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HYPERSUBSTITUTIONS FOR THE VARIETY OF STAR BANDS

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Abstract.

We determine a normal form for hypersubstitutions for the type (2,1) variety of star-bands, find the monoid of all star-band-proper hypersubstitutions, and show that although none of the basis identities for star-bands are hyperidentities, this variety does have some hyperidentities of any arity k.

1 Introduction A star-regular band, or star-band for short, is an algebra of type $(2, 1)$ with a binary multiplication indicated by juxtaposition and a unary operation *, which satisfies the following identities:

$$x(yz) \approx (xy)z,\ x^{**} \approx x,\ (xy)^* \approx y^*x^*,\ xx^*x \approx x, \text{ and } xx \approx x.$$

We will use the notation StB for the variety of all star-bands. The lattice of all subvarieties of StB is countably infinite, and has been studied by Adair ([Ad]) and Petrich ([Pet]).

In this paper, we apply the theory of hypersubstitutions and hyperidentities to the variety StB and its subvarieties. Section 2 provides the necessary background on hyperidentities and hypersubstitutions, in particular the concepts of proper and normal form hypersubstitutions. Section 3 summarizes the results we need about the varieties of star-bands and their identities. In Section 4, we produce a normal form for hypersubstitutions for StB, and use this to find the monoid $P(StB)$ of all proper hypersubstitutions for StB. Finally in Section 5 we show that although StB satisfies some general iterative identities as hyperidentities, none of the five defining identities for StB given above are hyperidentities.

2 Hyperidentities and Hypersubstitutions In this paper we will be interested in varieties of type $(2, 1)$, with two operation symbols. But first we present information on hypersubstitutions for the most general setting, an arbitrary type τ. We let τ be a fixed type, with fundamental operation symbols f_i, $i \in I$. An identity $s \approx t$ of type τ is called a hyperidentity of a variety V if for every substitution of terms of V (of appropriate arity) for the operation symbols in $s \approx t$, the resulting identity holds in V. To make this precise, we use the idea of a map σ which associates to every operation symbol f_i of the given type τ a term $\sigma(f_i)$ of type τ, of the same arity as f_i. Any such map σ is called a hypersubstitution (of type τ).

Let $W_\tau(X)$ be the set of all terms of type τ on an alphabet $X = \{x_1, x_2, x_3, \ldots\}$. Any hypersubstitution σ can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ inductively as follows:

(i) if $t = x_i$ for some $i \geq 1$, then $\hat{\sigma}[t] = x_i$;

(ii) if $t = f(t_1, \ldots, t_n)$ for some n-ary operation symbol f and some terms $t_1, \ldots, t_n$, then $\hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$.

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Here the left side of (ii) means the composition of the term $\sigma(f)$ and the terms $\hat{\sigma}[t_1]$, ..., $\hat{\sigma}[t_n]$. We can define a binary operation $\circ$ on the set $Hyp(\tau)$ of all hypersubstitutions of type τ, by taking $\sigma_1 \circ \sigma_2$ to be the hypersubstitution which maps each fundamental operation symbol f_i to the term $\hat{\sigma}_1[\sigma_2(f_i)]$. The set $Hyp(\tau)$ of all hypersubstitutions of type τ is closed under this associative binary operation. $Hyp(\tau)$ then is a monoid, since the identity hypersubstitution σ_{id} (mapping every f_i to $f_i(x_1, \ldots, x_{n_i})$) acts as an identity element.

Now let M be any submonoid of $Hyp(\tau)$. An identity $u \approx v$ of a variety V is called an M-hyperidentity of V if for every hypersubstitution $\sigma \in M$, the identity $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ holds in V. A variety V is called M-solid if every identity of V is an M-hyperidentity of V. In the special case that M is all of $Hyp(\tau)$, we speak of a hyperidentity and a solid variety.

Denecke and Reichel in [DR] connected submonoids of $Hyp(\tau)$ with sublattices of the lattice of all varieties of type τ. If M is a submonoid of $Hyp(\tau)$, then the collection of all M-solid varieties of type τ is a complete sublattice of the lattice of all varieties of type τ. There is then a Galois correspondence between submonoids of $Hyp(\tau)$ and complete sublattices of the lattice of all varieties of type τ. Thus studying submonoids of $Hyp(\tau)$ may give a method for studying the complete sublattices of this lattice.

Our goal is to study, in the particular context of star-bands, two concepts for hypersubstitutions defined by Płonka.

Definition 2.1 ([P]): Let V be a variety of type τ.
i) A hypersubstitution σ of type τ is called a V-proper hypersubstitution if for every identity $s \approx t$ of V, the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in V. We use $P(V)$ for the set of all V-proper hypersubstitutions of type τ.
ii) Two hypersubstitutions σ_1 and σ_2 of type τ are called V-equivalent iff $\sigma_1(f_i) \approx \sigma_2(f_i)$ is an identity in V for all $i \in I$. In this case we write $\sigma_1 \sim_V \sigma_2$.

Płonka showed that $P(V)$ is a submonoid of $Hyp(\tau)$, the largest monoid M for which V is M-solid. The relation $\sim_V$ is always an equivalence relation on $Hyp(\tau)$, and is sometimes but not always a congruence. (Denecke and Marszałek ([DM]) and Wismath ([W]) have characterized which varieties V have $\sim_V$ a congruence.) The relation $\sim_V$ also has three other important properties, which we shall refer to as the Płonka properties.

Lemma 2.2 *([P]): Płonka properties of $\sim_V$:*
Let V be a variety of type τ, and let σ_1 and $\sigma_2 \in Hyp(\tau)$. If $\sigma_1 \sim_V \sigma_2$, then
(i) for every term t of type τ, the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity of V;
(ii) σ_1 is a V-proper hypersubstitution iff σ_2 is a V-proper hypersubstitution;
(iii) for all $s, t \in W_\tau(X)$, the equation $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in V iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in V.

The second of these properties tells us that the submonoid $P(V)$ of $Hyp(\tau)$ is in fact a union of equivalence classes of the relation $\sim_V$. This is also true when we restrict our attention to a submonoid M of $Hyp(\tau)$, and to the relation $\sim_V |_M$. This is significant for the testing of hyperidentities of V, since it allows us to reduce the number of hypersubstitutions we need to consider.

Definition 2.3 Let M be a monoid of hypersubstitutions of type τ, and let V be a variety of type τ. Let ϕ be a choice function which chooses from M one hypersubstitution from each equivalence class of the relation $\sim_V |_M$, and let N_M^ϕ be the set of hypersubstitutions so chosen. Thus N_M^ϕ is a set of distinguished hypersubstitutions from M, which we shall call normal form hypersubstitutions.

It follows from Płonka property (ii) that to test whether an identity $s \approx t$ is an M-hyperidentity of V, it suffices to consider $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ for each σ from N_M^{ϕ}. Thus a variety V is M-solid if the result of applying any one of the distinguished hypersubstitutions from N_M^{ϕ} to an identity of V is still an identity of V.

Although such normal-form hypersubstitutions suffice for testing for M-hyperidentities, Denecke and Arworn have shown in [DA] that the set N_M^{ϕ} is no longer a monoid. When we compose two hypersubstitutions σ and ρ in N_M^{ϕ}, according to the usual composition in $Hyp(\tau)$, the result need not be in N_M^{ϕ}, although it is equivalent to some element in N_M^{ϕ}. If we define a new product $*$ on N_M^{ϕ} which assigns this equivalent element to $\sigma * \rho$, we get a groupoid structure on N_M^{ϕ}, but the operation $*$ is not always associative.

3 Identities and Varieties of Star Bands In this section we present background information on star-bands. These are type $(2,1)$ algebras, having both a binary and a unary operation. We shall denote these operation symbols in two ways, depending upon the context: formally, and when we consider hyperidentities, we denote the binary operation by f and the unary operation by st; informally, when we consider identities, we often replace f by juxtaposition and $st(x)$ by x^*.

The variety StB of all star-bands is the type $(2,1)$ variety defined by the following five identities:

(I1) $x(yz) \approx (xy)z$, (associativity law)
(I2) $x^{**} \approx x$, (involution law)
(I3) $(xy)^* \approx y^*x^*$, (product law)
(I4) $xx^*x \approx x$, (absorption law) and
(I5) $xx \approx x$ (idempotent law).

The lattice of all subvarieties of StB has been described by Adair ([Ad]) and Petrich ([Pet]). This lattice consists of four special varieties, then a countably infinite chain of varieties V_n, with each variety being defined within StB by one additional identity. For $u \approx v$ an identity of type $(2,1)$, we will use the notation $V(u \approx v)$ for the subvariety of StB determined by $u \approx v$. For any term u of type $(2,1)$, we use the notation $\overline{u}$ for the left-to-right dual of u, so that for instance $\overline{x_1x_2^*x_2} = x_2x_2^*x_1$. The lattice of all varieties of star-bands is shown in Figure 1. We list here the special varieties which appear in this diagram:

$TR = V(x \approx y)$, the trivial variety,
$V_1 = SL = V(x^* \approx x) = V(x \approx xx^*) = V(xy \approx yx)$, the variety of semilattice star-bands,
$RB = V(xyx \approx x)$, the variety of rectangular star-bands,
$V_2 = NB = V(xy \approx xy^*xy) = V(xyzw \approx xzyw)$, the variety of normal star-bands,
$V_3 = RegB = V(xy \approx xx^*yxy) = V(xyzx \approx xyxzx)$, the variety of regular star-bands.

Above $n = 2$, the identities defining varieties V_n are defined inductively; for our purposes here it is not necessary to know these identities.

For any word w, the content $c(w)$ of w is the set of letter variables (ignoring stars) which occur in w. Note that $c(xx^*) = \{x\} = c(x^*)$. (The content is also sometimes denoted by $Var(w)$, the set of all variables which occur in the term w.) Many of our calculations for identities will use the following important fact about contents.

Lemma 3.1 *([Ad]) For any words u, v and w, if $c(v) \subseteq c(u) = c(w)$, then the identity $uvw \approx uw$ holds in every variety of star-bands.*

In the remainder of this section we summarize, without proof, Adair's notation and results about identities which hold in the various varieties of star-bands.

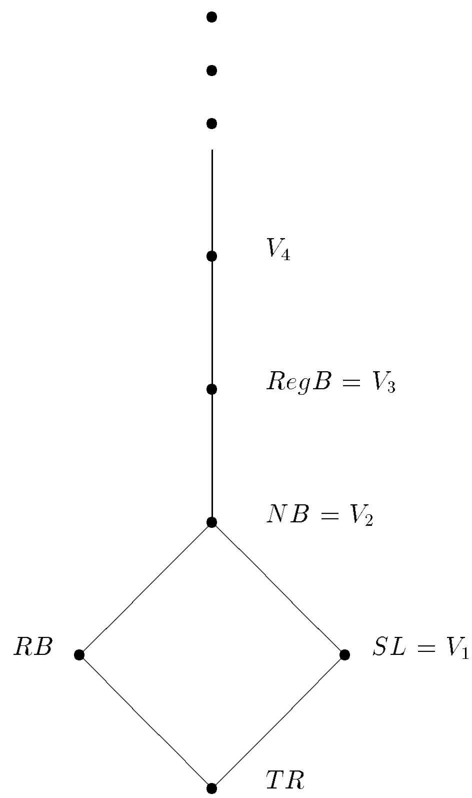

Figure 1: The lattice of varieties of star-bands

Definition 3.2 ([Ad]) Let X be a set of variable letters, with X^* a dual set of starred letters. Let p and q be words of type $(2,1)$ on the alphabet X. We will use the following notation:

i) An identity $p \approx q$ will be called homotypical if $c(p) = c(q)$; otherwise $p \approx q$ is called heterotypical.

ii) The head $h(p)$ of p is the element of $X \bigcup X^*$ which occurs first in the word p;
dually, the tail $t(p)$ of p is the element of $X \bigcup X^*$ which occurs last in p.

iii) The initial part $i(p)$ of p is the word obtained from p by keeping only the first occurrence of each letter in p, in the order in which they first occur in p;
dually, the final part $f(p)$ of p is the word obtained from p by keeping only the last occurrence of each letter in p, in the order in which they make their last occurrence.

iv) For n equal to the size of $c(p)$ and $2 \leq j \leq n$,
$\gamma_j(p)$ is the longest left cut of p with $j-1$ variables; and dually,
$\delta_j(p)$ is the longest right cut of p with $j-1$ variables.

v) When $p \approx q$ is a homotypical identity with content size n, we set
$I(p,q) = \{\gamma_j(p) \approx \gamma_j(q) : 2 \leq j \leq n\} \quad \bigcup \quad \{\delta_j(p) \approx \delta_j(q) : 2 \leq j \leq n\}$.

vi) An identity $p \approx q$ is called rectangular if $h(p) = h(q)$ and $t(p) = t(q)$; it is called initial (dually final) if $i(p) = i(q)$ (dually $f(p) = f(q)$).

Theorem 3.3 *([Ad]) i) Any heterotypical and rectangular identity defines the variety RB; any heterotypical and non-rectangular identity defines the trivial variety TR.*
ii) Any homotypical identity holds in SL; if it is non-rectangular then it defines SL, otherwise it also holds in NB.
iii) A homotypical, rectangular identity which is either not initial or not final defines NB.
iv) Let $n \geq 3$ and let $V_n = V(u_n \approx v_n)$. Let $r \approx s$ be an initial and final identity. Then $r \approx s$ is satisfied by V_n iff $I(r,s)$ holds in V_{n-2}.

4 Normal Form and StB-Proper Hypersubstitutions

Our first goal now is to describe a normal form for hypersubstitutions in $Hyp(2,1)$, modulo the relation $\sim_V$ induced by $V = StB$. We note first that any hypersubstitution σ in $Hyp(2,1)$ is completely determined by the two images $\sigma(f)$ and $\sigma(st)$. We shall sometimes denote a hypersubstitution σ as $\sigma_{u,v}$, to mean that $\sigma(f) = u$ and $\sigma(st) = v$.

To describe normal form hypersubstitutions for the variety StB, we make use of Płonka's relation $\sim_V$. However, $\sim_V$ requires that we consider all the operation symbols of the given type simultaneously. In our case, where we have two operation symbols to deal with in our type, it is more convenient to consider the two symbols separately. To this end, we introduce two related relations on the set of hypersubstitutions. Since Płonka's $\sim_V$ has not been used on types with more than one symbol before, we present our definitions for arbitrary type τ.

Definition 4.1 Let τ be a fixed type, with fundamental operation symbols f_i for $i \in I$. Let j be a fixed element of the index set I. We define relations R_j and $\sim_{j,V}$ on $Hyp(\tau)$ by setting

$\sigma_1 \; R_j \; \sigma_2$ iff $\sigma_1(f_j) \approx \sigma_2(f_j)$ is an identity of V and $\sigma_1(f_i) = \sigma_2(f_i)$ for all $i \neq j$;

and $\sigma_1 \sim_{j,V} \sigma_2$ iff $\sigma_1(f_j) \approx \sigma_2(f_j)$ is an identity of V.

In the special case where τ has only one operation symbol these new relations coincide with $\sim_V$, but otherwise they are different. We have $R_j \subseteq \sim_V \subseteq \sim_{j,V}$. It is easy to verify that the relations R_j have the Płonka properties from Lemma 2.2, making them useful in calculations of normal forms. The larger relation $\sim_{j,V}$ does not have the Płonka properties, but these relations are still useful since the intersection over all $j \in I$ of the $\sim_{j,V}$ is exactly $\sim_V$. Together, these relations mean that to describe $\sim_V$ for a particular type and variety V, we can proceed by examining one image $\sigma(f_j)$ at a time.

We illustrate these ideas with our variety StB. For clarity we shall refer to the relations R_j as R_f and R_{st}, and similarly for $\sim_{j,V}$. We look first at the second operation symbol of our type, the unary operation st. Any hypersubstitution σ must map st to some unary term v. It is clear from the identities for StB, especially the idempotent and absorption laws, that any such unary term v is equivalent to one of the four choices x, x^*, xx^* and x^*x. This proves the following:

Lemma 4.2 *For $V = StB$, the relation $\sim_{st,V}$ has exactly four equivalence classes; and for each choice of a binary term for $\sigma(f)$, there are four equivalence classes in R_{st}.*

Thus we think of partitioning $Hyp(2,1)$ thus far into four classes, each corresponding to one of the four choices available for $\sigma(st)$. Next we consider further partitioning according to the behaviour of $\sigma(f)$.

Lemma 4.3 *For $V = StB$, the relation $\sim_{f,V}$ has 264 equivalence classes.*

Proof. Any hypersubstitution σ must map f to a binary term of type $(2,1)$, and it follows from our definition of $\sim_{f,V}$ that we are essentially looking for equivalence classes of binary terms modulo the set of identities of StB. First, we note that among terms that use only one of the two possible content letters, there are 8 non-equivalent terms: x, x^*, xx^*, x^*x, y, y^*, yy^* and y^*y. Now consider any word w whose content contains both variable letters x and y. In order to use Theorem 3.3 in subsequent proofs, we shall be particularly interested in starts and ends of words, and longest left or right cuts using one letter only (since binary terms use only two variables). So we will consider for each word w what the left one-letter cut is, what the next letter is, and dually what the right one-letter cut is and what the preceding letter is. By the Content Lemma 3.1 and idempotence, we see that any such word w is equivalent to a word of the form $\alpha pq\beta$, where α and β are the longest one-letter left and right cuts, and p and q are single letters (possibly starred). That is, either $\alpha \in \{x, x^*, xx^*, x^*x\}$ and $p \in \{y, y^*\}$, or $\alpha \in \{y, y^*, yy^*$ and $y^*y\}$ and $p \in \{x, x^*\}$; and similarly for q and β. This shows that there are $8 \times 2 \times 2 \times 8 = 256$ non-equivalent binary terms using both variable letters. Together with the 8 terms of content size one, this gives our result. ∎

Corollary 4.4 *We have a normal form description of $Hyp(2,1)/\sim_{StB}$ using 1056 classes, based on 4 choices for $\sigma(st)$ and 264 choices for $\sigma(f)$, as described above.*

We now use our normal form result to show that the set $P(StB)$ of star-band-proper hypersubstitutions contains only two of these 1056 equivalence classes. As is customary, we do not usually distinguish between a normal form hypersubstitution and the equivalence class of $\sim_{StB}$ it represents.

Theorem 4.5 *The monoid $P(StB)$ of star-band-proper hypersubstitutions consists of two hypersubstitutions only, σ_{id} and $\sigma_{x*,yx}$.*

Proof. We need to investigate which of our normal form hypersubstitutions σ have the property that when applied to any identity $u \approx v$ of StB they always produce an identity $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ which holds in StB. It follows from [DR] that it suffices to look for σ which have this property for all of the five basis identities for StB. Thus we have the finite task of checking 1056 hypersubstitutions on five identities!

We proceed by a series of observations about the base identities which restrict the possibilities for a hypersubstitution σ to be in $P(StB)$. First, consider the identity (I2), $st(st(x)) \approx x$. Regardless of the value of $\sigma(f)$, if $\sigma(st) = xx^*$ or x^*x then application of $\hat{\sigma}$ to (I2) yields an identity which defines the variety SL. This means that any σ in $P(StB)$ must have $\sigma(st)$ equal to x or x^*.

Next we consider the identity (I3), $st(f(x, y)) \approx f(st(y), st(x))$. If $\sigma(st) = x$ and $\sigma(f)$ is a binary term w, application of $\hat{\sigma}$ yields the identity $w(x, y) \approx w(y, x)$. But for any binary term w this identity is non-rectangular, and holds at most in the varieties SL and TR. This means that for $\sigma \in P(StB)$, we must have $\sigma(st) = x^*$.

Next we consider (I3) again, this time with $\sigma(st) = x^*$ and $\sigma(f) = w$, for some binary term w. The identity that results when $\hat{\sigma}$ is applied to (I3) is $w(x, y)^* \approx w(y^*, x^*)$. If the term w uses only one of the input variables x and y, then our identity is non-rectangular and heterotypical, and hence holds only in the trivial variety. Thus w must use both letters x and y. If w contains no occurrences of the star operator, then it must be one of the six terms x, y, xy, yx, xyx and yxy. Testing each of these individually shows that only two of them yield an identity which holds in StB. This gives us two possible elements of $P(StB)$: the identity element $\sigma_{id} = \sigma_{xy,x^*}$ and σ_{yx,x^*}. The identity element is always in $P(StB)$, and it is easily verified that σ_{yx,x^*} is also in $P(StB)$, since all five base identities for StB are self-dual.

Thus we now have two elements in $P(StB)$, and we know that for any other σ in $P(StB)$ we must have $\sigma(st) = x^*$ and $\sigma(f)$ equal to a word w which uses both letters x and y and has at least one occurrence of the star operator. Now consider the identity (I5), $f(x, x) \approx x$. If the first or last symbol to occur in w is starred, then the result of applying $\hat{\sigma}$ to (I5) is an identity of the form $x^*\alpha \approx x$ or $\alpha x^* \approx x$, for some word α with content $\{x\}$, and such an identity holds only in the varieties SL and TR. Thus we conclude that w cannot have a star on its first or last symbol.

We return to (I3) again, where the result of applying $\hat{\sigma}$ is an identity $w(x, y)^* \approx w(y^*, x^*)$. If w has its first and last symbol the same, we see that the resulting identity is not rectangular, and hence cannot hold in StB.

This reduces our possibilities for σ in $P(StB)$ to the following. We must have $\sigma(st) = x^*$ and $\sigma(f) = w$, where w is a binary term using both letters x and y, at least one occurrence of a star, and with first symbol x and last symbol y or vice versa. Together with our normal form for terms, this allows us to completely describe the remaining possibilities for $\sigma \in P(StB)$. In normal form, the term $w = \alpha pq\beta$ can have $\alpha \in \{x, xx^*\}$, $p \in \{y, y^*\}$, $q \in \{x, x^*\}$ and $\beta \in \{y, y^*\}$, or dually if w has first symbol y. This gives 32 possible terms to test. We now show that other than the two hypersubstitutions already found in $P(StB)$, none of these 32 are in $P(StB)$. We do this by showing that when any such $\hat{\sigma}$ is applied to the associative identity (I1), the result is an identity which does not hold in $St(B)$.

We will use the notation $L_w \approx R_w$ for the identity which results from (I1) when the hypersubstitution with $\sigma(f) = w$ is applied. It is clear that $L_w \approx R_w$ is always going to be homotypical and rectangular, and thus holds in at least the variety NB and its subvarieties. From Theorem 3.3, we know that if it is not initial or not final, then the identity does not hold in any higher varieties, in particular in StB. If it is both initial and final, we use Theorem 3.3 to test whether it holds in the next highest variety, $RegB$. Since $L_w \approx R_w$ has content $\{x, y, z\}$, we need to consider longest left and right cuts of content sizes one and two. It follows from Theorem 3.3 part (iv) that $L_w \approx R_w$ holds in $RegB$ if and only if the two cut-identities in $I(L_w, R_w)$ both hold in SL, and moreover that if these two cut-identities define SL then $L_w \approx R_w$ holds in no higher varieties than $RegB$.

We now claim that any of our 32 possibilities for w results in an identity $L_w \approx R_w$ for which the cut identities $I(L_w, R_w)$ define SL. This can be checked by calculating each identity and its cuts, in a case by case analysis.

Case 1: $w = xyxy$: here w is equivalent modulo StB to xy, already considered.

Case 2: $w = xyxy^*y$: Here
$L_w = xyzyz^*zxz^*zy^*z^*y^*yzyz^*z$ and $R_w = xyxy^*yzxyxy^*yz^*z$.
Hence the right two-variable longest cut-identity is non-rectangular, and holds only in SL and TR.

Case 3: $w = xyx^*y$: Here $L_w = xyzy^*zx^*yzy^*z$ and $R_w = xyx^*yzy^*xy^*x^*z$. Again the right two-variable cut-identity is non-rectangular and heterotypical, so defines the trivial variety.

Case 4: $w = xyx^*y^*y$: Here
$L_w = xyzy^*z^*zx^*z^*zyz^*y^*yzy^*z^*z$ and $R_w = xyx^*y^*yzy^*yxy^*x^*z^*z$.
Again the right two-variable longest cut-identity is heterotypical and non-rectangular.

Cases 5 to 8: $w = xy^*uy$, where u is one of x, xy^*, x^* or x^*y^*: In all of these, the left two-variable longest cut has content $\{x, z\}$ in L_w and $\{x, y\}$ in R_w, and the identities are non-rectangular in each case. This means that the cut-identities hold in at most SL, and $L_w \approx R_w$ holds in at most $RegB$.

Cases 9 to 12: $w = xx^*yuy$, where u is one of x, xy^*, x^* or x^*y^*: In all of these, the left two-variable longest cut is not rectangular.

Cases 13 to 16: $w = xx^*y^*uy$, where u is one of x, xy^*, x^* or x^*y^*: In all of these, the left two-variable longest cuts have different contents.

Cases 17 to 32: These are the duals of the first 16, where now w has y as its first symbol and x as its last. These cases are handled similarly. ∎

We have shown that the monoid $P(StB)$ is quite small, containing only two (equivalence classes of) hypersubstitutions. However, the monoid is not just the trivial submonoid. Denecke and Koppitz ([DK]) have called a variety unsolid if the only hypersubstitutions in $P(V)$ are those equivalent to the identity hypersubstitution, and completely unsolid when $P(V)$ consists only of σ_{id}. Our theorem thus shows that StB is neither unsolid nor completely unsolid.

In the proof of the previous theorem, and in Theorem 5.3 below, the variety SL of semilattices plays a special role. We conclude this section by finding $P(SL)$, the monoid of proper hypersubstitutions, for this variety.

Theorem 4.6 *$P(SL)$ is the monoid of hypersubstitutions σ for which $\sigma(f)$ is a binary term which uses both letters x and y.*

Proof. From Theorem 3.3 we know that any identity $u \approx v$ for which $c(u) = c(v)$ holds in SL. Thus (I2), (I4) and (I5) are clearly hyperidentities for SL. For (I1) also, any choice of term to use for the symbol f results in a homotypical identity which holds in SL, so (I1) is a hyperidentity. For (I3) however, we see that the result of applying $\hat{\sigma}$ is a homotypical identity if and only if $\sigma(f)$ is a word w for which $c(w)$ is $\{x, y\}$. ∎

5 Hyperidentities for Star-Band Varieties

In this section we show that none of the identities in the usual basis for the variety StB, consisting of the five identities from Section 1, is a hyperidentity for StB. We show that there are however some identities for StB which are hyperidentities.

Theorem 5.1 *None of the five identities in the defining basis for StB is a hyperidentity for StB.*

Proof. For each of the five identities (I1) to (I5), we exhibit a hypersubstitution σ for which the result of applying $\hat{\sigma}$ to the identity is an identity which does not hold in StB. For (I1), application of a hypersubstitution which maps f to the term xyx yields the identity $xyzyx \approx xyxzxyx$, which is known to define the subvariety $RegB$. For (I2), using xx^* for the operation st yields $xx^* \approx x$, which is equivalent to SL. Use of any terms such as x or y for f in (I3) lead to identities of the form $x \approx y$, which only hold in the trivial variety TR. Finally, using x^* for f in (I4) or (I5) leads to the identity $x \approx x^*$, which defines SL. ∎

Corollary 5.2 *The identities (I2), (I5) and (I4) hold as hyperidentities in the varieties SL and TR only; the identity (I3) holds as a hyperidentity only in the trivial variety TR; and the identity (I1) holds in NB and its subvarieties, but in no higher varieties.*

Proof. The claim for (I3) follows from Theorem 5.1. The claims for (I2), (I5) and (I4) follow from Theorems 4.6 and 5.1. For (I5), we saw in the proof of Theorem 4.5 that this identity holds in at least the variety NB; but case 3, using $w = xys^*y$ gives a hypersubstitution σ whose application to (I1) results in a non-final identity, showing that this identity cannot hold in any higher variety. ∎

Although none of the basis identities for StB are hyperidentities, there are some hyperidentities known for this variety. In [KW], Koppitz and Wismath showed that StB satisfies some iterative hyperidentities, of arity one and two. We summarize their results here, and show that they may be extended to arbitrary arities. Thus we now consider as possible hyperidentities a family of identities with one operation symbol G of some arity $n \geq 1$. In this case, hypersubstitution amounts to substitution of an n-ary term w of StB for the operation symbol G. As discussed in the previous section, we may reduce the possibilities for this term modulo the identities of StB; in particular, we may use associativity to write w as a "word" in the variables or letters x_1, …, x_n.

We define $G^2(x_1, \ldots, x_n) = G(G(x_1, \ldots, x_n), x_2, \ldots, x_n)$, and then inductively, $G^{k+1}(x_1, \ldots, x_n) = G(G^k(x_1, \ldots, x_n), x_2, \ldots, x_n)$, for $k \geq 2$. An n-ary iterative hyperidentity is a hyperidentity of the form $G^a(x_1, \ldots, x_n) \approx G^b(x_1, \ldots, x_n)$, for some $b > a \geq 1$. This concept of iterative hyperidentities was studied in [HMT]. The following theorem was proved for the special cases $n = 1$ and $n = 2$ in [KW].

Theorem 5.3 *(i) If $b > a \geq 1$ and a and b have opposite parity, then the hyperidentity $G^a(x_1, \ldots, x_n) \approx G^b(x_1, \ldots, x_n)$ holds only in SL and TR.*

(ii) If $b \geq 3$ with b odd, then $G(x_1, \ldots, x_n) \approx G^b(x_1, \ldots, x_n)$ holds only in SL and TR.

(iii) If $b \geq a \geq 2$ have the same parity, then the hyperidentity $G^a(x_1, \ldots, x_n) \approx G^b(x_1, \ldots, x_n)$ is satisfied in every variety of star-bands.

Proof. (i) Using x_1^* for G forces $x_1^* \approx x_1$, and hence SL. It is clear that the hyperidentity does hold in TR and SL.

(ii) It is easy to verify inductively that for b odd, the result of using $x_1^* x_2^* \ldots y^*$ for G in $G^b(x, y)$ is the word $x_n x_{n-1} \ldots x_2 x_1^* x_2^* \ldots x_n^*$. Thus a hypersubstitution which uses the term $x_1^* x_2^* \ldots y^*$ results in a non-rectangular identity which by Theorem 3.3 defines the variety SL.

(iii) Let w be any n-ary term. We will let w_k denote the result of hypersubstituting w for G in the term $G^k(x_1, \ldots, x_n)$, for $k \geq 1$. We must show that when $b > a \geq 2$ have the same parity, the identity $w_a \approx w_b$ holds in every variety of star-bands. Since this identity is clearly homotypical and contains (some of) the n variables x_1, …,x_n, we know from Theorem 3.3 that it suffices to consider the longest left and right cuts for each content size from 1 to $n-1$. We claim that these cuts are the same for each of w_a and w_b, which will establish the desired result. This is because the cuts for any w_k, for $k \geq 2$, depend on the cuts for $w = w_1$, in an inductive manner determined according to the following key observation. If $G^k(x_1, \ldots, x_n)$ has produced a word w_k, then we can obtain the word w_{k+1} from it by going through w_k and replacing every occurrence of x_1 or x_1^* by w_k or w_k^* respectively, while leaving each x_i or x_i^* alone, for $2 \leq i \leq n$. This means that while occurrences of the variable x_1 or x_1^* are significant in w_1, occurrences of the other variables are not particularly important. If w_1 contains no occurrences of x_1 or x_1^*, then clearly $w_a = w_b$ always holds. So we assume that w_1 contains at least one occurrence of x_1 or x_1^*, and we can write $w = \alpha p \beta q \gamma$, where $p, q \in \{x, x^*\}$, β is any (possibly empty) word with $c(\beta) \subseteq \{x_1, \ldots, x_n\}$, and α and γ are any (possibly empty) words with content a subset of $\{x_2, \ldots, x_n\}$. This gives us four cases to consider, depending on the choices of p and q.

Case 1: $w = w_1 = \alpha x_1 \beta x_1 \gamma$.
Note that this means that $\alpha w_1 \approx w_1$ and $w_1 \gamma \approx w_1$. Let us inductively define $\beta = \beta_1$ and β_{k+1} as the result of using w_k for the x_1-inputs and x_i as the x_i-input, for $2 \leq i \leq n$, in the subword β_k. Then we have $w_2 \approx \alpha w_1 \beta_2 w_1 \gamma \approx w_1 \beta_2 w_1$, and for each $k \geq 2$, $w_{k+1} \approx \alpha w_k \beta_{k+1} w_k \gamma$. Thus by induction, for all $k \geq 2$ we have w_k starting with $\alpha w_1 \approx w_1$ and ending with $w_1 \gamma \approx w_1$. In particular, this means that w_a and w_b have the same left and right cuts of each content size, since they are all the same as the cuts for w_1.

Case 2: $w = w_1 = \alpha x_1 \beta x_1^* \gamma$.
Then $w_2 \approx \alpha w_1 \beta_2 w_1^* \gamma$, and for each $k \geq 2$, $w_{k+1} \approx \alpha w_k \beta_{k+1} w_k^* \gamma$. By induction, we get that for $k \geq 2$, the word w_k starts with $\alpha w_1 \approx w_1$, and so has the same left cuts of each content size as w_1. This also means that for $k \geq 2$, w_k ends with $w_1^* \gamma$, and so has the same right cuts as $w_1^* \gamma$. In particular, w_a and w_b have the same right cuts.

Case 3: $w= w_1 = \alpha x_1^* \beta x_1 \gamma$.
This case is dual to Case 2, and is proved in a similar way.

Case 4: $w = w_1 = \alpha x_1^* \beta x_1^* \gamma$.
Then $w_1^* \approx \gamma^* x_1 \beta_1^* x_1 \alpha^*$, so that $\gamma^* w_1^* \approx w_1^* \approx w_1^* \alpha^*$. We have $w_2 \approx \alpha w_1^* \beta_2 w_1^* \gamma$, and for any $k \geq 1$, $w_{k+1} \approx \alpha w_k^* \beta_{k+1} w_k^* \gamma$, and
$w_{k+2} \approx \alpha w_{k+1}^* \beta_{k+2} w_{k+1}^* \gamma \approx \alpha \gamma^* w_k \beta_{k+1}^* w_k \alpha^* \beta_{k+2} \gamma^* w_k \beta_{k+1}^* w_k \alpha^* \gamma$.

Therefore by induction, for $k \geq 3$ and odd, the word w_k starts with $\alpha \gamma^* w_1$ and ends with $w_1 \alpha^* \gamma$. Thus if $b \geq a$ are both odd, w_a and w_b have the same left and right cuts of each content size. Similarly, for $k \geq 4$ and even, we have w_k starting with $\alpha \gamma^* \alpha w_1^*$ and ending with $w_1^* \gamma \alpha^* \gamma$, and $w_a \approx w_b$ holds. Finally, for the special case $a = 2$ and $b \geq 2$ is even, we note that t_2 also starts with $\alpha \gamma^* \alpha w_1^*$, since $\alpha w_1^* \approx \alpha \gamma^* w_1^* \approx \alpha \gamma^* \alpha \gamma^* w_1^* \approx \alpha \gamma^* \alpha w_1^*$; and similarly for the ends. ∎

Although the iterative identities presented in Theorem 5.3 are all iterated or nested on the left-most position, for the variable x_1, it is clear that we could do the iteration on any fixed variable x_i, for $1 \leq i \leq n$.

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