(QUASI-)SEMI-APPROACH UNIFORM CONVERGENCE SPACES

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ABSTRACT. To be able to consider convenient hulls of (q)**AUnif**, the category of (quasi-) approach uniform spaces and uniform contractions, one needs to have (in particular finally dense) topological universe extensions of these categories available. To this end, the categories (q)**SAUConv**, (q)**SAULim** and (q)**PsAULim** are introduced and are shown to be topological universes and appropriate generalizations (quantifications) of the categories (q)**SUConv**, (q)**SULim** and (q)**PsULim** introduced earlier by Preuß and Behling ([2]) (as extensions of (q)**Unif**, the category of (quasi-)uniform spaces and uniformly continuous maps), in the same way that (q)**AUnif** generalizes (q)**Unif**. By also describing the final hulls of (q)**AUnif** in the previously mentioned topological universes, some finally dense topological universe extensions are eventually obtained.

1. INTRODUCTION.

On the one hand, in [5], Cook and Fisher introduced the category of uniform convergence spaces (and uniformly continuous maps) as a generalization of the category **Unif** of uniform spaces (and uniformly continuous maps) in the sense of Weil [27]. This category, when modified as suggested by Wyler [29] and then denoted **ULim**, turned out to be a cartesian closed topological category (Lee [11]), but not an extensional (= hereditary) topological category (cf. Behling [2]). In [2], [23] and [24], Behling and Preuß further extended this category to obtain the topological universes **SULim** and **SUConv** of respectively semi-uniform limit spaces and semi-uniform convergence spaces (and corresponding quasi-versions by dropping a symmetry requirement), where a topological universe (= topological quasitopos) is a cartesian closed and extensional topological construct.

On the other hand, in [14], Lowen and Windels introduced the category **AUnif** of approach uniform spaces (and uniform contractions) to contain both uniform and metric spaces and thereby allowing a quantified view on uniform properties ([15]) and completion ([16]). Also, an "approach" extension **AUCS** (also denoted **AULim** in the following) of **ULim** was considered (see e.g. [28]) which, unlike **ULim**, is not cartesian closed topological.

In this paper, the categories (q)**SAULim** and (q)**SAUConv** of respectively (quasi-) semi-approach uniform limit spaces and (quasi-)semi-approach uniform convergence spaces are introduced as "approach" extensions of (q)**SULim** and (q)**SUConv** and are shown to be topological universes (and therefore convenient extensions of (q)**AUnif**) such that a non-quantified construct is bireflectively bicoreflectively embedded in its "approach" extension (among other various relations shown to hold).

Next, some particular attention is given to an example lifted from [17], that is, the category (q)**PsAULim** $\subset$ (q)**SAULim** of (quasi-)pseudo-approach uniform limit spaces (and uniform contractions), which is an "approach" extension of the category (q)**PsULim** $\subset$

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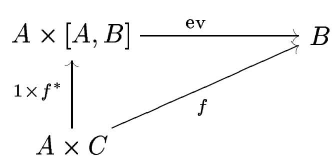
(q)**SULim** of (quasi-)pseudo-uniform limit spaces as described in [2], which are both topological universes determined by means of axioms similar to the one's of Choquet's pseudotopological spaces ([4]) and Lowen's pseudo-approach spaces ([13]).
Also obtained in this paper are the final hulls of (q)**AUnif** in the topological universes (q)**PsAULim**, (q)**SAULim** and (q)**SAUConv** (and as an extra, also in (q)**AULim**), which are most useful in describing smallest possible convenient extensions (= hulls) of (q)**AUnif**, such as a cartesian closed topological hull (see [19]) or a topological universe (= quasitopos) hull (which turns out to be the final hull of (q)**AUnif** in (q)**PsAULim**, see [20]). Similar results are also obtained in non-quantified versions, which is useful in describing the CCT hull of (q)**Unif** ([18]) and in obtaining an alternative, internal characterization of the topological universe hull of (q)**Unif** (which is analogously the final hull of (q)**Unif** in (q)**PsULim**, see also [20]).

2. Preliminaries.

Since topological categories (or constructs) will be considerably used, first note that a *topological* construct will stand for a concrete category over **Set** which is a *well-fibred topological c-construct* in the sense of [1], i.e. each structured source has an initial lift, every set carries only a set of structures and each constant map (or empty map) between two objects is a morphism.
A topological construct **A** is called *CCT (cartesian closed topological)* if **A** has *canonical function spaces*, i.e. for every pair (A, B) of **A**-objects the set $\hom(A, B)$ can be supplied with the structure of an **A**-object, denoted by $[A, B]$, such that

(a) the evaluation map $\mathrm{ev} : A \times [A, B] \longrightarrow B$ is an **A**-morphism,
(b) for each **A**-object C and **A**-morphism $f : A \times C \longrightarrow B$, the map $f^* : C \longrightarrow [A, B]$ defined by $f^*(c)(a) = f(a, c)$ is an **A**-morphism (f^* is called the *transpose* of f). Note that given $f : A \times C \longrightarrow B$, the transpose $f^* : C \longrightarrow [A, B]$ is the map which makes the following diagram commute:



A topological construct **A** is called *extensional* (or *hereditary*) if it has *representable partial morphisms* (to all **A**-objects), where

- a *partial morphism* from A to C is a morphism $f : B \longrightarrow C$, whose domain B is a subspace (= initial subobject) of A, and
- *partial morphisms to* C *are representable*, provided C can be embedded via the addition of a single point ∞_C into an **A**-object $C^\#$ (called *one-point extension* of C) such that for every partial morphism $f : B \longrightarrow C$, the map $f^A : A \longrightarrow C^\#$ defined by $f^A(x) = f(x)$ if $x \in |B|$, $f^A(x) = \infty_C$ if $x \in |A| \setminus |B|$, is an **A**-morphism.

In general, categorical concepts and terminology used in this paper (and possibly not recalled here), in particular regarding categorical topology can be found in [1] and [22]. Furthermore, a functor shall always be assumed to be concrete (unless this is clearly not the case from its definition) and subcategories to be full and isomorphism-closed.

Next, let us turn to introducing some notations and recalling some necessities regarding (approach) uniform spaces (and variations thereof). Given a set X, $\mathbf{F}(X)$ stands for the set of all filters on X; if $\mathcal{F} \in \mathbf{F}(X)$, then $\mathbf{U}(\mathcal{F})$ stands for the set of all ultrafilters on X finer than $\mathcal{F}$. In particular, $\mathbf{U}(X) := \mathbf{U}(\{X\})$ stands for the set of all ultrafilters on X. Given

$A \subset X$, we recall that $\operatorname{stack} A := \{B \subset X \mid A \subset B\}$ and if A consists of a single point a, we also denote $\dot{a} := \operatorname{stack} a := \operatorname{stack} A$.

If $\mathcal{F} \in \mathbf{F}(X^2)$, then $\mathcal{F}^{-1}$ denotes the filter generated by $\{F^{-1} \mid F \in \mathcal{F}\}$, where, given $F \subset X^2$, it holds that $F^{-1} := \{(y,x) \mid (x,y) \in F\}$. If $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2)$, then $\mathcal{F} \circ \mathcal{G}$ (the *composite* of $\mathcal{F}$ and $\mathcal{G}$) is defined to be the filter on X^2 generated by the filterbasis $\{F \circ G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$, where $F \circ G := \{(x,z) \in X^2 \mid \exists y \in X : (x,y) \in G \text{ and } (y,z) \in F\}$. Besides the "normal" product of sets, maps, filters, ... , we also define the following product of filters. If $\mathcal{F} \in \mathbf{F}(X^2)$ and $\mathcal{G} \in \mathbf{F}(Y^2)$, then $\mathcal{F} \otimes \mathcal{G}$ denotes the filter generated by $\{F \otimes G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$, where, given $F \subset X^2$ and $G \subset Y^2$, the set $F \otimes G$ is given by $F \otimes G := \{((x,y),(x',y')) \mid (x,x') \in F, (y,y') \in G\}$. Also, given a set X, Δ_X denotes the diagonal of X^2, that is, the set $\{(x,x) \mid x \in X\}$.

Given $F \subset X$, we let

$$\mathbf{S}_q(X,F) := \{\mathcal{F} \in \mathbf{F}(X^2) \mid \mathcal{F} \subset \operatorname{stack} \Delta_F \text{ and } F \times F \in \mathcal{F}\}$$
$$\text{and } \mathbf{S}(X,F) := \{\mathcal{F} \in \mathbf{S}_q(X,F) \mid \mathcal{F}^{-1} = \mathcal{F}\},$$

elements of which are called *quasi-semi-uniformities (on F)* and *semi-uniformities (on F)* respectively. Also let $\mathbf{S}_q(X) := \cup_{F \subset X} \mathbf{S}_q(X,F)$ and $\mathbf{S}(X) := \cup_{F \subset X} \mathbf{S}(X,F)$ denote the collection of quasi-semi-uniformities (in X) and semi-uniformities (in X) respectively, and observe that the set $F \subset X$ such that $\mathcal{F} \in \mathbf{S}_q(X,F)$ is uniquely determined by $\mathcal{F} \in \mathbf{S}_q(X)$, i.e. $\mathbf{S}_q(X,F) \cap \mathbf{S}_q(X,G) = \emptyset$ whenever $F \neq G$. Indeed, if $\mathcal{F} \in \mathbf{S}_q(X,F)$, $\mathcal{G} \in \mathbf{S}_q(X,G)$ and $\mathcal{F} \subset \mathcal{G}$, then it follows that $\Delta_G \subset F \times F$, hence $G \subset F$. Consequently, $\mathbf{S}_q(X,F) \cap \mathbf{S}_q(X,G) \neq \emptyset$ implies that $F = G$.

A *semi-uniform convergence space* is a pair $(X, \mathbb{L})$, where X is a set and $\mathbb{L}$ is a *semi-uniform convergence structure (on X)*, i.e. a set of filters on $X \times X$ such that the following conditions are satisfied:

(SUC_1) $\forall x \in X : \dot{x} \times \dot{x} \in \mathbb{L}$.

(SUC_2) $\forall \mathcal{F} \in \mathbb{L}, \forall \mathcal{G} \in \mathbf{F}(X^2) : \mathcal{F} \subset \mathcal{G} \Rightarrow \mathcal{G} \in \mathbb{L}$.

(SUC_3) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \mathcal{F} \in \mathbb{L} \Rightarrow \mathcal{F}^{-1} \in \mathbb{L}$.

A semi-uniform convergence space $(X, \mathbb{L})$ is called a *semi-uniform limit space* provided that the following is satisfied:

(SUL) $\forall \mathcal{F}, \mathcal{G} \in \mathbb{L} : \mathcal{F} \cap \mathcal{G} \in \mathbb{L}$.

A semi-uniform limit space $(X, \mathbb{L})$ is called a *pseudo-uniform limit space* provided that the following is satisfied:

(PsUL) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \mathcal{F} \in \mathbb{L} \iff \mathbf{U}(\mathcal{F}) \subset \mathbb{L}$.

A semi-uniform limit space $(X, \mathbb{L})$ is called a *uniform limit space* provided that the following is satisfied:

(UL) $\forall \mathcal{F}, \mathcal{G} \in \mathbb{L} : \mathcal{F} \circ \mathcal{G} \in \mathbb{L}$.

It is also possible to consider related concepts of the foregoing by omitting the symmetry-like axiom (SUC_3), which shall be indicated by using the prefix *quasi* (and observe that leaving out the triangular inequality-like axiom (UL) is indicated by the prefix *semi*).

A map $f : (X, \mathbb{L}_X) \longrightarrow (Y, \mathbb{L}_Y)$ between semi-uniform convergence spaces is said to be *uniformly continuous* provided that $\forall \mathcal{F} \in \mathbb{L}_X : (f \times f)(\mathcal{F}) \in \mathbb{L}_Y$.

Semi-uniform convergence spaces and uniformly continuous maps form the objects and morphisms of a construct, denoted by **SUConv**, and its full subconstructs of semi-uniform limit spaces, pseudo-uniform limit spaces and uniform limit spaces is denoted by **SULim**, **PsULim** and **ULim** respectively, wheras the quasi-variants are denoted by q**SUConv**, q**SULim**, q**PsULim** and q**ULim** respectively.

The category $(q)(s)\mathbf{Unif}$ of (quasi-)(semi-)uniform spaces and uniformly continuous maps (in the sense of Weil [27], Császár [6] and Čech [3] (see also Fletcher and Lindgren [7] and Künzi [9], [10]) can be nicely embedded into the category $(q)\mathbf{SULim}$, as $(q)(s)\mathbf{Unif}$ is isomorphic to the full subcategory of $(q)\mathbf{SULim}$ whose objects consist of all principal (quasi-)(semi-)uniform limit spaces, where a (quasi-)(semi-)uniform limit space $(X, \mathbb{L})$ is called a *principal* provided that it satisfies

(PrSUL) For any family $(\mathcal{F}_j)_{j\in J} \in \prod_{j\in J} \mathbb{L} : \bigcap_{j\in J} \mathcal{F}_j \in \mathbb{L}$.

Indeed, observe that a (quasi-)semi-uniform limit space $(X, \mathbb{L})$ is principal if and only if there exists a (quasi-)semi-uniformity $\mathcal{U}$ on X such that $\mathbb{L} = \{\mathcal{F} \in \mathbf{F}(X^2) \mid \mathcal{U} \subset \mathcal{F}\}$. Furthermore, the elements of $\mathcal{U}$ are called *entourages* and we shall also let $(X, \mathcal{U})$ refer to $(X, \mathbb{L})$, which satisfies (UL) if and only if $\mathcal{U}$ is even a (quasi-)uniformity, meaning that $\mathcal{U}$ additionally satisfies $\forall U \in \mathcal{U}, \exists V \in \mathcal{U} : V \circ V \subset U$.

Using the previous identifications of (concretely) isomorphic constructs, the following properties hold (see e.g. Lee [11, 12], Behling [2] and Preuß [23, 24]).

2.1. Proposition. *$(q)\mathbf{SUConv}$ is a topological universe.*
Moreover, given a source $(f_i : X \longrightarrow (X_i, \mathbb{L}_i))_{i\in I}$, one obtains the initial lift $\mathbb{L}_X$ by

$$\mathbb{L}_X := \{\mathcal{F} \in \mathbf{F}(X^2) \mid \forall i \in I : (f_i \times f_i)(\mathcal{F}) \in \mathbb{L}_i\}.$$

Also, given $(X, \mathbb{L}_X), (Y, \mathbb{L}_Y) \in (q)\mathbf{SUConv}$, the function space $[(X, \mathbb{L}_X), (Y, \mathbb{L}_Y)]$ (in $(q)\mathbf{SUConv}$) is given by $(\mathrm{hom}((X, \mathbb{L}_X), (Y, \mathbb{L}_Y)), \mathbb{L})$, where

$$\mathbb{L} := \{\Psi \in \mathbf{F}(\mathrm{hom}((X, \mathbb{L}_X), (Y, \mathbb{L}_Y))^2) \mid \forall \mathcal{F} \in \mathbb{L}_X : \Psi(\mathcal{F}) \in \mathbb{L}_Y\}$$

(where $\Psi(\mathcal{F}) := (\mathrm{ev} \times \mathrm{ev})(\mathcal{F} \times \Psi)$ and $\mathrm{ev} : X \times \mathrm{hom}((X, \mathbb{L}_X), (Y, \mathbb{L}_Y)) \longrightarrow Y$).
Next, let $(X, \mathbb{L}_X) \in (q)\mathbf{SUConv}$, then the $(q)\mathbf{SUConv}$-one point extension $(Z, \mathbb{L}_X^{\#}) := (X, \mathbb{L}_X)^{\#}$ is given by $Z := X^{\#} := X \cup \{\infty_X\}$ and

$$\begin{aligned}\mathbb{L}_X^{\#} := \{\mathcal{F} \in \mathbf{F}(Z^2) \mid &(\mathcal{F} \text{ has no trace on } X^2) \\ &\text{or } (\mathcal{F} \text{ has a trace on } X^2 \text{ and } \mathcal{F}_{|X^2} \in \mathbb{L}_X)\}.\end{aligned}$$

The following relations hold (where r (c) : $\mathbf{A} \longrightarrow \mathbf{B}$ means that $\mathbf{A}$ is a bi(co)reflective subconstruct of $\mathbf{B}$):

$$\begin{array}{ccccccccc}
q\mathbf{Unif} & \xrightarrow{r} & qs\mathbf{Unif} & \xrightarrow{r} & q\mathbf{PsULim} & \xrightarrow{r} & q\mathbf{SULim} & \xrightarrow{r} & q\mathbf{SUConv} \\
r\uparrow c & & r\uparrow c & & r\uparrow c & & r\uparrow c & & r\uparrow c \\
\mathbf{Unif} & \xrightarrow{r} & s\mathbf{Unif} & \xrightarrow{r} & \mathbf{PsULim} & \xrightarrow{r} & \mathbf{SULim} & \xrightarrow{r} & \mathbf{SUConv}
\end{array}$$

In particular, all indicated constructs are topological constructs, and furthermore, $(q)\mathbf{SULim}$ and $(q)\mathbf{PsULim}$ are closed under formation of initial lifts, function spaces and one-point extensions in $(q)\mathbf{SUConv}$, hence, $(q)\mathbf{SULim}$ and $(q)\mathbf{PsULim}$ are topological universes.
■

An *approach uniform space* is a pair $(X, (\mathcal{U}_\epsilon)_{\epsilon\in\mathbb{R}^+})$, where $(\mathcal{U}_\epsilon)_{\epsilon\in\mathbb{R}^+}$ is a *uniform tower* on X, meaning a family of filters $(\mathcal{U}_\epsilon)_{\epsilon\in\mathbb{R}^+}$ on $X \times X$ such that

(UT1) $\forall \epsilon \in \mathbb{R}^+, \forall U \in \mathcal{U}_\epsilon : \Delta_X \subset U$.

(UT2) $\forall \epsilon \in \mathbb{R}^+, \forall U \in \mathcal{U}_\epsilon : U^{-1} \in \mathcal{U}_\epsilon$.

(UT3) $\forall \epsilon, \epsilon' \in \mathbb{R}^+ : \mathcal{U}_\epsilon \circ \mathcal{U}_{\epsilon'} \supset \mathcal{U}_{\epsilon+\epsilon'}$.

(UT4) $\forall \epsilon \in \mathbb{R}^+ : \mathcal{U}_\epsilon = \bigcup_{\alpha>\epsilon} \mathcal{U}_\alpha$.

Thus, a uniform tower is a stack of semi-uniformities satisfying (UT3) and (UT4).
A map $f : (X, (\mathcal{U}_\epsilon)_{\epsilon \in \mathbb{R}^+}) \longrightarrow (Y, (\mathcal{U}'_\epsilon)_{\epsilon \in \mathbb{R}^+})$ between approach uniform spaces is called a *uniform contraction* if it fulfills the property that

$$\forall \epsilon \in \mathbb{R}^+ : f : (X, \mathcal{U}_\epsilon) \longrightarrow (Y, \mathcal{U}'_\epsilon) \text{ is uniformly continuous (i.e. } \mathcal{U}'_\epsilon \subset (f \times f)(\mathcal{U}_\epsilon)).$$

The category **AUnif** of approach uniform spaces and uniform contractions is a topological construct and is extensively studied and described in [28].
As before, one also considers related categories in [28] by leaving out the triangular inequality-like axiom (UT3) (indicated by using the prefix *semi-*) and/or the symmetry-like axiom (UT2) (indicated by using the prefix *quasi-*), which leads to the categories qs**AUnif**, q**AUnif** and s**AUnif**.

Note: For convenience of the reader, it should also be noted that there is a diagram at the end of this paper which presents an overview of most of the categories to be introduced in the sequel.

3. Convenient extensions of (q)AUnif.

3.1. Definition. A map $\eta : \mathbf{F}(X^2) \longrightarrow [0, \infty]$ is called a *semi-approach uniform convergence structure (on X)* if it satisfies:

(SAUCS$_1$) $\forall x \in X : \eta(\dot{x} \times \dot{x}) = 0$.

(SAUCS$_2$) $\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2) : \mathcal{F} \subset \mathcal{G} \Rightarrow \eta(\mathcal{G}) \leq \eta(\mathcal{F})$.

(SAUCS$_3$) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \eta(\mathcal{F}^{-1}) = \eta(\mathcal{F})$.

The pair (X, η) is called a *semi-approach uniform convergence space.*
It is called a *semi-approach uniform limit space* if it additionally satisfies

(SAULS) $\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2) : \eta(\mathcal{F} \cap \mathcal{G}) \leq \eta(\mathcal{F}) \vee \eta(\mathcal{G})$.

A semi-approach uniform convergence space (X, η) is called a *pseudo-approach uniform limit space* provided it satisfies:

(PsAULS) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \eta(\mathcal{F}) \leq \sup\limits_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \eta(\mathcal{U})$.

A semi-approach uniform limit space (X, η) is called an *approach uniform limit space* if it additionally satisfies

(AULS) $\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2) : \eta(\mathcal{F} \circ \mathcal{G}) \leq \eta(\mathcal{F}) + \eta(\mathcal{G})$.

Further, a semi-approach uniform limit space (X, η) is called *principal* if it satisfies

(PrSAULS) For any family $(\mathcal{F}_j)_{j \in J} \in \prod\limits_{j \in J} \mathbf{F}(X^2) : \eta\left(\bigcap\limits_{j \in J} \mathcal{F}_j\right) \leq \sup\limits_{j \in J} \eta(\mathcal{F}_j)$.

Again, it is possible to consider related concepts with the symmetry-like property (SAUCS$_3$) left out (indicated by the prefix *quasi-*). Furthermore, any meaningful combination of the foregoing is also acceptable, such as for instance a *principal quasi-approach uniform limit space*. Also observe that because of (SAUCS$_2$), equivalent properties are obtained by replacing inequality in (SAULS), (PsAULS) and (PrSAULS) with equality.

It should be noted here that an approach uniform limit space was introduced in Windels [28] (and was called an *approach uniform convergence space*) to be able to express for every filter $\mathcal{F}$ on $X \times X$ *to what extent* it belongs to the structure, rather than just having a collection to which a filter either belongs or not. The change in terminology is intended to have a corresponding one to the one recalled earlier (in the preliminaries).

As in [28], equivalent descriptions of the previous structures can be considered.

3.2. Proposition. *Let $(\mathbb{L}_\epsilon)_{\epsilon\in\mathbb{R}^+}$ (also shortly denoted by $(\mathbb{L}_\epsilon)_\epsilon$ or even $\mathbb{L}$) be a (quasi-) semi-uniform convergence tower (on X), i.e. a family of (quasi-)semi-uniform convergence structures such that*

(UCT) $\mathbb{L}_\epsilon = \bigcap_{\alpha>\epsilon} \mathbb{L}_\epsilon$, *i.e.* $(1_X : (X,\mathbb{L}_\epsilon) \longrightarrow (X,\mathbb{L}_\alpha))_{\alpha>\epsilon}$ *is initial in* q**SUConv**,

then $\eta_{\mathbb{L}}(\mathcal{F}) := \min\{\alpha \in \mathbb{R}^+ \mid \mathcal{F} \in \mathbb{L}_\alpha\}$ defines a (quasi-)semi-approach uniform convergence structure $\eta_{\mathbb{L}}$ on X. Conversely, if η is a (quasi-)semi-approach uniform convergence structure on X, then $\mathbb{L}_{\eta,\epsilon} := \{\mathcal{F} \in \mathbf{F}(X^2) \mid \eta(\mathcal{F}) \le \epsilon\}$ ($\epsilon \in \mathbb{R}^+$) defines a (quasi-)semi-uniform convergence tower $(\mathbb{L}_{\eta,\epsilon})_{\epsilon\in\mathbb{R}^+}$.
Moreover, this establishes a one-one correspondence between (quasi-)semi-approach uniform convergence structures and (quasi-)semi-uniform convergence towers such that

- *η satisfies (SAULS) if and only if*
  (SULT) $\forall \epsilon \in \mathbb{R}^+ : \mathbb{L}_\epsilon$ *is a (quasi-)semi-uniform limit structure.*
- *η satisfies (PsAULS) if and only if*
  (PsULT) $\forall \epsilon \in \mathbb{R}^+ : \mathbb{L}_\epsilon$ *is a (quasi-)pseudo-uniform limit structure,*
- *η satisfies (PrSAULS) if and only if*
  (PrSULT) $\forall \epsilon \in \mathbb{R}^+ : \mathbb{L}_\epsilon$ *is a principal (quasi-)semi-uniform limit structure.*
- *η satisfies (AULS) if and only if*
  (ULT) $\forall \epsilon,\epsilon' \in \mathbb{R}^+, \forall \mathcal{F},\mathcal{G} \in \mathbf{F}(X^2) : \mathcal{F} \in \mathbb{L}_\epsilon$ *and* $\mathcal{G} \in \mathbb{L}_{\epsilon'} \Rightarrow \mathcal{F}\circ\mathcal{G} \in \mathbb{L}_{\epsilon+\epsilon'}$.

Proof. First of all, observe that it follows from (UCT) that the mentioned minimum definitely exists, and it is subsequently a straightforward verification to check that the defined operations indeed yield a (quasi-)semi-approach uniform convergence structure and a (quasi-)semi-uniform convergence tower respectively.
Let η be a (quasi-)semi-approach uniform convergence structure η, then it holds for every $\mathcal{F} \in \mathbf{F}(X^2)$ that

$$\eta_{\mathbb{L}_\eta}(\mathcal{F}) = \min\{\alpha \in \mathbb{R}^+ \mid \mathcal{F} \in \mathbb{L}_{\eta,\alpha}\} = \min\{\alpha \in \mathbb{R}^+ \mid \eta(\mathcal{F}) \le \alpha\} = \eta(\mathcal{F}).$$

On the other hand, let $(\mathbb{L}_\epsilon)_{\epsilon\in\mathbb{R}^+}$ be a tower, then it holds for every $\epsilon \in \mathbb{R}^+$ that

$$\begin{aligned}\mathbb{L}_{\eta_{\mathbb{L}},\epsilon} &= \{\mathcal{F} \in \mathbf{F}(X^2) \mid \eta_{\mathbb{L}}(\mathcal{F}) \le \epsilon\}\\ &= \{\mathcal{F} \in \mathbf{F}(X^2) \mid \min\{\alpha \in \mathbb{R}^+ \mid \mathcal{F} \in \mathbb{L}_\alpha\} \le \epsilon\} = \mathbb{L}_\epsilon.\end{aligned}$$

The remaining claims regarding the one-one correspondence of the appropriate properties are then easily verified. ■

Having considered several descriptions of (quasi-)semi-approach uniform convergence spaces (and variations thereof), it remains to discuss morphisms between them.

3.3. Proposition. *Let (X,η_X) and (Y,η_Y) be quasi-semi-approach uniform convergence spaces and let $f : X \longrightarrow Y$ be a map, then the following are equivalent.*

(1) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \eta_Y((f\times f)(\mathcal{F})) \le \eta_X(\mathcal{F})$.

(2) $\forall \epsilon \in \mathbb{R}^+ : f : (X,\mathbb{L}_{\eta_X,\epsilon}) \longrightarrow (Y,\mathbb{L}_{\eta_Y,\epsilon})$ *is uniformly continuous.*

Proof. This is an easy verification (cf. previous proposition). ■

3.4. Definition. A map $f : (X,\eta_X) \longrightarrow (Y,\eta_Y)$ between quasi-semi-approach uniform convergence spaces is called a a *uniform contraction* if it satisfies the foregoing equivalent conditions.
The category (construct) of (quasi-)semi-approach uniform convergence spaces and uniform contractions is denoted by (q)**SAUConv**, and its full subconstruct of (quasi-)semi-approach

uniform limit spaces, (quasi-)pseudo-approach uniform limit spaces and (quasi-)approach uniform limit spaces is denoted by (q)**SAULim**, (q)**PsAULim** and (q)**AULim** respectively. Also note that by (PsAULS) a pseudo-approach uniform limit is completely determined by its restriction to ultrafilters and that it is equivalent to define (q)**PsAULim** as the full subconstruct of (q)**SAUConv** whose objects (X, η) satisfy (PsAULS), since (PsAULS) implies (SAULS) (indeed, recall that $\mathbf{U}(\mathcal{F} \cap \mathcal{G}) = \mathbf{U}(\mathcal{F}) \cup \mathbf{U}(\mathcal{G})$ $(\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2))$.

The following result justifies that the full subconstruct of q**SAUConv** consisting of principal (quasi-)(semi-)approach uniform limit spaces is denoted by $(q)(s)$**AUnif**.

3.5. Proposition.

(1) Given a set X and a principal (quasi-)(semi-)uniform limit tower $(\mathbb{L}_\epsilon)_{\epsilon \in \mathbb{R}^+}$ on X, $(\mathcal{U}_{\mathbb{L},\epsilon})_{\epsilon \in \mathbb{R}^+}$ defined by

$$\mathcal{U}_{\mathbb{L},\epsilon} := \bigcap_{\mathcal{F} \in \mathbb{L}_\epsilon} \mathcal{F} \quad (\forall \epsilon \in \mathbb{R}^+)$$

is a (quasi-)(semi-)uniform tower on X, and vice versa, if $(\mathcal{U}_\epsilon)_{\epsilon \in \mathbb{R}^+}$ is a (quasi-)(semi-) uniform tower on X, then $(\mathbb{L}_{\mathcal{U},\epsilon})_{\epsilon \in \mathbb{R}^+}$ defined by

$$\mathbb{L}_{\mathcal{U},\epsilon} := \{\mathcal{F} \in \mathbf{F}(X^2) \mid \mathcal{U}_\epsilon \subset \mathcal{F}\} \quad (\forall \epsilon \in \mathbb{R}^+)$$

is a principal (quasi-)(semi-)uniform limit tower on X, such that

$$\mathbb{L}_{\mathcal{U}_{\mathbb{L}}} = \mathbb{L} \text{ and } \mathcal{U}_{\mathbb{L}_{\mathcal{U}}} = \mathcal{U}.$$

(2) If $(X, (\mathbb{L}^X_\epsilon)_{\epsilon \in \mathbb{R}^+})$ and $(Y, (\mathbb{L}^Y_\epsilon)_{\epsilon \in \mathbb{R}^+})$ are principal quasi-semi-approach uniform limit spaces, then the following are equivalent for a map $f : X \longrightarrow Y$:

(1) $f : (X, (\mathbb{L}^X_\epsilon)_{\epsilon \in \mathbb{R}^+}) \longrightarrow (Y, (\mathbb{L}^Y_\epsilon)_{\epsilon \in \mathbb{R}^+})$ is a uniform contraction.

(2) $f : (X, (\mathcal{U}_{\mathbb{L}^X,\epsilon})_{\epsilon \in \mathbb{R}^+}) \longrightarrow (Y, (\mathcal{U}_{\mathbb{L}^Y,\epsilon})_{\epsilon \in \mathbb{R}^+})$ is a uniform contraction.

Proof. Let $(\mathbb{L}_\epsilon)_\epsilon$ be a principal (quasi-)(semi-)uniform limit tower, then it is easily verified that (UT1) and (UT2) (if required) are satisfied. If $(\mathbb{L}_\epsilon)_{\epsilon \in \mathbb{R}^+}$ satisfies (ULT), then it holds for any $\epsilon, \epsilon' \in \mathbb{R}^+$ that $\mathcal{U}_{\mathbb{L},\epsilon} \circ \mathcal{U}_{\mathbb{L},\epsilon'} \in \mathbb{L}_{\epsilon+\epsilon'}$, hence $\mathcal{U}_{\mathbb{L},\epsilon+\epsilon'} \subset \mathcal{U}_{\mathbb{L},\epsilon} \circ \mathcal{U}_{\mathbb{L},\epsilon'}$ (by construction). Next, let $\epsilon \in \mathbb{R}^+$, then $\cup_{\alpha > \epsilon} \mathcal{U}_{\mathbb{L},\alpha} \in \cap_{\alpha > \epsilon} \mathbb{L}_\alpha = \mathbb{L}_\epsilon$, consequently $\mathcal{U}_{\mathbb{L},\epsilon} \subset \cup_{\alpha > \epsilon} \mathcal{U}_{\mathbb{L},\alpha}$. Since the reverse inclusion obviously holds as well, it has been shown that $(\mathcal{U}_{\mathbb{L},\epsilon})_\epsilon$ is a (quasi-)(semi-) uniform tower, whereas the straightforward verification of the remaining claims is left to the reader. ∎

Now it is time to establish the convenience of these extensions.

3.6. Proposition. *q**SAUConv** is a topological construct.*
More precisely,

(1) given a source $(f_i : X \longrightarrow (X_i, \eta_i))_{i \in I}$, one obtains the initial lift η (on X) by

$$\eta(\mathcal{F}) := \sup_{i \in I} (f_i \times f_i)(\mathcal{F}),$$

(2) given a source $(f_i : X \longrightarrow (X_i, (\mathbb{L}^i_\epsilon)_{\epsilon \in \mathbb{R}^+}))_{i \in I}$, the initial lift $(X, (\mathbb{L}_\epsilon)_{\epsilon \in \mathbb{R}^+})$ is obtained in a "per level" way, i.e. such that

$$\forall \epsilon \in \mathbb{R}^+ : (f_i : (X, \mathbb{L}_\epsilon) \longrightarrow (X_i, \mathbb{L}^i_\epsilon))_{i \in I} \text{ is initial (in } q\mathbf{SUConv}).$$

Proof. This is easily verified. ∎

3.7. Proposition. *q**SAUConv** is a cartesian closed topological construct.*
*More precisely, let $(X, \eta_X), (Y, \eta_Y) \in q$**SAUConv**, then the function space $(Z, \eta) := [(X, \eta_X), (Y, \eta_Y)]$ (in q**SAUConv**) is given by (where $\Psi \in \mathbf{F}(Z^2)$)*

$$\begin{aligned} \eta(\Psi) &:= \min\{\alpha \in [0, \infty] \mid \forall \mathcal{F} \in \mathbf{F}(X^2) : \eta_Y(\Psi(\mathcal{F})) \leq \eta_X(\mathcal{F}) \vee \alpha\} \\ &= \sup\{\eta_Y(\Psi(\mathcal{F})) \mid \mathcal{F} \in \mathbf{F}(X^2) \text{ and } \eta_Y(\Psi(\mathcal{F})) > \eta_X(\mathcal{F})\}. \end{aligned}$$

Proof. First of all, one easily finds that the indicated minimum exists and that both formulas are indeed equal. It is also evident that $\eta(\dot{f} \times \dot{f}) = 0$ (for any uniform contraction f) and that $\Psi \subset \Phi$ implies that $\eta(\Phi) \leq \eta(\Psi)$ $(\Phi, \Psi \in \mathbf{F}(Z^2))$, hence η is a quasi-semi-approach uniform convergence structure.
Next, it is needed that $\mathrm{ev} : (X, \eta_X) \times (Z, \eta) \longrightarrow (Y, \eta_Y)$ is a uniform contraction. To this end, let $\mathcal{F} \in \mathbf{F}((X \times Z)^2)$ and $\mathcal{G} := (\mathrm{pr}_X \times \mathrm{pr}_X)(\mathcal{F})$ and $\Psi := (\mathrm{pr}_Z \times \mathrm{pr}_Z)(\mathcal{F})$. As $\mathcal{G} \otimes \Psi \subset \mathcal{F}$, it follows that $\eta_Y((\mathrm{ev} \times \mathrm{ev})(\mathcal{F})) \leq \eta_Y((\mathrm{ev} \times \mathrm{ev})(\mathcal{G} \otimes \Psi)) = \eta_Y(\Psi(\mathcal{G})) \leq \eta_X(\mathcal{G}) \vee \eta(\Psi) = (\eta_X \times \eta)(\mathcal{F})$ (where the last inequality holds by the first formula of η).
Further, let $f : (X, \eta_X) \times (W, \eta_W) \longrightarrow (Y, \eta_Y)$ be a uniform contraction. To show the uniform contractivity of $f^* : (W, \eta_W) \longrightarrow (Z, \eta_Z)$, let $\mathcal{G} \in \mathbf{F}(W^2)$. For any $\mathcal{F} \in \mathbf{F}(X^2)$, observe that $\eta_Y(((f^* \times f^*)(\mathcal{G}))(\mathcal{F})) = \eta_Y((f \times f)(\mathcal{F} \otimes \mathcal{G})) \leq (\eta_X \times \eta_W)(\mathcal{F} \otimes \mathcal{G}) = \eta_X(\mathcal{F}) \vee \eta_W(\mathcal{G})$, consequently, by definition of η, $\eta((f^* \times f^*)(\mathcal{G})) \leq \eta_W(\mathcal{G})$. ■

3.8. Proposition. *q**SAUConv** is an extensional topological construct.*
*More precisely, let $(X, \eta) \in q\mathbf{SAUConv}$, then the one-point extension $(Z, \eta^\#) := (X, \eta)^\#$ (in q**SAUConv**) is given by $Z := X^\# := X \cup \{\infty_X\}$ and*

$$\forall \mathcal{F} \in \mathbf{F}(Z^2) : \eta^\#(\mathcal{F}) := \begin{cases} 0 & \text{if } \mathcal{F} \text{ has no trace on } X^2 \\ \eta\left(\mathcal{F}_{|X^2}\right) & \text{otherwise.} \end{cases}$$

Proof. Evidently, $(Z, \eta^\#)$ is a quasi-semi-approach uniform convergence space having (X, η) as a subspace (see proposition 3.6).
Furthermore, let $f : (Y', \eta') \longrightarrow (X, \eta)$ be a partial morphism from (Y, η_Y). To show that $f^Y : (Y, \eta_Y) \longrightarrow (Z, \eta^\#)$ is a uniform contraction, let $\mathcal{F} \in \mathbf{F}(Y^2)$. Either $\mathcal{F}$ has a trace on $(Y')^2$, hence $(f^Y \times f^Y)(\mathcal{F})$ has a trace on X^2, and since $(f^Y \times f^Y)(\mathcal{F})_{|X^2} = (f \times f)\left(\mathcal{F}_{|(Y')^2}\right)$, it follows that

$$\begin{aligned} \eta^\#((f^Y \times f^Y)(\mathcal{F})) &= \eta\left((f^Y \times f^Y)(\mathcal{F})_{|X^2}\right) = \eta\left((f \times f)\left(\mathcal{F}_{|(Y')^2}\right)\right) \\ &\leq \eta'\left(\mathcal{F}_{|(Y')2}\right) = \eta_Y\left(\mathcal{F}_{|(Y')^2}\right) \leq \eta_Y(\mathcal{F}). \end{aligned}$$

In case $\mathcal{F}$ has no trace on $(Y')^2$, it follows that $(f^Y \times f^Y)(\mathcal{F}) = \dot{\infty}_X \times \dot{\infty}_X$, hence $\eta^\#((f^Y \times f^Y)(\mathcal{F})) = 0 \leq \eta(\mathcal{F})$. ■

It remains to properly indicate the relation to "classical" constructs before presenting an overview.

3.9. Proposition. *q**SUConv** is concretely isomorphic to the subconstruct of q**SAUConv** whose objects (X, η) satisfy*

(SUC) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \eta(\mathcal{F}) \in \{0, \infty\}$.

Proof. It is easily verified that

$$q\mathbf{SUConv} \longrightarrow q\mathbf{SAUConv} : (X, \mathbb{L}) \mapsto (X, (\mathbb{L})_{\epsilon \in \mathbb{R}^+}) \quad \text{(meaning: } \mathbb{L} \text{ on every level)}$$

is a concrete full embedding onto the desired subconstruct of q**SAUConv** (note that the corresponding quasi-semi-approach uniform convergence structure is given by

$$\eta_{\mathbb{L}} : \mathbf{F}(X^2) \longrightarrow [0, \infty] : \mathcal{F} \mapsto \begin{cases} 0 & \text{if } \mathcal{F} \in \mathbb{L} \\ \infty & \text{otherwise}). \end{cases} \quad ■$$

3.10. Proposition. *q**SUConv** is bicoreflective in q**SAUConv** and*

$$\mathcal{C}_0 : q\mathbf{SAUConv} \longrightarrow q\mathbf{SUConv} : (X, \eta) = (X, (\mathbb{L}_\epsilon)_{\epsilon \in \mathbb{R}^+}) \mapsto (X, \eta_0) = (X, \mathbb{L}_0)$$

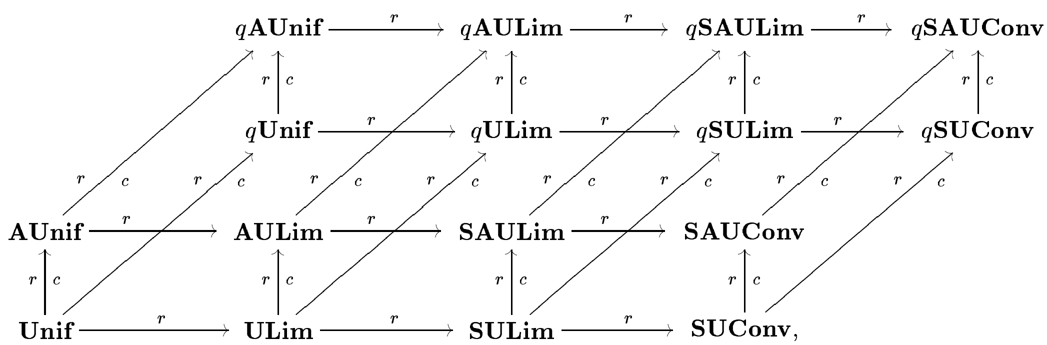
is the bicoreflection.

Proof. Observing first (using proposition 3.2) that

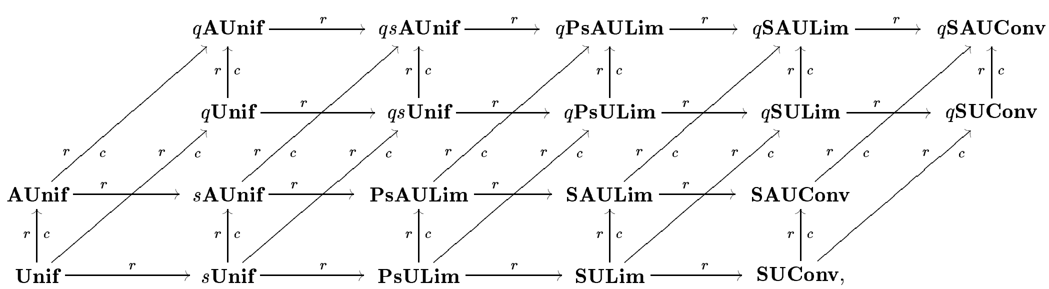
$$\eta_0 : \mathbf{F}(X^2) \longrightarrow [0,\infty] : \mathcal{F} \mapsto \begin{cases} 0 & \text{if } \eta(\mathcal{F}) = 0 \\ \infty & \text{otherwise,} \end{cases}$$

it follows that $(X, \eta_0) \in q\mathbf{SUConv}$ and $1_X : (X, \eta_0) \longrightarrow (X, \eta)$ is a uniform contraction. Furthermore, let $(Y, \eta_Y) \in q\mathbf{SUConv}$ and let $f : (Y, \eta_Y) \longrightarrow (X, \eta)$ be a uniform contraction. To show the uniform contractivity of $f : (Y, \eta_Y) \longrightarrow (X, \eta_0)$, let $\mathcal{F} \in \mathbf{F}(Y^2)$. Either $\eta_Y(\mathcal{F}) = \infty$ and then clearly, $\eta_0((f \times f)(\mathcal{F})) \leq \eta_Y(\mathcal{F})$. Otherwise $\eta_Y(\mathcal{F}) = 0$ (since $(Y, \eta_Y) \in q\mathbf{SUConv}$), hence $\eta((f \times f)(\mathcal{F})) \leq \eta_Y(\mathcal{F}) = 0$, consequently also $\eta_0((f \times f)(\mathcal{F})) = 0$ (by construction of η_0). ∎

3.11. Proposition. *The following relations hold:*



and



where all constructs on the bottom level are fully embedded in the top level by means of the functor

$$q\mathbf{SUConv} \longrightarrow q\mathbf{SAUConv} : (X, \mathbb{L}) \mapsto (X, (\mathbb{L})_\epsilon) \text{ (meaning: } \mathbb{L} \text{ on every level)}$$

such that every lower level construct is the restriction of the corresponding top level one to $q\mathbf{SUConv}$.
Also, all vertical bicoreflectors from the top level to the bottom level are restrictions of the bicoreflector

$$\mathfrak{C}_0 : q\mathbf{SAUConv} \longrightarrow q\mathbf{SUConv} : (X, (\mathbb{L}_\epsilon)_\epsilon) \mapsto (X, \mathbb{L}_0)$$

and all diagonal bicoreflectors from the back wall to the forward wall are restrictions of the bicoreflector

$$\mathfrak{C}_s : q\mathbf{SAUConv} \longrightarrow \mathbf{SAUConv} : (X, \eta) \mapsto (X, \eta_s)$$
$$\text{where } \forall \mathcal{F} \in \mathbf{F}(X) : \eta_s(\mathcal{F}) = \eta(\mathcal{F}) \vee \eta(\mathcal{F}^{-1}).$$

Regarding convenience, the three right most diagonal rectangles of the latter diagram consist of topological universes that are closed under formation of function spaces and one-point extensions in q**SAUConv**.

Proof. The claims regarding the embeddings, restrictions and $\mathcal{C}_0$-bicoreflector are easily seen to follow from the foregoing two propositions and proposition 3.2. The vertical bireflectivenesses follow from the easy verification that the property (SUC) is stable under formation of initial lifts (described in proposition 3.6) and from the horizontal bireflectivenesses, which can be argued by means of initial closedness. This follows at once for (q)**SAULim** from the description of initial lifts (in (q)**SAUConv**) given in proposition 3.6, which also coincides with initial lifts in (q)**AULim** as given in [28] and with a "per level (tower)" construction of initial lifts in $(q)(s)$**AUnif** also given in [28]. The bireflectiveness of (q)**PsAULim** in (q)**SAULim** has been shown in [17] in the symmetric case, and the same argumentations work for the quasi case.

The diagonal bireflectivenesses also follow directly from the description of initial lifts given in proposition 3.6, just as the descriptions given in propositions 3.7 and 3.8 show that **SAUConv** is closed under function spaces and one-point extensions in q**SAUConv** (observe for instance that $\forall \Psi \in \mathbf{F}(Z^2), \forall \mathcal{F} \in \mathbf{F}(X^2) : \Psi^{-1}(\mathcal{F}) = (\Psi(\mathcal{F}^{-1}))^{-1}$ and $\forall \mathcal{F} \in \mathbf{F}((X^{\#})^2)$: $\mathcal{F}$ has a trace on $X^2 \Leftrightarrow \mathcal{F}^{-1}$ has a trace on X^2) and so is (q)**SAULim** in (q)**SAUConv**. This claim regarding function spaces in case of (q)**PsAULim** has also been shown in [17] in the symmetric case, and again, analogy holds for the quasi case. As for one-point extensions in this case, let $(X, \eta) \in (q)$**PsAULim** and let $\mathcal{F} \in \mathbf{F}((X^{\#})^2)$. Either $\mathcal{F}$ has no trace on X^2 and then clearly, $\eta^{\#}(\mathcal{F}) \leq \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \eta^{\#}(\mathcal{U})$. Otherwise, it holds that $\mathcal{F} \subset \mathcal{F}_{|X^2}$, hence $\mathbf{U}(\mathcal{F}_{|X^2}) \subset \mathbf{U}(\mathcal{F})$, and for any $\mathcal{U} \in \mathbf{U}(\mathcal{F}_{|X^2})$, we have that $\mathcal{U}$ has a trace on X^2 and moreover $\mathcal{U}_{|X^2} = \mathcal{U}$, consequently

$$\eta^{\#}(\mathcal{F}) = \eta(\mathcal{F}_{|X^2}) = \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F}_{|X^2})} \eta(\mathcal{U}) = \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F}_{|X^2})} \eta^{\#}(\mathcal{U}) \leq \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \eta^{\#}(\mathcal{U}).$$

As (SAUC_2) implies the reverse inequality, it follows that $(X, \eta)^{\#} \in (q)$**PsAULim**.

As for the claims regarding the diagonal bicoreflectivenesses, η_s clearly yields a semi-approach uniform convergence structure such that $\eta_s \geq \eta$, hence $1_X : (X, \eta_s) \longrightarrow (X, \eta)$ is a uniform contraction.

Also, let $f : (Y, \eta_Y) \longrightarrow (X, \eta)$ be a uniform contraction where $(Y, \eta_Y) \in$ **SAUConv**. To show that also $f : (Y, \eta_Y) \longrightarrow (X, \eta_s)$ is a uniform contraction, let $\mathcal{F} \in \mathbf{F}(Y^2)$, then $\eta_s((f \times f)(\mathcal{F})) = \eta((f \times f)(\mathcal{F})) \vee \eta((f \times f)(\mathcal{F})^{-1}) \leq \eta_Y(\mathcal{F}) \vee \eta_Y(\mathcal{F}^{-1}) = \eta_Y(\mathcal{F})$.

It is easily seen that the bicoreflector $\mathcal{C}_s$ preserves the properties (SUC), (SAULS), (PrSAULS), (PsAULS) and (AULS). Indeed, regarding (PsAULS), assume that $(X, \eta) \in q$**SAULim** satisfies (PsAULS), then it follows for $\mathcal{F} \in \mathbf{F}(X^2)$ that

$$\begin{aligned} \eta_s(\mathcal{F}) &= \eta(\mathcal{F}) \vee \eta(\mathcal{F}^{-1}) = \left(\sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \eta(\mathcal{U})\right) \vee \left(\sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F}^{-1})} \eta(\mathcal{U})\right) \\ &= \left(\sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \eta(\mathcal{U})\right) \vee \left(\sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \eta(\mathcal{U}^{-1})\right) = \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \left(\eta(\mathcal{U}) \vee \eta(\mathcal{U}^{-1})\right) \\ &= \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \eta_s(\mathcal{U}), \end{aligned}$$

hence (X, η_s) also satisfies (PsAULS). Lastly, let $(X, \eta) \in q$**SAULim** satisfy (AULS), then it follows for $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2)$ that

$$\begin{aligned} \eta_s(\mathcal{F} \circ \mathcal{G}) &= \eta(\mathcal{F} \circ \mathcal{G}) \vee \eta(\mathcal{G}^{-1} \circ \mathcal{F}^{-1}) \\ &\leq (\eta(\mathcal{F}) + \eta(\mathcal{G})) \vee (\eta(\mathcal{F}^{-1}) + \eta(\mathcal{G}^{-1})) \end{aligned}$$

$$\begin{aligned} &\leq (\eta(\mathcal{F}) \vee \eta(\mathcal{F}^{-1})) + (\eta(\mathcal{G}) \vee \eta(\mathcal{G}^{-1})) \\ &= \eta_s(\mathcal{F}) + \eta_s(\mathcal{G}), \end{aligned}$$

hence (X, η_s) also satisfies (AULS). ∎

4. Final hulls of (q)**(A)Unif**.

4.1. Definition. Define **qsaug-qSAUConv**, also shortly denoted by **g-qSAUConv**, to be the full subconstruct of q**SAUConv** whose objects (X, η) are *quasi-semi-approach uniformly generated*, i.e. satisfy

(qsaug) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \eta(\mathcal{F}) = \inf_{\substack{\mathcal{H} \in \mathbf{S}_q(X), \\ \mathcal{H} \subset \mathcal{F}}} \eta(\mathcal{H})$.

(saug_Δ) $\forall H \subset X : \eta(\text{stack}\,\Delta_H) < \infty \Rightarrow \eta(\text{stack}\,\Delta_H) = 0$.

4.2. Proposition. **g-qSAUConv** *is bicoreflective in* q**SAUConv** *and*

$$\mathcal{C}_g : q\mathbf{SAUConv} \longrightarrow \mathbf{g}\text{-}q\mathbf{SAUConv} : (X, \eta) \mapsto (X, \eta_g)$$

$$\textit{where } \eta_g(\mathcal{F}) := \inf_{\substack{\mathcal{H} \in \mathbf{S}_q(X,\eta), \\ \mathcal{H} \subset \mathcal{F}}} \eta(\mathcal{H}) \quad (\forall \mathcal{F} \in \mathbf{F}(X^2)),$$

$$\mathbf{S}_q(X, H, \eta) := \{\mathcal{H} \in \mathbf{S}_q(X, H) \mid \eta(\text{stack}\,\Delta_H) = 0\}$$

$$\textit{and } \mathbf{S}_q(X, \eta) := \bigcup_{H \subset X} \mathbf{S}_q(X, H, \eta),$$

is the corresponding bicoreflector.

Proof. Evidently, $(X, \eta_g) \in q\mathbf{SAUConv}$.
Furthermore, first observe that $\mathcal{G} \in \mathbf{S}_q(X, G)$, $\mathcal{H} \in \mathbf{S}_q(X, H, \eta)$ and $\mathcal{H} \subset \mathcal{G}$ implies that also $\mathcal{G} \in \mathbf{S}_q(X, G, \eta)$. Indeed, recall that $\mathcal{H} \subset \mathcal{G}$ implies that $G \subset H$, hence $\text{stack}\,\Delta_H \subset \text{stack}\,\Delta_G$ and therefore $\eta(\text{stack}\,\Delta_G) \leq \eta(\text{stack}\,\Delta_H) = 0$. Now to show ($\text{saug}_\Delta$), let $G \subset X$ be such that $\eta_g(\text{stack}\,\Delta_G) < \infty$, consequently there exists $\mathcal{H} \in \mathbf{S}_q(X, \eta)$ such that $\mathcal{H} \subset \text{stack}\,\Delta_G$, hence (by the observation), $\text{stack}\,\Delta_G \in \mathbf{S}_q(X, \eta)$, which implies that $\eta_g(\text{stack}\,\Delta_G) = 0$.
Next, it follows from (SAUCS_2) that $\eta_g(\mathcal{H}) = \eta(\mathcal{H})$ whenever $\mathcal{H} \in \mathbf{S}_q(X, \eta)$, consequently, for any $\mathcal{F} \in \mathbf{F}(X^2)$, it holds that

$$\eta_g(\mathcal{F}) = \inf_{\substack{\mathcal{H} \in \mathbf{S}_q(X,\eta), \\ \mathcal{H} \subset \mathcal{F}}} \eta(\mathcal{H}) \geq \inf_{\substack{\mathcal{H} \in \mathbf{S}_q(X), \\ \mathcal{H} \subset \mathcal{F}}} \eta_g(\mathcal{H}).$$

Since (SAUCS_2) again implies the reverse inequality and also $\eta_g \geq \eta$, it follows that $(X, \eta_g) \in \mathbf{g}\text{-}q\mathbf{SAUConv}$ and that $1_X : (X, \eta_g) \longrightarrow (X, \eta)$ is a uniform contraction.
Lastly, let $f : (Y, \eta_Y) \longrightarrow (X, \eta)$ be a uniform contraction such that $(Y, \eta_Y) \in \mathbf{g}\text{-}q\mathbf{SAUConv}$. To show that $f : (Y, \eta_Y) \longrightarrow (X, \eta_g)$ is also a uniform contraction, let $\eta_Y(\mathcal{F}) < \alpha$ $(0 \leq \alpha < \infty)$. Hence, by (qsaug), there exists $\mathcal{H} \in \mathbf{S}_q(X, H)$ $(H \subset X)$ such that $\mathcal{H} \subset \mathcal{F}$ and $\eta(\mathcal{H}) < \alpha$. Consequently, by $\mathcal{H} \subset \text{stack}\,\Delta_H$ and (saug_Δ), $\eta_Y(\text{stack}\,\Delta_H) = 0$ and therefore $\eta((f \times f)(\text{stack}\,\Delta_H)) = \eta(\text{stack}\,\Delta_{f(H)}) = 0$. It follows that $(f \times f)(\mathcal{H}) \in \mathbf{S}_q(X, f(H), \eta)$ and since $\eta((f \times f)(\mathcal{H})) < \alpha$, it can be concluded that also $\eta_g((f \times f)(\mathcal{F})) < \alpha$ (by construction).
∎

4.3. Definition. Define **g-SAUConv** to be the full subconstruct of **SAUConv** whose objects (X, η) are *semi-approach uniformly generated*, i.e. satisfy

(saug) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \eta(\mathcal{F}) = \inf_{\substack{\mathcal{H} \in \mathbf{S}(X), \\ \mathcal{H} \subset \mathcal{F}}} \eta(\mathcal{H})$.

(saug_Δ) $\forall H \subset X : \eta(\text{stack}\,\Delta_H) < \infty \Rightarrow \eta(\text{stack}\,\Delta_H) = 0$.

4.4. Proposition. **g-SAUConv** *is bicoreflective in* **SAUConv**.

Proof. One obtains the required bicoreflector and corresponding argumentation by replacing $\mathbf{S}_q$ by $\mathbf{S}$ and (qsaug) by (saug) in the foregoing result. ∎

4.5. Proposition. *$(q)s$**AUnif** is contained in the final hull of (q)**AUnif** in (q)**SAUConv**.*

Proof. The proof is slightly different whether we consider the quasi case or not. Therefore, in the following, to obtain the proof in the quasi case, disregard pieces indicated by [[...]] and use pieces indicated by [...], while in the symmetric (non-quasi) case, just reverse the previous convention.
Let $(X, (\mathcal{U}_\epsilon)_{\epsilon\in\mathbb{R}^+})$ be a principal [quasi-]semi-approach uniform limit space, where $(\mathcal{U}_\epsilon)_{\epsilon\in\mathbb{R}^+}$ is the corresponding [quasi-]semi uniform tower. Define $Z := \{(x,y,i) \mid x,y \in X,\ i \in \{1,2\}\}$ and $f : Z \longrightarrow X : (x,y,i) \mapsto f(x,y,i)$, where $f(x,y,i) := x$ if $i = 1$ and $f(x,y,i) := y$ if $i = 2$, hence, f is a surjective map. Further, for $\epsilon \in \mathbb{R}^+$ and $U_\epsilon \in \mathcal{U}_\epsilon$, let $\hat{U}'_\epsilon := \{((x,y,1),(x,y,2)) \mid (x,y) \in U_\epsilon\}$ and $[\hat{U}_\epsilon := \hat{U}'_\epsilon \cup \Delta_Z]$ $[[\hat{U}_\epsilon := \hat{U}'_\epsilon \cup \Delta_Z \cup (\hat{U}'_\epsilon)^{-1}]]$.
It is then easily verified that $\hat{U}_\epsilon^2 = \hat{U}_\epsilon$ (and clearly $\Delta_Z \subset \hat{U}_\epsilon$ $[[= \hat{U}_\epsilon^{-1}]]$), hence $\{\hat{U}_\epsilon \mid U_\epsilon \in \mathcal{U}_\epsilon\}$ is a filterbasis (easily checked) that generates a [quasi-]semi-uniformity $\hat{\mathcal{U}}_\epsilon$ such that $\hat{\mathcal{U}}_\epsilon \circ \hat{\mathcal{U}}_\epsilon = \hat{\mathcal{U}}_\epsilon$. Consequently, for $\epsilon, \epsilon' \in \mathbb{R}^+$, it follows that $\hat{\mathcal{U}}_{\epsilon+\epsilon} \subset \hat{\mathcal{U}}_{\epsilon\vee\epsilon'} = \hat{\mathcal{U}}_{\epsilon\vee\epsilon'} \circ \hat{\mathcal{U}}_{\epsilon\vee\epsilon'} \subset \hat{\mathcal{U}}_\epsilon \circ \hat{\mathcal{U}}_{\epsilon'}$. Note that the inclusions hold because of (UT4), which is also satisfied (this follows immediately from (UT4) for $(\mathcal{U}_\epsilon)_{\epsilon\in\mathbb{R}^+}$), hence $(\hat{\mathcal{U}}_\epsilon)_{\epsilon\in\mathbb{R}^+}$ is a [quasi-]uniform tower.
Also, [[if $U_\epsilon = U_\epsilon^{-1}$, then]] $(f \times f)(\hat{U}_\epsilon) = U_\epsilon$ $(U_\epsilon \in \mathcal{U}_\epsilon)$, hence $(f \times f)(\hat{\mathcal{U}}_\epsilon) = \mathcal{U}_\epsilon$ [[since $\mathcal{U}_\epsilon$ has a filterbasis consisting of symmetric sets (by UT2)]], which easily implies that $f : (Z, (\hat{\mathcal{U}})_{\epsilon\in\mathbb{R}^+}) \longrightarrow (X, (\mathcal{U})_{\epsilon\in\mathbb{R}^+})$ is a $[q]$**SAUConv**-quotient. ∎

4.6. Proposition. **g-**(q)**SAUConv** *is the final hull of* (q)**AUnif** *in* (q)**SAUConv**.

Proof. Let $(X, \eta) \in qs$**AUnif**, then it follows from propositions 3.2 and 3.5 that there exists a quasi-semi-uniform tower $(\mathcal{U}_\epsilon)_{\epsilon\in\mathbb{R}^+}$ such that $\forall \mathcal{F} \in \mathbf{F}(X^2) : \eta(\mathcal{F}) = \min\{\epsilon \in \mathbb{R}^+ \mid \mathcal{U}_\epsilon \subset \mathcal{F}\}$. This immediately implies (qsaug) and $\eta(\text{stack}\,\Delta_H) = 0$ $(\forall H \subset X)$, hence, qs**AUnif** $\subset$ **g-**q**SAUConv**, consequently, by proposition 4.2, also the final hull of q**AUnif** in q**SAUConv** is contained in **g-**q**SAUConv**.
Conversely, let $(X, \eta) \in$ **g-**q**SAUConv** and let $\mathcal{F} \in \mathbf{F}(X^2)$ such that $\eta(\mathcal{F}) < \infty$. By (qsaug) and (saug$_\Delta$), for any $\delta > 0$, there exists $\mathcal{H} \in \mathbf{S}_q(X, H)$ $(H \subset X)$ such that $\mathcal{H} \subset \mathcal{F}$, $\eta(\mathcal{H}) < \eta(\mathcal{F}) + \delta$ $(*)$ and $\eta(\text{stack}\,\Delta_H) = 0$ $(**)$. Define the quasi-semi-approach uniform space $(X, \eta)_{\mathcal{F},\delta} = (H, (\mathcal{U}_\epsilon^H)_{\epsilon\in\mathbb{R}^+})$ by $\mathcal{U}_\epsilon^H := \text{stack}\,\Delta_H$ $(\epsilon < \eta(\mathcal{F}) + \delta)$ and $\mathcal{U}_\epsilon^H := \mathcal{H}$ $(\epsilon \geq \eta(\mathcal{F}) + \delta)$. Also, $f_{\mathcal{F},\delta} = 1_H : (X, \eta)_{\mathcal{F},\delta} \longrightarrow (X, \eta)$ is a uniform contraction, which follows from propositions 3.2 and 3.5 and $(*)$ and $(**)$. It now suffices to show that

$$(f_{\mathcal{F},\delta} : (X, \eta)_{\mathcal{F},\delta} \longrightarrow (X, \eta))_{\mathcal{F}\in\mathbf{F}(X^2), \eta(\mathcal{F})<\infty, \delta>0}$$

is a final (epi-)sink in q**SAUConv**. To this end, let $g : X \longrightarrow (Y, \eta_Y)$ be a map (and $(Y, \eta_Y) \in q$**SAUConv**) such that all $g \circ f_{\mathcal{F},\delta}$ are uniform contractions. To show the uniform contractivity of $g : (X, \eta) \longrightarrow (Y, \eta_Y)$, let $\mathcal{F} \in \mathbf{F}(X^2)$. Either $\eta(\mathcal{F}) = \infty$ and then clearly, $\eta_Y((g \times g)(\mathcal{F})) \leq \eta(\mathcal{F})$. Otherwise, let $\delta > 0$, then the uniform contractivity of $g \circ f_{\mathcal{F},\delta}$ implies that

$$\eta_Y((g \times g)(\mathcal{F})) = \eta_Y(((g \circ f_{\mathcal{F},\delta}) \times (g \circ f_{\mathcal{F},\delta}))(\mathcal{F})) \leq \eta(\mathcal{F}) + \delta.$$

The desired result for the quasi case then follows from the arbitrariness of δ and from the foregoing proposition.
The symmetric case is analogous to the foregoing argumentation (just note that one should now use $\mathcal{H} \in \mathbf{S}(X, H)$, which allows to construct a symmetric tower). ∎

Next in line is to consider final hulls in even smaller topological universes.

4.7. Definition. Define **g-**q**SAULim** := **g-**q**SAUConv** $\cap$ q**SAULim** and **g-SAULim** := **g-SAUConv** $\cap$ **SAULim**.

4.8. Proposition. **g-qSAULim** *is bicoreflective in* q**SAULim** *and the bicoreflector is the restriction of the bicoreflector* $\mathcal{C}_g : q\mathbf{SAUConv} \longrightarrow \mathbf{g}\text{-}q\mathbf{SAUConv}$.

Proof. Let $(X, \eta) \in q\mathbf{SAULim}$, then it suffices to show that also $(X, \eta_g) \in q\mathbf{SAULim}$. To this end, let $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(X^2)$ and let $\eta_g(\mathcal{F}_1) \vee \eta_g(\mathcal{F}_2) < \alpha$ $(0 < \alpha < \infty)$. Hence, by definition of η_g, there exist $\mathcal{H}_i \in \mathbf{S}_q(X, H_i, \eta)$ $(i = 1, 2)$ such that $\mathcal{H}_i \subset \mathcal{F}_i$ and $\eta(\mathcal{H}_i) < \alpha$. It then follows from $(X, \eta) \in q\mathbf{SAULim}$ that $\mathcal{H}_1 \cap \mathcal{H}_2 \in \mathbf{S}_q(X, H_1 \cup H_2, \eta)$ and $\eta(\mathcal{H}_1 \cap \mathcal{H}_2) < \alpha$. Since also $\mathcal{H}_1 \cap \mathcal{H}_2 \subset \mathcal{F}_1 \cap \mathcal{F}_2$, it follows from definition of η_g that also $\eta_g(\mathcal{F}_1 \cap \mathcal{F}_2) < \alpha$. ■

4.9. Proposition. **g-qSAULim** *is the final hull of* q**AUnif** *in* q**SAULim**.

Proof. It follows immediately from proposition 4.6 and the definition of **g-qSAULim** that the latter contains q**AUnif** and is contained in the final hull of it in q**SAULim**. By the previous proposition, **g-qSAULim** is also finally closed in q**SAULim**, which proves the desired result. ■

4.10. Proposition. **g-SAULim** = **g-qSAULim** ∩ **SAULim**.

Proof. Clearly, the inclusion $\subset$ holds.
Conversely, let (X, η) belong to the right-hand side, then only (saug) remains to be shown. To this end, let $\mathcal{F} \in \mathbf{F}(X^2)$ and $\eta(\mathcal{F}) < \alpha \in \mathbb{R}^+$, then it follows from (qsaug) that there exists $\mathcal{H} \in \mathbf{S}_q(X)$ such that $\mathcal{H} \subset \mathcal{F}$ and $\eta(\mathcal{H}) < \alpha$. Consequently, $\mathcal{H} \cap \mathcal{H}^{-1} \in \mathbf{S}(X)$ such that $\mathcal{H} \cap \mathcal{H}^{-1} \subset \mathcal{F}$ and $\eta(\mathcal{H} \cap \mathcal{H}^{-1}) = \eta(\mathcal{H}) \vee \eta(\mathcal{H}^{-1}) = \eta(\mathcal{H}) \leq \alpha$ (as $(X, \eta) \in \mathbf{SAULim}$).
■

4.11. Proposition. **g-SAULim** *is bicoreflective in* **SAULim** *and the bicoreflector can be obtained as*

(1) the restriction of the bicoreflector $\mathcal{C}_g : q\mathbf{SAULim} \longrightarrow \mathbf{g}\text{-}q\mathbf{SAULim}$.

(2) the restriction of the bicoreflector $\mathbf{SAUConv} \longrightarrow \mathbf{g}\text{-}\mathbf{SAUConv}$.

Proof. (1): Let $(X, \eta) \in \mathbf{SAULim}$. By the previous proposition, it will suffice to show that also $(X, \eta_g) \in \mathbf{SAULim}$. This is indeed the case, since $\big(\mathcal{H} \subset \mathcal{F}$ and $\mathcal{H} \in \mathbf{S}_q(X, \eta)\big)$ if and only if $\big(\mathcal{H}^{-1} \subset \mathcal{F}^{-1}$ and $\mathcal{H}^{-1} \in \mathbf{S}_q(X, \eta)\big)$. It is then an easy consequence of the definition of η_g (and $(X, \eta) \in \mathbf{SAULim}$) that also $(X, \eta_g) \in \mathbf{SAULim}$.
(2): Consider proposition 4.4 (and its proof) and carry out the corresponding adaptations in the proof of proposition 4.8 (i.e. replace $\mathbf{S}_q$ by $\mathbf{S}$). ■

4.12. Proposition. **g-SAULim** *is the final hull of* **AUnif** *in* **SAULim**.

Proof. The argument is like the one of proposition 4.9. ■

4.13. Definition. Define **g-qAULim** := **g-qSAULim** ∩ q**AULim** and **g-AULim** := **g-SAULim** ∩ **AULim**.

4.14. Proposition. **g-AULim** = **g-qAULim** ∩ **AULim**.

Proof. This follows from

$$\begin{aligned} \mathbf{g}\text{-}\mathbf{AULim} &= \mathbf{g}\text{-}\mathbf{SAULim} \cap \mathbf{AULim} \\ &= \mathbf{g}\text{-}q\mathbf{SAULim} \cap \mathbf{SAULim} \cap \mathbf{AULim} \text{ (by proposition 4.10)} \\ &= \mathbf{g}\text{-}q\mathbf{AULim} \cap \mathbf{AULim} \text{ (by definition).} \quad \blacksquare \end{aligned}$$

4.15. Proposition. **g-(q)AULim** *is bicoreflective in* (q)**AULim** *and the bicoreflector is a restriction of the bicoreflector* $\mathcal{C}_g : q\mathbf{SAULim} \longrightarrow \mathbf{g}\text{-}q\mathbf{SAULim}$.

Proof. Let $(X, \eta) \in q\mathbf{AULim}$, then it will suffice to show that also $(X, \eta_g) \in q\mathbf{AULim}$. To this end, let $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(X^2)$ and let $\eta_g(\mathcal{F}_i) < \alpha_i$ $(0 < \alpha_i < \infty)$ $(i = 1, 2)$. Hence, by definition of η_g, there exist $\mathcal{H}_i \in \mathbf{S}_q(X, H_i, \eta)$ such that $\mathcal{H}_i \subset \mathcal{F}_i$ and $\eta(\mathcal{H}_i) < \alpha_i$. But then also $\mathcal{H}'_i := \mathcal{H}_i \cap \operatorname{stack}\Delta_{H_1} \cap \operatorname{stack}\Delta_{H_2} \in \mathbf{S}_q(X, H_1 \cup H_2, \eta)$, $\mathcal{H}'_i \subset \mathcal{F}_i$ and $\eta(\mathcal{H}'_i) = \eta(\mathcal{H}_i)$. Consequently, $\mathcal{H}'_1 \circ \mathcal{H}'_2 \in \mathbf{S}_q(X, H_1 \cup H_2, \eta)$, $\mathcal{H}'_1 \circ \mathcal{H}'_2 \subset \mathcal{F}_1 \circ \mathcal{F}_2$ and, by (AULS), $\eta(\mathcal{H}'_1 \circ \mathcal{H}'_2) < \alpha_1 + \alpha_2$. Hence, by definition of η_g and arbitrariness of α_i, it follows that $\eta_g(\mathcal{F}_1 \circ \mathcal{F}_2) \leq \eta_g(\mathcal{F}_1) + \eta_g(\mathcal{F}_2)$. ∎

4.16. Proposition. **g-**(q)**AULim** *is the final hull of* (q)**AUnif** *in* (q)**AULim**.

Proof. This is analogous to the argumentation in proposition 4.9. ∎

4.17. Definition. Let **g-**q**PsAULim** := **g-**q**SAULim**$\cap q$**PsAULim** and **g-PsAULim** := **g-SAULim** $\cap$ **PsAULim**.

4.18. Lemma. *If* $\mathcal{F} \in \mathbf{F}(X)$ *and* $\Psi \subset \mathbf{F}(X)$, *then the following are equivalent:*

(1) $\forall \mathcal{W} \in \mathbf{U}(\mathcal{F}), \exists \mathcal{G} \in \Psi : \mathcal{G} \subset \mathcal{W}$.

(2) For any family $(\sigma(\mathcal{G}))_{\mathcal{G}\in\Psi}$ *such that* $\sigma(\mathcal{G}) \in \mathcal{G}$ $(\mathcal{G} \in \Psi)$, *there exists a finite set* $\Psi' \subset \Psi$ *such that* $\bigcup_{\mathcal{G}\in\Psi'} \sigma(\mathcal{G}) \in \mathcal{F}$.

Proof. $\boxed{1 \Rightarrow 2}$ Let $(\sigma(\mathcal{G}))_{\mathcal{G}\in\Psi}$ be a family such that $\sigma(\mathcal{G}) \in \mathcal{G}$ $(\mathcal{G} \in \Psi)$. Suppose the conclusion does not hold, then it follows that the family $\mathcal{F} \cup \{X \setminus \sigma(\mathcal{G}) \mid \mathcal{G} \in \Psi\}$ has the finite intersection property and is therefore contained in some ultrafilter $\mathcal{W} \in \mathbf{U}(\mathcal{F})$. By (1), there exists $\mathcal{G} \in \Psi$ such that $\mathcal{G} \subset \mathcal{W}$. This implies that both $\sigma(\mathcal{G}) \in \mathcal{G} \subset \mathcal{W}$ and $X \setminus \sigma(\mathcal{G}) \in \mathcal{W}$, which is a contradiction.
$\boxed{2 \Rightarrow 1}$ Suppose (1) does not hold, then there exists some $\mathcal{W} \in \mathbf{U}(\mathcal{F})$ such that $\forall \mathcal{G} \in \Psi$: $\mathcal{G} \not\subset \mathcal{W}$, which implies that $\forall \mathcal{G} \in \Psi, \exists \sigma(\mathcal{G}) \in \mathcal{G} : \sigma(\mathcal{G}) \notin \mathcal{W}$ $(*)$. Applying (2) on the family $(\sigma(\mathcal{G}))_{\mathcal{G}\in\Psi}$ yields a finite set $\Psi' \subset \Psi$ such that $\cup_{\mathcal{G}\in\Psi'}\sigma(\mathcal{G}) \in \mathcal{F}$. As $\mathcal{F} \subset \mathcal{W}$ and $\mathcal{W}$ is an ultrafilter, there is some $\mathcal{G} \in \Psi' : \sigma(\mathcal{G}) \in \mathcal{W}$, which contradicts $(*)$. ∎

4.19. Lemma. *Let* $(X, \eta) \in$ **g-**q**SAULim**, *then the following are equivalent:*

(1) $(X, \eta) \in q$**PsAULim**.

(2) $\forall \mathcal{H} \in \mathbf{S}_q(X) : \eta(\mathcal{H}) = \sup_{\mathcal{U}\in\mathbf{U}(\mathcal{H})} \eta(\mathcal{U})$.

(3) $\forall \mathcal{H} \in \mathbf{S}_q(X, H) : \big(\eta(\operatorname{stack}\Delta_H) = 0 \Rightarrow \eta(\mathcal{H}) = \sup_{\mathcal{U}\in\mathbf{U}(\mathcal{H})} \eta(\mathcal{U})\big)$.

Proof. Clearly, $\boxed{1 \Rightarrow 2}$ and $\boxed{2 \Rightarrow 3}$.
$\boxed{3 \Rightarrow 1}$ Let $\mathcal{F} \in \mathbf{F}(X^2)$. Since it follows from (SAUCS_2) that $\sup_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \eta(\mathcal{U}) \leq \eta(\mathcal{F})$, it suffices to consider the case where $\sup_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \eta(\mathcal{U}) < \infty$.
Consequently, by (qsaug), for any $\mathcal{U} \in \mathbf{U}(\mathcal{F})$, there exists $\mathcal{H}_{\mathcal{U}} \in \mathbf{S}_q(X, H_{\mathcal{U}})$ such that $\mathcal{H}_{\mathcal{U}} \subset \mathcal{U}$ and $\eta(\mathcal{H}_{\mathcal{U}}) < \infty$, hence, by ($\mathrm{saug}_\Delta$), $\eta(\operatorname{stack}\Delta_{H_{\mathcal{U}}}) = 0$. Letting $\Psi := \mathbf{U}(\mathcal{F})$, it follows that (1) of the previous lemma is satisfied, hence, applying (2) of the previous lemma to the family $(H^2_{\mathcal{U}})_{\mathcal{U}\in\mathbf{U}(\mathcal{H})}$ leads to $n \in \mathbb{N}_0$ and $H^2_{\mathcal{U}_1} \cup \ldots \cup H^2_{\mathcal{U}_n} \in \mathcal{F}$. Letting $H := H_{\mathcal{U}_1} \cup \ldots \cup H_{\mathcal{U}_n}$, it follows that $\eta(\operatorname{stack}\Delta_H) = \eta(\operatorname{stack}\Delta_{H_{\mathcal{U}_1}} \cap \ldots \cap \operatorname{stack}\Delta_{H_{\mathcal{U}_n}}) = 0$ and $H \times H \in \mathcal{F}$. Consequently, $\eta(\mathcal{F} \cap \operatorname{stack}\Delta_H) = \eta(\mathcal{F})$ and $\mathcal{F} \cap \operatorname{stack}\Delta_H \in \mathbf{S}_q(X, H)$. Hence, by (3),

$$\eta(\mathcal{F}) = \eta(\mathcal{F} \cap \operatorname{stack}\Delta_H) = \sup_{\mathcal{U}\in\mathbf{U}(\mathcal{F}\cap\operatorname{stack}\Delta_H)} \eta(\mathcal{U}) = \sup_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \eta(\mathcal{U}).$$

Note that the latter equality follows from $\mathbf{U}(\mathcal{F} \cap \operatorname{stack}\Delta_H) = \mathbf{U}(\mathcal{F}) \cup \mathbf{U}(\operatorname{stack}\Delta_H)$ and $\forall \mathcal{U} \in \mathbf{U}(\operatorname{stack}\Delta_H) : \eta(\mathcal{U}) \leq \eta(\operatorname{stack}\Delta_H) = 0$. ∎

4.20. Proposition. **g-PsAULim** = **g-**q**PsAULim** $\cap$ **PsAULim**.

Proof. This follows from

g-PsAULim = **g-SAULim** $\cap$ **PsAULim**

$$= \textbf{g-}q\textbf{SAULim} \cap \textbf{SAULim} \cap \textbf{PsAULim} \text{ (by proposition 4.10)}$$
$$= \textbf{g-}q\textbf{PsAULim} \cap \textbf{PsAULim} \text{ (by definition).} \quad ■$$

4.21. Proposition. $\textbf{g-}(q)\textbf{PsAULim}$ *is bicoreflective in* $(q)\textbf{PsAULim}$ *and the bicoreflector is a restriction of the bicoreflector* $\mathcal{C}_g : q\textbf{SAULim} \longrightarrow \textbf{g-}q\textbf{SAULim}$.

Proof. Let $(X, \eta) \in q\textbf{PsAULim}$, then it will suffice to show that also $(X, \eta_g) \in q\textbf{PsAULim}$. To this end, let $\mathcal{H} \in \textbf{S}_q(X, H)$ such that $\eta_g(\text{stack}\,\Delta_H) = 0$. Consequently, $\eta(\text{stack}\,\Delta_H) = 0$ (as $\eta_g \geq \eta$) and therefore $\mathcal{H} \in \textbf{S}_q(X, H, \eta)$, hence $\eta(\mathcal{H}) = \eta_g(\mathcal{H})$ (cf. proof of proposition 4.2). It then follows from $(X, \eta) \in q\textbf{PsAULim}$ that

$$\eta_g(\mathcal{H}) = \eta(\mathcal{H}) = \sup_{\mathcal{U} \in \textbf{U}(\mathcal{H})} \eta(\mathcal{U}) \leq \sup_{\mathcal{U} \in \textbf{U}(\mathcal{H})} \eta_g(\mathcal{U}).$$

Since the reverse inequality follows from (SAUCS_2) (and $(X, \eta_g) \in \textbf{g-}q\textbf{SAULim}$), it has been shown that (3) of the previous lemma is satisfied, consequently, $(X, \eta_g) \in q\textbf{PsAULim}$. ■

4.22. Proposition. $\textbf{g-}(q)\textbf{PsAULim}$ *is the final hull of* $(q)\textbf{AUnif}$ *in* $q\textbf{PsAULim}$.

Proof. This is analogous to the $q\textbf{SAULim}$-case argumentation (proposition 4.9). ■

It remains to consider some final hulls in a more "classical" (non-quantified) setting.

4.23. Definition. Define $\textbf{g-}(q)\textbf{SUConv} := \textbf{g-}(q)\textbf{SAUConv} \cap q\textbf{SUConv}$, that is, $\textbf{g-}(q)\textbf{SUConv}$ is the full subconstruct of $(q)\textbf{SUConv}$ whose objects $(X, \mathbb{L})$ satisfy

((q)sug) $\forall \mathcal{F} \in \textbf{F}(X^2), \exists \mathcal{H} \in \textbf{S}_{(q)}(X) \cap \mathbb{L} : \mathcal{H} \subset \mathcal{F}$.

4.24. Definition. Let

$$\begin{aligned} \textbf{g-}(q)\textbf{SULim} &:= \textbf{g-}(q)\textbf{SAULim} \cap q\textbf{SUConv}, \\ \textbf{g-}(q)\textbf{PsULim} &:= \textbf{g-}(q)\textbf{PsAULim} \cap q\textbf{SUConv}, \\ \text{and } \textbf{g-}(q)\textbf{ULim} &:= \textbf{g-}(q)\textbf{AULim} \cap q\textbf{SUConv}. \end{aligned}$$

4.25. Lemma. *Let* $(X, \eta) \in \textbf{g-}(q)\textbf{SAULim}$, *then* (X, η) *satisfies the property*

((q)saug′) $\forall \mathcal{F} \in \textbf{F}(X^2) : \eta(\mathcal{F}) = \min_{\substack{\mathcal{H} \in \textbf{S}_{(q)}(X), \\ \mathcal{H} \subset \mathcal{F}}} \eta(\mathcal{H})$.

Proof. Let $\mathcal{F} \in \textbf{F}(X^2)$. If $\eta(\mathcal{F}) = \infty$, then it suffices to choose $\mathcal{H} := \text{stack}\, X \times X \subset \mathcal{F}$ (and clearly then also $\eta(\text{stack}\, X \times X) = \infty$).
If $\eta(\mathcal{F}) < \infty$, then there exists some $\mathcal{H} \in \textbf{S}_q(X, H)$ $(H \subset X)$ such that $\mathcal{H} \subset \mathcal{F}$ and $\eta(\mathcal{H}) < \infty$, hence, by ($\text{saug}_\Delta$), $\eta(\text{stack}\,\Delta_H) = 0$. As $\mathcal{H} \subset \mathcal{F}$, it follows that $H \times H \in \mathcal{F}$ and therefore $\mathcal{G} := \mathcal{F} \cap \text{stack}\,\Delta_H \in \textbf{S}_q(X, H)$ such that $\mathcal{G} \subset \mathcal{F}$ and $\eta(\mathcal{G}) = \eta(\mathcal{F})$, which proves the claim in the quasi-case.
In the symmetric case, it suffices to consider $\mathcal{G} := \mathcal{F} \cap \mathcal{F}^{-1} \cap \text{stack}\,\Delta_H$ in the previous argumentation to show the required. ■

4.26. Proposition. *The following hold:*

(1) $\textbf{g-}(q)\textbf{SULim}$ *is bicoreflective in* $\textbf{g-}(q)\textbf{SAULim}$.

(2) $\textbf{g-}(q)\textbf{PsULim}$ *is bicoreflective in* $\textbf{g-}(q)\textbf{PsAULim}$.

(3) $\textbf{g-}(q)\textbf{ULim}$ *is bicoreflective in* $\textbf{g-}(q)\textbf{AULim}$.

Furthermore, each bicoreflector is a restriction of the bicoreflector $\mathcal{C}_0 : q\textbf{SAULim} \longrightarrow q\textbf{SULim}$.

Proof. It follows immediately from the foregoing lemma that $\mathcal{C}_0(X,\eta) = (X,\eta_0) \in$ **g**-(q)**SAULim** whenever $(X,\eta) \in$ **g**-(q)**SAULim**. Combining this with proposition 3.11 then shows the required. ∎

4.27. Proposition. *The following hold:*

(1) **g**-(q)**SULim** *is the final hull of* (q)**Unif** *in* (q)**SULim**.

(2) **g**-(q)**PsULim** *is the final hull of* (q)**Unif** *in* (q)**PsULim**.

(3) **g**-(q)**ULim** *is the final hull of* (q)**Unif** *in* (q)**ULim**.

In particular, each of the far left mentioned constructs is bicoreflective in the the far right mentioned construct and the bicoreflectors are restrictions of the bicoreflector $\mathcal{C}_g$: (q)**SAULim** $\longrightarrow$ **g**-(q)**SAULim**.

Proof. The latter claim follows immediately from the definitions of the constructs involved and the results of the previous section regarding the bicoreflector $\mathcal{C}_g$, which also preserves the property (SUC) (given in proposition 3.9), as can easily be seen from its description in proposition 4.2.

Using proposition 4.6, it follows that (q)**Unif** $\subset$ **A** $\subset$ **B** (where **A** and **B** are the appropriate far left and far right side constructs), consequently, by the latter claim, also the final hull of (q)**Unif** in **B** is contained in **A**.

Conversely, considering the constructions used in proving proposition 4.6, one observes that starting from a space (X,η) satisfying (SUC) results into spaces also satisfying (SUC) (modulo a few changes) (in short, the construction preserves (SUC)). The few changes are in the proof of proposition 4.6; replace what is indicated by $(*)$ with $\eta(\mathcal{H}) = \eta(\mathcal{F})$ and just consider the quasi-semi-(approach) uniform space $(H,\mathcal{H})$. ∎

To conclude, let us gather some of the "most useful" constructs that have been considered into a diagram indicating the appropriate relations.

4.28. Proposition. *The following relations hold (for diagram; see slightly further). Furthermore, all bicoreflectors from the 2-most left sides to the 2-most right sides are restrictions of the bicoreflector* $\mathcal{C}_0$: q**SAULim** $\longrightarrow$ q**SULim**, *all bicoreflectors from the 2-most back walls to the 2-most forward walls are restrictions of the bicoreflector* $\mathcal{C}_s$: q**SAULim** $\longrightarrow$ **SAULim** *and all bicoreflectors from the inner rectangle to the outer rectangle are restrictions of the bicoreflector* $\mathcal{C}_g$: q**SAULim** $\longrightarrow$ **g**-q**SAULim**.

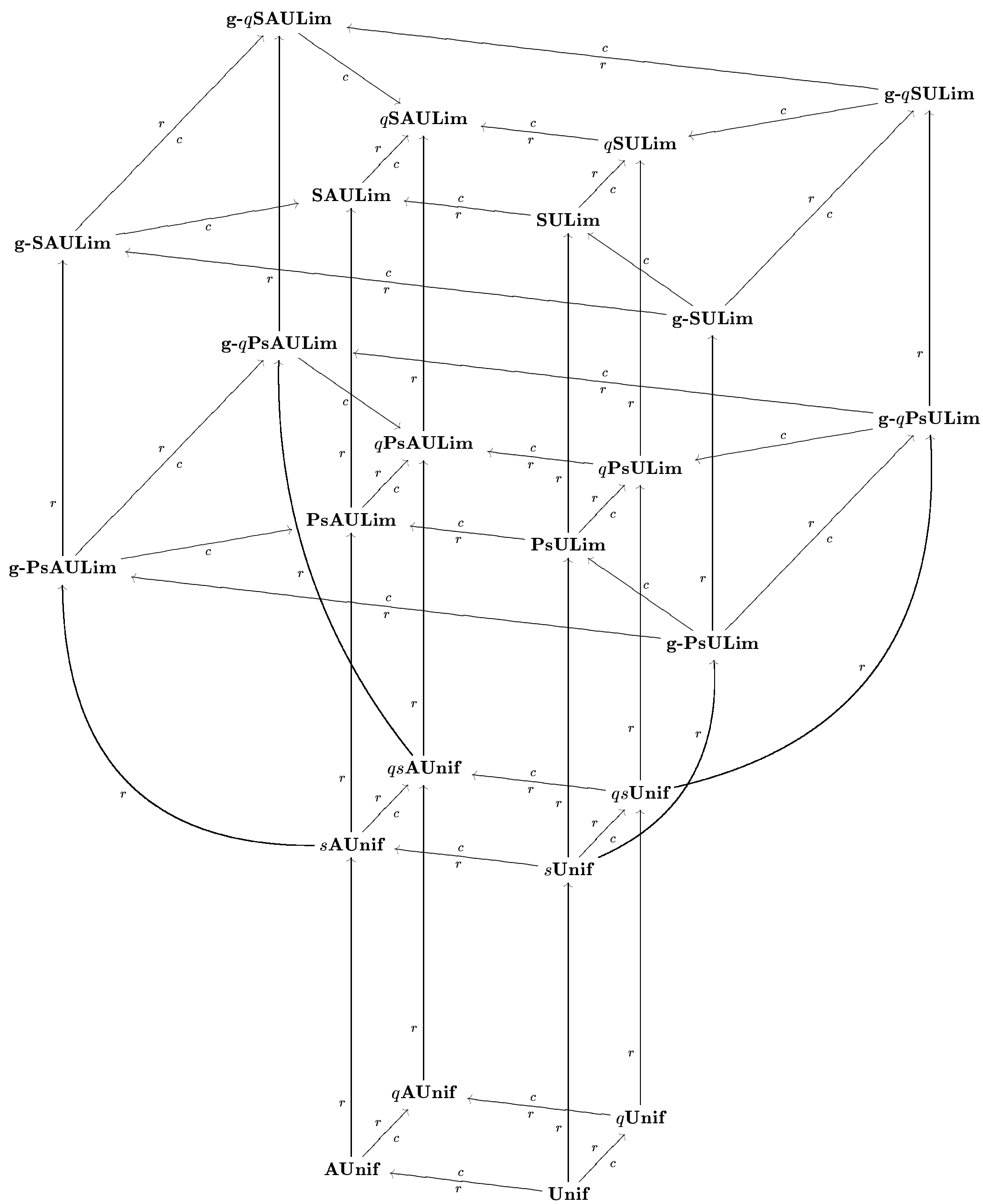
All constructs in the 2-most top levels are topological universes, the inner ones being closed under formation of function spaces and one-point extensions in q**SAULim** *and the outer one so closed in* **g**-q**SAULim**.

Proof. The required behaviour of the $\mathcal{C}_0$-bicoreflector in the inner rectangle follows immediately from proposition 3.11 and for the outer rectangle from proposition 4.26, whereas the desired behaviour of $\mathcal{C}_g$ has already been shown in the foregoing.

As for the $\mathcal{C}_s$-bicoreflector, this follows from proposition 3.11 and the fact that (SUC), (saug$_\Delta$) and (qsaug) are preserved by $\mathcal{C}_s$. Indeed, the former are clear and as for the latter, let $(X,\eta) \in$ **g**-q**SAULim**, then it suffices to show that (X,η_s) also satisfies (qsaug). To this end, let $\mathcal{F} \in \mathbf{F}(X^2)$ and $\eta(\mathcal{F}) < \alpha \in \mathbb{R}^+$, then there exists $\mathcal{H} \in \mathbf{S}_q(X,H)$ such that $\eta(\mathcal{H}) < \alpha$ and $\mathcal{H} \subset \mathcal{F}$. In particular, $H \times H \in \mathcal{F}$ and $\eta(\operatorname{stack}\Delta_H) = 0$. Consequently, $\mathcal{F} \cap \operatorname{stack}\Delta_H \in \mathbf{S}_q(X,H)$ and $\mathcal{F} \cap \operatorname{stack}\Delta_H \subset \mathcal{F}$ such that $\eta_s(\mathcal{F} \cap \operatorname{stack}\Delta_H) = \eta(\mathcal{F} \cap \operatorname{stack}\Delta_H) \vee \eta(\mathcal{F}^{-1} \cap \operatorname{stack}\Delta_H) = \eta(\mathcal{F}) \vee \eta(\mathcal{F}^{-1}) = \eta_s(\mathcal{F}) < \alpha$.

The convenience of the final hulls follows at once from results of Nel [21], Herrlich [8] and Schwarz [25, 26], and so does the preserving of special structures $(*)$ (that is, initial lifts (bireflectiveness), function spaces and one-point extensions) as follows.

Let **A** be a top level construct on the outer level, **A**$'$ the corresponding inner construct, and let **B** and **B**$'$ be obtained by descending one level, then it suffices to observe that $(*)$ is



formed in $\mathbf{B}$ (respectively $\mathbf{A}$) by first forming it in $\mathbf{B}'$ (respectively $\mathbf{A}'$) and then applying the $\mathbf{B}$-bicoreflector in $\mathbf{B}'$ (respectively the $\mathbf{A}$-bicoreflector in $\mathbf{A}'$), and to note that the former bicoreflector is a restriction of the latter and that $\mathbf{B}$ is closed under formation of $(*)$ in $\mathbf{A}$.

The remaining preservation of $(*)$ (in particular bireflectivenesses) within each of the top levels can be shown analogously, which justifies all such claims as stated. ∎

4.29. Remark. The results obtained here show the necessity of correcting a result in Zhang [30], specifically, the author there considers tower extensions $\mathbf{C}(L)$ of a topological construct $\mathbf{C}$ indexed by a completely distributive lattice L, examples of which are found in this paper in e.g. $s\mathbf{AUnif} = \mathbf{Unif}([0,\infty]^{op})$ and $\mathbf{PsAULim} = \mathbf{PsULim}([0,\infty]^{op})$ (see also proposition 3.2). It is then (a.o.) claimed in [30] that $\mathbf{A}(L)$ is finally dense in $\mathbf{C}(L)$ whenever $\mathbf{A}$ is so in $\mathbf{C}$. However, it is necessary to add the minor condition that *discrete* $\mathbf{C}$*-objects are* $\mathbf{A}$*-objects*. If this is not satisfied (as is the case in our uniform setting), additional conditions may be needed to obtain final density, as was (saug_Δ) needed here in addition to the "tower version" (saug) of (sug).

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