

Indeed, if $\omega(z) = \sum_{n=2}^{\infty} b_n z^n$ then by (13), $z^2\left(\frac{1}{z} - \frac{1}{f(z)}\right)' = \alpha \sum_{n=2}^{\infty} \frac{b_n z^n}{n}$, and on dividing by z^2 and integrating we obtain, because $a_2 = 0$,

$$\frac{1}{z} - \frac{1}{f(z)} = \alpha \sum_{n=2}^{\infty} \frac{b_n z^{n-1}}{n(n-1)} = \frac{\alpha}{z} \int_0^1 \frac{\omega(tz)}{t^2}(1-t) dt$$

which is (14).

As $|\omega(z)| \le |z|^2$, (14) gives

$$\left|\frac{z}{f(z)} - 1\right| \le \frac{\alpha}{2}|z|^2. \tag{15}$$

Since $0 < \alpha \le 2$ this yields

$$Re\ \frac{z}{f(z)} \ge 1 - \frac{\alpha}{2}. \tag{16}$$

which is sharp in view of (5). Further, (15) is equivalent to

$$\left|\frac{f(z)}{z} - \frac{1}{1 - \frac{\alpha^2}{4}|z|^4}\right| \le \frac{\alpha\frac{|z|^2}{2}}{1 - \frac{\alpha^2}{4}|z|^4}.$$

This yields

$$Re\ \frac{f(z)}{z} \ge \frac{1}{1 + \frac{\alpha}{2}|z|^2} \ge \frac{1}{1 + \frac{\alpha}{2}} \tag{17}$$

which establishes (i).

From (13) we obtain

$$z\frac{f'(z)}{f(z)} = \frac{f(z)}{z}(1 + \alpha\omega_1(z)), \quad \omega_1(z) = \int_0^1 \frac{\omega(tz)}{t}dt$$

which leads to

$$\begin{aligned} \left|arg\left(z\frac{f'(z)}{f(z)}\right)\right| &= \left|arg\frac{f(z)}{z} + arg(1 + \alpha\omega_1(z))\right| \\ &\le \left|arg\frac{f(z)}{z}\ |+|\ arg(1 + \alpha\omega_1(z))\right| \\ &\le 2\sin^{-1}\left(\alpha\frac{|z|^2}{2}\right) \qquad (18) \end{aligned}$$

because of (15) and the fact that $|\omega_1(z)| \le \frac{|z|^2}{2}$.

As $Re\ z\frac{f'(z)}{f(z)} > 0 \iff \left|arg\left(z\frac{f'(z)}{f(z)}\right)\right| \le \frac{\pi}{2}$, we obtain from (18)

$$Re\ z\frac{f'(z)}{f(z)} > 0 \quad if \quad |z|^2 \le \frac{\sqrt{2}}{\alpha}.$$

This establishes (ii).

Once again, from (13) we get

$$f'(z) = \left(\frac{f(z)}{z}\right)^2 (1 + \alpha\omega_1(z))$$

and a similar argument gives

$$|arg\ f'(z)| \le 3\sin^{-1}\left(\alpha\frac{|z|^2}{2}\right)$$

which gives (iii). This completes the proof of the theorem.

It may be mentioned that in [2] the class of functions $f(z)$ satisfying $\left|z^2\frac{f'(z)}{f^2(z)} - 1\right| \le \mu$ had been considered and in the present situation $\mu = \frac{\alpha}{2}$.

References

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