SOME ORDER PRESERVING OPERATOR INEQUALITY VIA FURUTA INEQUALITY

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Received March 9, 2001

Abstract. In this paper, we shall show the following: If $A_1^{\delta} \geq A_2^{\delta}$ for some $\delta > 0$ and $\log A_2 \geq \log B$, then for each α such that $0 \leq \alpha \leq \delta$,

$$(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} \geq (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}}$$

holds for all p, t and r such that $\delta \geq p \geq \frac{\alpha}{2}$, $t \geq \max\{\alpha, p\}$ and $r \geq 0$. And also we shall discuss a relation between $A_1^{\alpha} \geq A_2^{\alpha}$ and $(A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \geq A_2^{p-\beta}$ for $0 \leq \beta \leq p < t$. These results are extensions of Yang's recent ones.

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is said to be strictly positive (in symbol: $T > 0$) if T is positive and invertible.

First of all, we recall the following Theorem F. We remark that Theorem F yields the celebrated Löwner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$"* by putting $r = 0$ in (i) or (ii) of Theorem F.

Theorem F (Furuta inequality [7]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$.

Figure

Alternative proofs of Theorem F are given in [2] and [17] and also an elementary one page proof in [8]. Tanahashi [22] showed that the domain drawn for p, q and r in the Figure is the best possible one for Theorem F.

As stated in [17], Theorem F was arranged as the following Theorem F' by using notion of α-power mean $\sharp_{\alpha}$ for $\alpha \in [0,1]$ introduced by Kubo-Ando [20], that is,

$$A \sharp_{\alpha} B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\alpha} A^{\frac{1}{2}} \quad \text{for } A > 0 \text{ and } B \geq 0.$$

2000 *Mathematics Sabject Classification.* Primary 47A63, 47A64.

Key words and phrases. Positive operator, Löwner-Heinz theorem and Furuta inequality.

Theorem F'. *If $A \geq B > 0$, then*

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$$

holds for $p \geq 1$ and $r \geq 0$.

On the other hand, in [4][18][23] and others, many reseachers discussed Furuta-type inequalities with negative powers (see section 2). Related to these matters, Yang showed the following results in [26]. We remark that we use the notation $A \natural_s B$ for $s \in \mathbb{R}$ defined by $A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^s A^{\frac{1}{2}}$.

Theorem A.1 ([26]). *Let $1 < p < t$, $A_1 > 0$ and $A_2 \geq B > 0$. If*

$$A_1^t \natural_{\frac{1-t}{p-t}} A_2^p \leq A_2, \tag{1.1}$$

then

$$(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} \geq (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} \tag{1.2}$$

holds for any $0 \leq \alpha \leq \min\{2p-1, t\}$ and $r \geq 0$.

Theorem A.2 ([26]). *Let $1 \leq p < 2p < t$, $A_1 > 0$ and $A_2 \geq B > 0$. If*

$$A_1^t \natural_{\frac{2p-t}{p-t}} A_2^p \geq A_2^{2p}, \tag{1.3}$$

then

$$(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} \geq (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} \tag{1.2}$$

holds for any $0 \leq \alpha \leq 2p$ and $r \geq 0$.

In this paper, firstly we shall discuss a relation between $A_1^\alpha \geq A_2^\alpha$ and (1.1) (or (1.3)). Secondly we shall show an extension of Theorem A.1 and Theorem A.2.

2. Furuta-type inequalities with negative powers

The following Theorem B means that the inequality in Theorem F remains valid for some negative numbers $r = -t < 0$.

Theorem B ([4][18][23][27]). *If $A \geq B \geq 0$ with $A > 0$, then the following inequalities hold:*

(I) $A^{1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}}$ *for $1 \geq p > t \geq 0$ with $p \geq \frac{1}{2}$.*

(II) $A^{-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-t}{p-t}}$ *for $1 \geq t > p \geq 0$ with $\frac{1}{2} \geq p$.*

(III) $A^{2p-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t}{p-t}}$ *for $\frac{1}{2} \geq p > t \geq 0$.*

(IV) $A^{2p-1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-1-t}{p-t}}$ *for $1 \geq t > p \geq \frac{1}{2}$.*

Firstly, Yoshino [27] initiated an attempt to extend the domain in which Furuta-type inequality holds. Afterwards, the domain given by him was enlarged to (I) by Fujii-Furuta-Kamei [4]. Kamei [18] gave simplified proofs of (I) and (III). Tanahashi [23] showed all the inequalities in Theorem B and proved that the outside exponents of (I),(II) and (IV) are best possible. But we remark that it has not been proved yet whether the outside exponents of (III) is best possible or not. Furuta-Yamazaki-Yanagida [13] showed the equivalence relation among (I), (II), (III) and (IV). Extensions and related results to Theorem B are shown in [6][11][14][15][16][19] and others.

At the end of this section, we note the following: In [18], (I) and (III) of Theorem B were arranged as the following (i) and (ii) of Theorem B', respectively.

Theorem B'. *If $A \geq B \geq 0$ with $A > 0$, then the following inequalities hold:*
(i) $A^t \natural_{\frac{1-t}{p-t}} B^p \leq B \leq A$ *for* $0 \leq t < p$ *and* $\frac{1}{2} \leq p \leq 1$.
(ii) $A^t \natural_{\frac{2p-t}{p-t}} B^p \leq B^{2p} \leq A^{2p}$ *for* $0 \leq t < p \leq \frac{1}{2}$.

3. A relation between $A_1^\alpha \geq A_2^\alpha$ and $(A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \geq A_2^{p-\beta}$ for $0 \leq \beta \leq p < t$

As corollaries of Theorem A.1 and Theorem A.2, the following were obtained in [26] by putting $r = 0$.

Corollary A.3 ([26]). *Let $1 < p < 2p - 1 < t$, $A_1 > 0$ and $A_2 > 0$. If*

$$A_1^t \natural_{\frac{1-t}{p-t}} A_2^p \leq A_2, \tag{1.1}$$

then $A_1^\alpha \geq A_2^\alpha$ holds for any $0 \leq \alpha \leq 2p - 1$.

Corollary A.4 ([26]). *Let $1 \leq p < 2p < t$, $A_1 > 0$ and $A_2 > 0$. If*

$$A_1^t \natural_{\frac{2p-t}{p-t}} A_2^p \geq A_2^{2p}, \tag{1.3}$$

then $A_1^\alpha \geq A_2^\alpha$ holds for any $0 \leq \alpha \leq 2p$.

We remark that (1.1) is the same form to the first inequality of (i) in Theorem B', and also (1.3) is the opposite inequality of the first inequality of (ii) in Theorem B'. It was also pointed out in [26] that $A^\alpha \geq B^\alpha$ for $\alpha \in [0, 2p-1]$ (resp. $\alpha \in [0, 2p]$) does not always ensure (1.1) for $1 < p < 2p - 1 < t$ (resp. (1.3) for $1 \leq p < 2p < t$) in general.

In this section, we shall show the following result as a generalization of Corollary A.3 and Corollary A.4.

Proposition 1. *Let $0 \leq \beta < p < t$, $A_1 > 0$ and $A_2 > 0$. If*

$$(A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \geq A_2^{p-\beta}, \tag{3.1}$$

then $A_1^\delta \geq A_2^\delta$ holds for $\delta = \min\{2p - \beta, t\}$.

Here we prepare the following Lemma F.

Lemma F ([10]). *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Proposition 1 yields Corollary A.3 by putting $\beta = 1$ since

$$A_1^t \natural_{\frac{1-t}{p-t}} A_2^p = A_1^{\frac{t}{2}}(A_1^{\frac{-t}{2}} A_2^p A_1^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A_1^{\frac{t}{2}} \leq A_2 \quad \text{for } 1 < p < t \tag{1.1}$$

holds if and only if

$$A_2^{\frac{p}{2}}(A_2^{\frac{p}{2}} A_1^{-t} A_2^{\frac{p}{2}})^{\frac{1-p}{p-t}} A_2^{\frac{p}{2}} \leq A_2$$

holds if and only if

$$(1.1') \qquad (A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-1}{t-p}} \geq A_2^{p-1} \quad \text{for } 1 < p < t$$

holds by Lemma F. Similarly, Proposition 1 yields Corollary A.4 by putting $\beta = 0$ since

$$(1.3) \qquad A_1^t \natural_{\frac{2p-t}{p-t}} A_2^p = A_1^{\frac{t}{2}} (A_1^{\frac{-t}{2}} A_2^p A_1^{\frac{-t}{2}})^{\frac{2p-t}{p-t}} A_1^{\frac{t}{2}} \geq A_2^{2p} \quad \text{for } 1 \leq p < 2p < t$$

holds if and only if

$$A_2^{\frac{p}{2}} (A_2^{\frac{p}{2}} A_1^{-t} A_2^{\frac{p}{2}})^{\frac{p}{p-t}} A_2^{\frac{p}{2}} \geq A_2^{2p}$$

holds if and only if

$$(1.3') \qquad (A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p}{t-p}} \geq A_2^p \quad \text{for } 1 \leq p < 2p < t$$

holds by Lemma F.

Proof of Proposition 1. Let $0 \leq \beta < p < t$.

(i) Case $2p - \beta \geq t$, i.e., $\frac{p-\beta}{t-p} \geq 1$. Applying Löwner-Heinz theorem to (3.1) since $\frac{t-p}{p-\beta} \in (0, 1]$, we have $A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}} \geq A_2^{t-p}$, that is, $A_1^t \geq A_2^t$.

(ii) Case $2p - \beta \leq t$, i.e., $0 \leq \frac{p-\beta}{t-p} \leq 1$. Put $S = (A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}}$ and $T = A_2^{p-\beta}$, then $S \geq T > 0$ by (3.1). Applying Theorem F to $S \geq T > 0$, we have

$$(T^{\frac{r_1}{2}} S^{p_1} T^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}} \geq T^{1+r_1} \quad \text{for } p_1 \geq 1 \text{ and } r_1 \geq 0.$$

Put $p_1 = \frac{t-p}{p-\beta} \geq 1$ and $r_1 = \frac{p}{p-\beta} \geq 0$. Then

$$(A_2^{\frac{p}{2}} \cdot A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}} \cdot A_2^{\frac{p}{2}})^{\frac{2p-\beta}{t}} \geq A_2^{2p-\beta},$$

that is, we have $A_1^{2p-\beta} \geq A_2^{2p-\beta}$.

Hence the proof of Proposition 1 is complete. □

Next, related to Proposition 1, we have the following result.

Proposition 2. *Let $0 \leq \beta < p < t$, $A_1 > 0$ and $A_2 > 0$. If $A_1^\gamma \geq A_2^\gamma$ for $\gamma = \max\{2p-\beta, t\}$, then the following inequality holds:*

$$(3.1) \qquad (A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \geq A_2^{p-\beta}.$$

Proof of Proposition 2. Let $0 \leq \beta < p < t$.

(i) Case $2p - \beta \geq t$, i.e., $\frac{p-\beta}{t-p} \geq 1$. By replacing S with B^{-1} and T with A^{-1} in (I) of Theorem B, we have the following:

$$(3.2) \qquad S \geq T > 0 \text{ ensures } (T^{\frac{-t}{2}} S^p T^{\frac{-t}{2}})^{\frac{1-t}{p-t}} \geq T^{1-t} \text{ for } 1 \geq p > t \geq 0 \text{ and } p \geq \tfrac{1}{2}.$$

Put $S = A_1^{2p-\beta}$ and $T = A_2^{2p-\beta}$. Then $S \geq T > 0$ by the assumption, and we have

$$(3.3) \qquad (A_2^{(2p-\beta)\frac{-t_1}{2}} A_1^{(2p-\beta)p_1} A_2^{(2p-\beta)\frac{-t_1}{2}})^{\frac{1-t_1}{p_1-t_1}} \geq A_2^{(2p-\beta)(1-t_1)}$$

for $1 \geq p_1 > t_1 \geq 0$ and $p_1 \geq \frac{1}{2}$ by (3.2). Put $p_1 = \frac{t}{2p-\beta}$ and $t_1 = \frac{p}{2p-\beta}$ in (3.3) since $1 \geq \frac{t}{2p-\beta} > \frac{p}{2p-\beta} \geq 0$ and $\frac{t}{2p-\beta} \geq \frac{1}{2}$, then the following desired inequality holds:

$$(3.1) \qquad (A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \geq A_2^{p-\beta}.$$

(ii) Case $2p-\beta \le t$, i.e., $0 \le \frac{p-\beta}{t-p} \le 1$. If $A_1^t \ge A_2^t$, by Löwner-Heinz theorem for $\frac{p-\beta}{t-p} \in [0,1]$, we obtain

$$(A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \ge A_2^{p-\beta}. \tag{3.1}$$

Hence the proof of Proposition 2 is complete. □

We can summarize Proposition 1 and Proposition 2 as follows: For $0 \le \beta < p < t$,

$$\begin{aligned} & A_1^\gamma \ge A_2^\gamma \quad \text{for } \gamma = \max\{2p-\beta, t\} \\ \Longrightarrow\ & (A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \ge A_2^{p-\beta} \quad \cdots (3.1) \qquad \text{by Proposition 2} \\ \Longrightarrow\ & A_1^\delta \ge A_2^\delta \quad \text{for } \delta = \min\{2p-\beta, t\}. \qquad \text{by Proposition 1} \end{aligned}$$

4. An extension of Theorem A.1 and Theorem A.2

In this section, we shall show an extension of Theorem A.1 and Theorem A.2.

Theorem 3. *Let $A_1, A_2, B > 0$. If $A_1^\delta \ge A_2^\delta$ for some $\delta > 0$ and $\log A_2 \ge \log B$, then for each α such that $0 \le \alpha \le \delta$,*

$$(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} \ge (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} \tag{1.2}$$

holds for all p, t and r such that $\delta \ge p \ge \frac{\alpha}{2}$, $t \ge \max\{\alpha, p\}$ and $r \ge 0$.

Proposition 1 and Theorem 3 ensure the following Corollary 4.

Corollary 4. *Let $0 \le \beta < p < t$ and $A_1, A_2, B > 0$. If*

$$(A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \ge A_2^{p-\beta} \tag{3.1}$$

and $\log A_2 \ge \log B$, then for each α such that $0 \le \alpha \le \min\{2p-\beta, t\}$,

$$(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} \ge (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} \tag{1.2}$$

holds for all $r \ge 0$.

For $A, B > 0$, it is well known that chaotic order $\log A \ge \log B$ is weaker than usual order $A \ge B$ since $\log t$ is an operator monotone function. Therefore Corollary 4 yields Theorem A.1 by putting $\beta = 1$ since (1.1) is equivalent to (1.1') in the previous section. Similarly, Corollary 4 yields Theorem A.2 by putting $\beta = 0$ since (1.3) is equivalent to (1.3').

We need the following theorem on chaotic order to prove Theorem 3.

Theorem C ([3][5][9][25]). *Let $A, B > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \ge \log B$.
(ii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \ge B^r$ *for all $p \ge 0$ and $r \ge 0$.*
(iii) $A^r \ge (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all $p \ge 0$ and $r \ge 0$.*
(iv) *For each $q \ge 0$ and $r \ge 0$, $f(s) = (B^{\frac{r}{2}} A^s B^{\frac{r}{2}})^{\frac{q+r}{s+r}}$ is increasing for $s \ge q$.*
(v) *For each $q \ge 0$ and $r \ge 0$, $g(s) = (A^{\frac{r}{2}} B^s A^{\frac{r}{2}})^{\frac{q+r}{s+r}}$ is decreasing for $s \ge q$.*

We remark that Theorem C is an extension of Ando's result in [1] that $\log A \geq \log B$ is equivalent to $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ for all $p \geq 0$.

Proof of Theorem 3. Let $f(s) = (B^{\frac{r}{2}} A_1^s B^{\frac{r}{2}})^{\frac{\alpha+r}{s+r}}$. The assumptions $A_1^\delta \geq A_2^\delta$ for $\delta > 0$ and $\log A_2 \geq \log B$ ensure $\log A_1 \geq \log A_2 \geq \log B$.

(i) Case $\alpha \geq p$. By (iii) of Theorem C, we have

$$A_2^p \geq (A_2^{\frac{p}{2}} B^r A_2^{\frac{p}{2}})^{\frac{p}{r+p}} \quad \text{for } p \geq \tfrac{\alpha}{2} \geq 0 \text{ and } r \geq 0. \tag{4.1}$$

(4.1) ensures the following (4.2) by Löwner-Heinz theorem since $\frac{\alpha-p}{p} \in [0,1]$.

$$A_2^{\alpha-p} \geq (A_2^{\frac{p}{2}} B^r A_2^{\frac{p}{2}})^{\frac{\alpha-p}{p+r}}. \tag{4.2}$$

Therefore, for each α such that $0 \leq \alpha \leq \delta$, we have

$$\begin{aligned}
(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} &= f(t) \geq f(\alpha) = B^{\frac{r}{2}} A_1^\alpha B^{\frac{r}{2}} && \text{by (iv) of Theorem C for } t \geq \alpha \\
&\geq B^{\frac{r}{2}} A_2^\alpha B^{\frac{r}{2}} = B^{\frac{r}{2}} A_2^{\frac{p}{2}} A_2^{\alpha-p} A_2^{\frac{p}{2}} B^{\frac{r}{2}} && \text{by Löwner-Heinz theorem} \\
&\geq B^{\frac{r}{2}} A_2^{\frac{p}{2}} (A_2^{\frac{p}{2}} B^r A_2^{\frac{p}{2}})^{\frac{\alpha-p}{p+r}} A_2^{\frac{p}{2}} B^{\frac{r}{2}} && \text{by (4.2)} \\
&= (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} && \text{by Lemma F}
\end{aligned}$$

for all p, t and r such that $(\delta \geq)\alpha \geq p \geq \frac{\alpha}{2}$, $t \geq \alpha = \max\{\alpha, p\}$ and $r \geq 0$.

(ii) Case $p \geq \alpha$. For each α such that $0 \leq \alpha \leq \delta$, we have

$$\begin{aligned}
(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} &= f(t) \geq f(p) = (B^{\frac{r}{2}} A_1^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} && \text{by (iv) of Theorem C for } t \geq p \geq \alpha \\
&\geq (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}}
\end{aligned}$$

for all p, t and r such that $\delta \geq p \geq \alpha (\geq \frac{\alpha}{2})$, $t \geq p = \max\{\alpha, p\}$ and $r \geq 0$, and the last inequality holds by Löwner-Heinz theorem since $\frac{\alpha+r}{p+r} \in [0,1]$.

Hence the proof of Theorem 3 is complete. □

Proof of Corollary 4. Assume that

$$(A_2^{\frac{-p}{2}} A_1^t A_2^{\frac{-p}{2}})^{\frac{p-\beta}{t-p}} \geq A_2^{p-\beta} \tag{3.1}$$

holds for $0 \leq \beta < p < t$. By Proposition 1, (3.1) ensures $A_1^\delta \geq A_2^\delta$ for $\delta = \min\{2p-\beta, t\}$. Let $0 \leq \alpha \leq \delta$. Then the conditions

$$\delta = \min\{2p-\beta, t\} \geq p \geq \frac{1}{2}(2p-\beta) \geq \frac{1}{2}\min\{2p-\beta, t\} = \frac{\delta}{2} \geq \frac{\alpha}{2},$$

$$t \geq \min\{2p-\beta, t\} = \delta \geq \alpha \quad \text{and} \quad t \geq p$$

hold, so that

$$(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} \geq (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} \tag{1.2}$$

holds for all $r \geq 0$ by Theorem 3.

Hence the proof of Corollary 4 is complete. □

5. Concluding remarks

Remark 1. Olson [21] introduced spectral order $A \succ B$ among the self adjoint operators A and B as follows: Let $A = \int t dE_t$ and $B = \int t dF_t$. Then the spectral order $A \succ B$ holds if $E_t \leq F_t$ for all t. It was also shown in [21] that for positive operators A and B,

$$A \succ B \text{ if and only if } A^n \geq B^n \text{ for all positive integer } n. \tag{5.1}$$

We remark that Uchiyama [24] and Furuta [12] showed several properties of spectral order. Here, as a similar result to Theorem 3, we have the following Theorem 5 on spectral order.

Theorem 5. *Let $A_1, A_2, B > 0$. If $A_1 \succ A_2$ and $\log A_2 \geq \log B$, then for each $\alpha \geq 0$,*

$$(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} \geq (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} \tag{1.2}$$

holds for all p, t and r such that $p \geq \frac{\alpha}{2}$, $t \geq \max\{\alpha, p\}$ and $r \geq 0$.

Proof of Theorem 5. Since $A_1 \succ A_2$ if and only if $A_1^\delta \geq A_2^\delta$ for all $\delta > 0$ by (5.1) and Löwner-Heinz theorem, Theorem 3 ensures Theorem 5. □

Remark 2. Related to Theorem 3 and Theorem 5, one might conjecture the following: *Let $A_1, A_2, B > 0$. If $\log A_1 \geq \log A_2 \geq \log B$, then for each $\alpha \geq 0$,*

$$(B^{\frac{r}{2}} A_1^t B^{\frac{r}{2}})^{\frac{\alpha+r}{t+r}} \geq (B^{\frac{r}{2}} A_2^p B^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}} \tag{1.2}$$

holds for all p, t and r such that $p \geq \frac{\alpha}{2}$, $t \geq \max\{\alpha, p\}$ and $r \geq 0$.

But we have the following counterexample.

Let $A_1 = \begin{pmatrix} 4 & 0 \\ 0 & 19 \end{pmatrix}^4$, $A_2 = \begin{pmatrix} 3 & 3 \\ 3 & 10 \end{pmatrix}^4$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^4$. Then $\log A_1 \geq \log A_2 \geq \log B$ since $A_1^{\frac{1}{4}} \geq A_2^{\frac{1}{4}} \geq B^{\frac{1}{4}} > 0$.

Let $\alpha = 0$, $p = 1$, $t = 2$ and $r = 2$. Then

$$(BA_1^2B)^{\frac{1}{2}} - (BA_2B)^{\frac{2}{3}} = \begin{pmatrix} -1913.59\ldots & -1004.86\ldots \\ -1004.86\ldots & 130130.\ldots \end{pmatrix}.$$

Eigenvalues of $(BA_1^2B)^{\frac{1}{2}} - (BA_2B)^{\frac{2}{3}}$ are $130138.\ldots$ and $-1921.24\ldots$, so that $(BA_1^2B)^{\frac{1}{2}} \ngeq (BA_2B)^{\frac{2}{3}}$.

Acknowledgement. The author would like to express his deepest gratitude to Professor Changsen Yang for giving an opportunity to read his paper [26] before publication, and also the author would like to express his cordial thanks to Professor Takayuki Furuta for his kind suggestion.

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