AN EAKIN-NAGATA THEOREM FOR SEMIGROUPS

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Abstract. We prove an Eakin-Nagata Theorem for commutative semigroups.

The well-known Eakin-Nagata Theorem for commutative rings states the following,

Theorem 1 (Eakin-Nagata). *Let S be a ring, and let R be a subring of S. If S is a Noetherian ring, and if S is a finitely generated R-module, then R is also a Noetherian ring.*

The aim of this note is to prove an Eakin-Nagata Theorem for commutative semigroups (explicitly, for g-monoids).

A submonoid of a torsion-free abelian (additive) group is called a grading monoid or a g-monoid. Let T be a g-monoid, and let S be a submonoid of T. Then T is called an extension semigroup of S, and S is called a subsemigroup of T.

Let S be a g-monoid, and let M be a non-empty set so that for each pair of elements $s \in S$ and $x \in M$, there defined $s + x \in M$. If, for all $s_1, s_2 \in S$ and $x \in M$, $(s_1 + s_2) + x = s_1 + (s_2 + x)$ and $0 + x = x$, then M is called an S-module.

If there exists a finite number of elements $x_1, \cdots, x_n$ of M such that $M = \cup_1^n (S + x_i)$, then M is called a finitely generated S-module.

Let S be a g-monoid. If every ideal of S is finitely generated, then S is called a Noetherian semigroup.

Let S be a g-monoid. If there exists a finite number of elements $s_1, \cdots, s_n$ of S such that $S = \sum_1^n \mathbf{Z}_0 s_i$, then S is called a finitely generated g-monoid, where $\mathbf{Z}_0$ denotes the set of non-negative integers. If S is a finitely generated g-monoid, then S is a Noetherian semigroup.

Let S be a g-monoid, and let T be an extension semigroup of S. If T is a Noetherian semigroup, and if T is a finitely generated S-module, we will prove that S is also a Noetherian semigroup.

Lemma 1 ([1, Appendix]). *Let R be a ring, and let S be a g-monoid. Then the semigroup ring $R[X; S]$ of S over R is a Noetherian ring if and only if R is a Noetherian ring and S is a finitely generated g-monoid.*

Theorem 1 and Lemma 1 imply the following,

Proposition 1. *Let S be a g-monoid, T be an extension semigroup of S, and let k be a field. If the semigroup ring $k[X; T]$ of T over k is a Noetherian ring, and if $k[X; T]$ is a finitely generatef $k[X; S]$-module, then S is a finitely generated g-monoid.*

Let S be a g-monoid, T an extension semigroup of S, and let k be a field. Then $k[X; T]$ is a finitely generated $k[X; S]$-module if and only if T is a finitely generated S-module.

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Our result is the following,

Theorem 2. *Let S be a g-monoid, and let T be an extension semigroup. If T is a Noetherian semigroup, and if T is a finitely generated S-module, then S is a Noetherian semigroup.*

Proof. Since T is a finitely generated S-module, T is an integral extension of S, that is, for each element t of T, there exists a positive integer h such that $ht \in S$.

We may assume that T is generated by one element x over S, that is $T = S[x] = S+\mathbf{Z}_0 x$. There exists a positive integer h such that $hx \in S$.

Since T is a Noetherian semigroup, there exists only a finite number of irreducible elements of T that are mutually non-associated.

The case that x is a unit of T: Let $q_1, \cdots, q_m$ be the a complete representative system of irreducible elements of T. Then we may assume that, for each i, $q_i \in S$.

The case that x is a non-unit of T: Let $q_1, \cdots, q_m, x$ be a complete representative system of irreducible elements of T. Then we may assume that, for each i, $q_i \in S$.

We put $q_{m+1} = hx$. We have $q_i \in S$ for each i.

Next, S satisfies the ascending chain condition for principal ideals. For, let $(a_1) \subset (a_2) \subset (a_3) \subset \cdots$ be an ascending chain of principal ideals of S. There exists a positive integer n such that $a_n + T = a_{n+1} + T$. We have $a_n = a_{n+1} + s_1$ and $a_{n+1} = a_n + s_2 + kx$, where $s_1, s_2 \in S$ and $k \in \mathbf{Z}_0$. It follows that $0 = hs_1 + hs_2 + kq_{m+1}$, and hence s_1 is a unit of S. Therefore $(a_n) = (a_{n+1})$.

We suppose that S is not a Noetherian semigroup. Then there exists an infinite number of irreducible elements of S that are mutually non-associated in S. For, each prime ideal of S is generated by irreducible elements. Let $p_1, p_2, p_3, \cdots$ be irreducible elements of S that are mutually non-associated.

Each p_i may be expressd as follows: $p_i = k_1^i q_1 + \cdots + k_{m+1}^i q_{m+1} + k^i x + u_i$, where each $k_j^i \in \mathbf{Z}_0$, $0 \leq k^i < h$ and u_i is a unit of S.

We may re-take $p_1, p_2, p_3, \cdots$ so that $k = k^i$ is definite and does not depend on i.

Suppose that $\{k_j^i \mid i = 1, 2, 3, \cdots\}$ is bounded for some j. Then we may re-take $p_1, p_2, p_3, \cdots$ so that k_j^i is definite and does not depend on j.

Suppose that $\{k_j^i \mid i = 1, 2, 3, \cdots\}$ is not bounded. Then we may re-take $p_1, p_2, p_3, \cdots$ so that $k_j^1 < k_j^2 < k_j^3 < \cdots$.

Then we have $k_j^i \leq k_j^{i+1}$ for each i and j. Specifically, we have

$p_1 = k_1^1 q_1 + \cdots + k_{m+1}^1 q_{m+1} + kx + u_1,$

$p_2 = k_1^2 q_1 + \cdots + k_{m+1}^2 q_{m+1} + kx + u_2,$

where u_1, u_2 are units of S, and $k_j^1 \leq k_j^2$ for each j. We have

$p_2 = p_1 + \sum_j (k_j^2 - k_j^1) q_j + u_2 - u_1.$

If $k_j^1 = k_j^2$ for each j, then p_1 and p_2 are associated in S; a contradiction. If $k_j^1 < k_j^2$ for some j, then p_2 is not an irreducible element of S; a contradiction. The proof is compllete.

REFERENCE

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