MULTIPLE SELECTION PROBLEM AND OLA STOPPING RULE

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Abstract. The optimal stopping rules with multiple selections of $m \geq 1$ objects with the objective of maximizing the probability of obtaining the best object are studied for two problems with an unknown number of objects:the problem with a random number of objects, and the problem where the objects arrive according to a homogeneous Poisson process with unknown intensity λ. These two problems are variation of the so-called secretary problem. This article introduces an easier method based on the one-stage look-ahead function (defined herein) depending on m and its recursive relation to the number m, to find the optimal stopping rule for all m, without a direct solution of equations suggested by a common dynamic programming approach.

1. Introduction. A man observes a sequence of rankable objects in a completely random order. He must decide after each observation whether or not to select that object using only the relative ranks of the objects seen. He is allowed to select at most a predetermined number, $m \geq 1$, of the objects. His objective is to select the very best object (the object of absolute rank 1). When $m = 1$ and the number of objects to be seen is known beforehand, this is the classical secretary problem, whose history is reviewed in the papers of Ferguson (1989) and Samuels (1991). When the number of objects is known beforehand, the case of arbitrary m was treated in Gilbert and Mosteller (1966), and the case with a random number of objects with $m = 1$ was treated in Presman and Sonin (1972). The problem with a random number of objects and arbitrary m was introduced by Tamaki (1979) and solved for $m = 2$. In this paper, we treat the general problem of a random number of objects and arbitrary $m \geq 2$. Ano and Sakaguchi (1987) studied the setting of the random duration time. We also consider multiple selections in the related problem treated by Bruss (1987), where the objects arrive at times of a homogeneous Poisson process whose rate has a prior exponential distribution.

For two selections, Haggstrom (1967) has studied the general stopping problems, and various aspects of the secretary problem with a fixed number of objects have been treated by Nikolaev (1977), Sakaguchi (1979), Tamaki (1980) and Rose (1982). Various multiple selection secretary problems have been studied in Glaser et al. (1983), Preater (1993a,b), Stadje (1980,1985), Sakaguchi (1989) and Wilson (1991). The asymptotically optimal rule for a large fixed number of objects with $m(\geq 1)$ selections, as found by Gilbert and Mosteller, can be summarized as follows: stop and select the first relatively best object which appears on or after stage s_m^*, where s_m^* is a determined sequence of integers, non-increasing in m. For large n, we have $s_1^* \approx ne^{-1}$, $s_2^* \approx ne^{-3/2}$, $s_3^* \approx ne^{-47/24}$, and $s_4^* \approx ne^{-2761/1152}$, and

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for large n, the maximum probability of obtaining the best object with m selections under the optimal rule is $s_1^*/n + s_2^*/n + \cdots + s_m^*/n$.

Proving optimality of rules for the stopping rule problem with $m \geq 3$ selections seems to be quite difficult employing the one-stage look-ahead approach suggested by a standard dynamic programming principle. This article introduces an easier method based on the *one-stage look-ahead function* (defined later) depending on the number of selections, and its recursive relation to m to find the optimal stopping rules for all $m \geq 1$, without direct solutions of the optimality equations. Ano and Tamaki (1991) seems to have been the first to use this method.

In Section 2, we apply this method to the problem where the number of objects is a random variable with known distribution $\delta_k = P(N = k), k = 0, 1, \cdots$ and $\pi_0 = 1, \pi_k = \sum_{s \geq k} \delta_s$. In the case with a single selection, Presman and Sonin (1972) have shown that the one-stage look-ahead rule is optimal if the following condition holds,

$$\text{(PS): } d_i \geq 0 \text{ implies } d_{i+1} \geq 0.$$

where

$$d_i \equiv \delta_i - \sum_{j \geq i+1} \delta_j / j \quad \text{for } i = 0, 1, \cdots$$

and $d_{-1} \equiv -1$. In other words, Condition (PS) requires that $\{d_i\}_{i=-1}^{\infty}$ changes sign at most once from negative to non-negative. However, the sequence, d_i, cannot be negative for all i (since $\sum_0^\infty d_i = \delta_0 \geq 0$) so the condition is also that it change sign exactly once from negative to nonnegative. We show under condition (PS) that the optimal rule for the problem with m selections is the same form as the one for the no-information secretary problem with m selections. As an example, we investigate in detail the case in which the total number, N, of objects is uniformly distributed on $[1, N_0]$. In this case, we see that as $N_0 \to \infty$, $s_1^*/N_0 \to e^{-2} \approx .135335$, $s_2^*/N_0 \to e^{-(1+\sqrt{21}/3)} \approx .079856$ and $s_3^*/N_0 \to e^{-(1+(\sqrt{135+42\sqrt{21}})/9)} \approx .04951742$ and for large N_0 the maximum probability of obtaining the best under the optimal stopping rule is $-((s_1^*/N_0)\log(s_1^*/N_0) + (s_2^*/N_0)\log(s_2^*/N_0) + \cdots + (s_m^*/N_0)\log(s_m^*/N_0))$.

Section 3 contains consideration of another problem with an unknown number of objects. Here the objects arrive according to a homogeneous Poisson process with unknown intensity λ which has a prior exponential distribution, $a\exp\{-a\lambda\}I(\lambda > 0)$ where a is a known nonnegative parameter. The objective is to maximize the probability of obtaining the best object from those (if any) available in the given interval $[0, T]$. The no-information version with single selection is the problem studied by Bruss (1987), which complements results of Cowan and Zabczyk (1978) with known intensity λ. Bruss (1987) has shown that the optimal rule for single selection is stationary, accepting (if possible) the first relatively best object after time $(T + a)/e - a$. Using our approach, based on his developments and results for single selection to which we refer in detail, we see that the optimal stopping rules with multiple selections have the following stationary form: if there are m selections remaining, the optimal rule is to accept (if possible) the first relatively best object after time $s_m^* = (T + a)/e^{C^{(m)}} - a$, where the $C^{(m)}$ are constants. For $a = 0$, it is interesting to see $s_1^* = T/e$, $s_2^* = T/e^{3/2}$, $s_3^* = T/e^{47/24}, \ldots$ compared with the values n/e, $n/e^{3/2}$, $n/e^{47/24}$, $n/e^{2761/1152}$ of the no-information secretary problem.

2. Random number of objects. Let X_i be the relative rank of the ith object among the first i objects (rank 1 being best) under the assumption that the objects are observed sequentially in random order. Then the X_i are independent random variables and for $i = 1, 2, \ldots,$ the distribution of X_i is given by $P(X_i = j) = 1/i$ for $j = 1, 2, \cdots, i$.

For the problem with a random number, N, of objects, let $W_i^{(m)}$ be the maximum probability of obtaining the best object among all N objects when we confront a relatively best object at the ith observation, and we can make more m selections thereafter. Similarly, when we can make more m selections in the future, let $U_i^{(m)}$ (resp. $V_i^{(m)}$) be the corresponding probability when we accept (resp. reject) the relatively best object at the ith observation. Suppose that the ith object is a relatively best object ($X_i = 1$). Then the conditional probability that the ith object is best among N given $N \geq i$ is

$$\sum_{j \geq i} P(X_{i+1} > 1, \cdots, X_j > 1 | N = j) P(N = j | N \geq i) = \sum_{j \geq i} \frac{i}{j} \frac{\delta_j}{\pi_i}. \tag{2.1}$$

Therefore

$$U_i^{(m)} = \sum_{j \geq i} \frac{i\delta_j}{j\pi_i} + V_i^{(m-1)}, \tag{2.2}$$

where $V_i^{(0)} = 0$ for all i. The conditional probability that the jth object is the first relatively best object after the ith object given $N \geq i$ is $(i\pi_j)/(j(j-1)\pi_i)$, so that

$$V_i^{(m)} = \sum_{j > i} \frac{i\pi_j}{j(j-1)\pi_i} W_j^{(m)}. \tag{2.3}$$

By the principle of optimality, we get the dynamic programming equation

$$W_i^{(m)} = \max\{U_i^{(m)},\ V_i^{(m)}\}, \quad \text{for } i = 1, 2, \cdots, \text{ and } m \geq 1. \tag{2.4}$$

The one-stage look-ahead rule is the rule that calls for selecting when selecting immediately is at least as good as waiting for the next relatively best to appear and then selecting. Thus for $i = 1, \cdots, n-1$ and $m \geq 1$, it requires us to select the ith object if

$$g_i^{(m)} \equiv U_i^{(m)} - \sum_{j=i+1}^{N} \frac{i\pi_j}{j(j-1)\pi_i} U_j^{(m)} \geq 0. \tag{2.5}$$

We also define $g_i^{(0)} = 0$ for all i and $g_{-1}^{(m)} \equiv -1$ for all $m \geq 1$. We call $g_i^{(m)}$ the *one-stage look-ahead function*. It is well-known that if for fixed m, $\{g_i^{(m)}\}_{i=-1}^{\infty}$ changes sign exactly once from negative to non-negative, then the problem is monotone in the sense of Chow et al. (1971), and the one-stage look-ahead rule is optimal having the following form of a threshold stopping rule with threshold $s_m \equiv \min\{i \geq 1 : g_i^{(m)} \geq 0\}$ given any m:

$$\tau_{s_m}^{(m)} \equiv \min\{k \geq s_m : X_k = 1\}. \tag{2.6}$$

A stopping problem is defined as monotone if the sets for a fixed m, $G_i^{(m)} = \{g_i^{(m)} \geq 0\}$, are monotone non-decreasing, i.e., $G_0^{(m)} \subset G_1^{(m)} \subset \cdots$ a.s. When the PS condition holds, the Presman & Sonin problem with single selection is monotone, and the one-stage look-ahead rule, which is a threshold rule $\tau_{s_1^*}^{(1)}$ with threshold $s_1^* = \min\{i \geq 1 : g_i^{(1)} \geq 0\}$ is optimal. The following theorem tells us that under the PS condition, the Presman & Sonin problem with multiple selections is also monotone.

Theorem 1. *If the PS condition holds, the optimal rule for the problem with a random number of objects when we make m more selections is a threshold rule $\tau^{(m)}_{s^*_m}$, where s^*_m can be specified as $s^*_m = \min\{i \geq 1 : g^{(m)}_i \geq 0\}$. Moreover, s^*_m is non-increasing in m.*

Proof. We carry out an induction on m. The induction hypotheses consist of the two statements, (A1): $g^{(m)}_i \geq 0$ implies $g^{(m)}_{i+1} \geq 0$, and (A2): $g^{(m+1)}_i \geq g^{(m)}_i$ for all i. These are the hypotheses that $\tau^{(m)}_{s^*_m}$ is the optimal rule and $s^*_m \geq s^*_{m+1}$. They imply that

$$W^{(m)}_i = U^{(m)}_i, \qquad V^{(m)}_i = \sum_{j>i} (i\pi_j/(j(j-1)\pi_i))U^{(m)}_j \quad \text{for } i \geq s^*_m,$$

and

$$W^{(m)}_i = V^{(m)}_i \quad \text{for } i < s^*_m.$$

Hence,

$$\begin{aligned} W^{(m)}_i - V^{(m)}_i &= (U^{(m)}_i - \sum_{j>i} \frac{i\pi_j}{j(j-1)\pi_i} U^{(m)}_j) I(i \geq s^*_m) \\ &= g^{(m)}_i I(i \geq s^*_m) \qquad \text{for } i = 1, 2, \cdots, \end{aligned} \tag{2.7}$$

where $I(A)$ represents the indicator function of the event A. On the other hand, from (2.5)

$$\begin{aligned} g^{(m+1)}_i &= \sum_{j \geq i} \frac{i\delta_j}{j\pi_i} + \sum_{j>i} \frac{i\pi_j}{j(j-1)\pi_i} W^{(m)}_j - \sum_{j>i} \frac{i\pi_j}{j(j-1)\pi_i} \{\sum_{k \geq j} \frac{j\delta_k}{k\pi_j} + V^{(m)}_j\} \\ &= g^{(1)}_i + \sum_{j>i} \frac{i\pi_j}{j(j-1)\pi_i} \{W^{(m)}_j - V^{(m)}_j\}, \end{aligned}$$

where $g^{(1)}_i = \sum_{j \geq i} (id_j)/(j\pi_i)$. Inserting (2.7) into the above equation,

$$g^{(m+1)}_i = g^{(1)}_i + \sum_{j \geq \max(i+1, s^*_m)} \frac{i\pi_j}{j(j-1)\pi_i} g^{(m)}_j. \tag{2.8}$$

It is convenient for the induction to consider the function $h^{(m)}_i = (\pi_i/i)g^{(m)}_i$ for $i \geq 1$ and $m \geq 1$. The induction hypotheses then reduce to (AR1): $h^{(m)}_i \geq 0$ implies $h^{(m)}_{i+1} \geq 0$, and (AR2): for all $i \geq 1$, $h^{(m+1)}_i \geq h^{(m)}_i$. Note that s^*_m can be written as $s^*_m = \min\{i \geq 1 : h^{(m)}_i \geq 0\}$ and $s^*_m \geq s^*_{m+1}$. Now equation (2.8) reduces to

$$h^{(m+1)}_i = h^{(1)}_i + \sum_{j \geq \max(i+1, s^*_m)} \frac{1}{j-1} h^{(m)}_j, \tag{2.9}$$

where $h^{(1)}_i = \sum_{j \geq i} d_j/j$ and $h^{(m)}_{-1} \equiv -1$ for all $m \geq 1$.

When $m = 1$, since

$$h^{(1)}_{i+1} - h^{(1)}_i = -d_i/i, \tag{2.10}$$

Condition (PS) shows that the differences change from positive to non-positive at most once, so the $h_i^{(1)}$ are unimodal. Since $h_i^{(1)} \to 0$ as $i \to \infty$, hypothesis (AR1) holds. By virtue of (2.9) we find

$$h_i^{(2)} - h_i^{(1)} = \sum_{j \geq \max(i+1, s_1^*)} \frac{1}{j} h_j^{(1)} \geq 0, \tag{2.11}$$

because $h_j^{(1)}$ is non-negative for $j \geq s_1^*$. Hence, the hypothesis (AR2) holds for $m = 1$.

We continue the induction. We first show that the first hypotheses holds with m replaced by $m+1$. If $i + 1 > s_m^*$, then $h_i^{(m)} \geq 0$ from the definition of s_m^*, so $h_i^{(m+1)} \geq 0$ from AR2. Suppose that $i + 1 \leq s_m^*$. Then from (2.9)

$$\begin{aligned} h_i^{(m+1)} - h_{i-1}^{(m+1)} &= h_i^{(1)} + \sum_{j \geq s_m^*} \frac{1}{j-1} h_j^{(m)} - \{h_{i-1}^{(1)} + \sum_{j \geq s_m^*} \frac{1}{j-1} h_j^{(m)}\} \\ &= h_i^{(1)} - h_{i-1}^{(1)} = -d_{i-1}/(i-1). \end{aligned} \tag{2.12}$$

Therefore, we have from the PS condition, that $h_i^{(m+1)}$ is unimodal on $i + 1 \leq s_m^*$ by the argument of (2.10). Since it is positive from $i = s_m^*$ on, the first hypothesis is therefore satisfied. Also, s_{m+1}^* is such that $s_{m+1}^* = \min\{1 \leq i \leq s_m^* : h_i^{(m)} \geq 0\}$.

We now show that the second hypothesis holds with m replaced by $m+1$. From (2.9)

$$\begin{aligned} h_i^{(m+2)} - h_i^{(m+1)} &= \sum_{j \geq \max(i+1, s_{m+1}^*)} \frac{1}{j-1} h_j^{(m+1)} - \sum_{j \geq \max(i+1, s_m^*)} \frac{1}{j-1} h_j^{(m)} \\ &\geq \sum_{j \geq \max(i+1, s_{m+1}^*)} \frac{1}{j-1} \{h_j^{(m+1)} - h_j^{(m)}\} \geq 0, \end{aligned} \tag{2.13}$$

The first inequality follows from $s_m^* \geq s_{m+1}^*$, and the last one follows from the hypothesis (AR2). Hence, the proof is complete.

Poisson, geometric, and uniform distributions satisfy the PS condition (see Presman and Sonin [14]). As an example, we study the uniform distribution in detail.

Uniform case: The total number N of objects is assumed to be uniformly distributed on $[1, N_0]$. Thus for $k = 1, 2, \cdots, N_0$, $\delta_k = 1/N_0$ and $\pi_k = (N_0 - k + 1)/N_0$. Then the condition (PS) is easily verified, since $d_i = (1/N_0)(1 - \sum_{j=i+1}^{N_0} (1/j))$ is increasing in i for $i = 0, 1, \cdots, N_0$. To find approximate formulas for large N_0, We need another modification. Let $H_i^{(m)} = (N_0 \pi_i / i) g_i^{(m)} = N_0 h_i^{(m)}$ for all m. Then from (2.9)

$$H_i^{(m+1)} = H_i^{(1)} + \sum_{j=\max(i+1, s_m^*)}^{N_0} \frac{1}{j} H_j^{(m)}, \tag{2.14}$$

where $H_i^{(1)} = \sum_{j=i}^{N_0} (1/j)(1 - \sum_{k=j+1}^{N_0} (1/k))$. This expression for $H_i^{(1)}$ is a Riemann approximation to an integral. In particular, if we let $N_0 \to \infty$ and $i/N_0 \to x$, we have

$$H_i^{(1)} \to H^{(1)}(x) := \int_x^1 \frac{1}{y}(1 - \int_y^1 \frac{1}{z} dz) dy = -\frac{1}{2} \log^2 x - \log x. \tag{2.15}$$

Define $\bar{s}_1^* = \inf\{x > 0 : H^{(1)}(x) \geq 0\}$. Since $H^{(1)}(x)$ is strictly increasing to its maximum, it is negative to the left of $\bar{s}_1^*$ and positive to the right of $\bar{s}_1^*$. Therefore, $s_1^*/N_0 \to \bar{s}_1^*$ as $N_0 \to \infty$. By induction on m, we have as $N_0 \to \infty$ and $i/N_0 \to x$,

$$H_i^{(m+1)} \to H^{(m+1)}(x) := H^{(1)}(x) + \int_{\max(x,\bar{s}_m^*)}^1 \frac{1}{y} H^{(m)}(y)dy. \tag{2.16}$$

Moreover, each $H^{(m+1)}(x)$ is continuous and increasing on the interval $(0, \bar{s}_m^*]$, so that $s_{m+1}^*/N_0 \to \bar{s}_{m+1}^* := \inf\{x > 0 : H^{(m+1)}(x) \geq 0\}$. In fact, $\bar{s}_m^*$ is a unique solution of the equation $H^{(m)}(x) = 0$ in $[0, \bar{s}_{m-1}^*]$. From (2.15) and (2.16), its value may be expressed as

$$\bar{s}_m^* = \exp\{-(1 + \sqrt{1 + 2C^{(m)}})\}, \tag{2.17}$$

where $C^{(1)} \equiv 0$ and

$$C^{(m)} = \int_{\bar{s}_{m-1}^*}^1 \frac{1}{y} H^{(m-1)}(y)dy. \tag{2.18}$$

Therefore we have $\bar{s}_1^* = e^{-2} \approx .135335$ and

$$C^{(2)} = \int_{e^{-2}}^1 \frac{1}{y}(-\frac{1}{2}\log^2 y - \log y)dy = \frac{2}{3}. \tag{2.19}$$

Then by (2.17),we see $\bar{s}_2^* = e^{-(1+\sqrt{21}/3)} \approx .079856$. Using (2.15) and (2.16), we have

$$H^{(2)}(x) = \begin{cases} -\frac{1}{2}\log^2 x - \log x + \frac{2}{3}, & x \leq e^{-2} \\ \frac{1}{6}\log^3 x - \log x, & x \geq e^{-2}. \end{cases}$$

Inserting $H^{(2)}(x)$ into (2.18),

$$\begin{aligned} C^{(3)} &= \int_{\bar{s}_2^*}^{e^{-2}} \frac{1}{y}(-\frac{1}{2}\log^2 y - \log y + \frac{2}{3})dy + \int_{e^{-2}}^1 \frac{1}{y}(\frac{1}{6}\log^3 y - \log y)dy \\ &= \frac{1}{3} + \frac{7}{27}\sqrt{21}, \end{aligned} \tag{2.20}$$

where we use the relation $-(1/2)\log^2 \bar{s}_2^* - \log \bar{s}_2^* + 2/3 = 0$. By (2.17), we have $\bar{s}_3^* = \exp\{-(1 + (\sqrt{35 + 42\sqrt{21}})/9)\} \approx .04951742$.

Corollary 2. *When the total number of objects has a uniform distribution on* $[1, N_0]$, *the limiting maximum probability of obtaining the best object under the optimal rule for the problem with* m *selections is given by* $-(\bar{s}_1^* \log \bar{s}_1^* + \bar{s}_2^* \log \bar{s}_2^* + \cdots + \bar{s}_m^* \log \bar{s}_m^*)$.

Proof. Let $v_i^{(m)} = (N_0\pi_i/i) = ((N_0 - i + 1)/i)V_i^{(m)}$ and $u_i^{(m)} = (N_0\pi_i/i) = ((N_0 - i + 1)/i)U_i^{(m)}$, then we have

$$v_i^{(m)} = \sum_{j=i+1}^{N_0} \frac{1}{j-1} \max\{u_j^{(m)}, v_j^{(m)}\} \text{ and } u_i^{(m)} = \sum_{j=i}^{N_0} \frac{1}{j} + v_i^{(m-1)}. \tag{2.21}$$

The optimal rule with a threshold s_m^* gives

$$v_i^{(m)} = \begin{cases} \sum_{j=i+1}^{s_m^*-1} \frac{1}{j-1} v_j^{(m)} + \sum_{j=s_m^*}^{N_0} \frac{1}{j} u_j^{(m)}, & i \le s_m^* - 1, \\ \sum_{j=s_m^*}^{N_0} \frac{1}{j} u_j^{(m)}, & i \ge s_m^* - 1. \end{cases}$$

Then we can show

$$v_1^{(m)} = 2v_2^{(m)} = 3v_3^{(m)} = \cdots = (s_m^* - 1) v_{s_m^*-1}^{(m)}. \tag{2.22}$$

Thus the maximum probability, $V_i^{(m)}$, is given by

$$V_1^{(m)} = \frac{1}{N_0} v_1^{(m)} = \frac{s_m^* - 1}{N_0} v_{s_m^*-1}^{(m)}. \tag{2.23}$$

By the same approximation for $v_i^{(m)}$ as $H_i^{(m)}$ of (2.16), we have

$$v^{(m)}(x) = \begin{cases} \int_x^{\bar{s}_m^*} \frac{1}{y} v^{(m)}(y) dy + \int_{\bar{s}_m^*}^1 \frac{1}{y} u^{(m)}(y) dy, & x \le \bar{s}_m^*, \\ \int_{\bar{s}_m^*}^1 \frac{1}{y} u^{(m)}(y) dy, & x \ge \bar{s}_m^*, \end{cases}$$

where

$$u^{(m)}(x) = \int_x^1 \frac{1}{y} dy + v^{(m-1)}(x).$$

From (2.23), the limiting probability is given by $\bar{s}_m^* v^{(m)}(\bar{s}_m^*) (\equiv a^{(m)}, a^{(0)} \equiv 0)$ and so we have

$$a^{(m)} = \bar{s}_m^* \int_{\bar{s}_m^*}^1 \frac{1}{y} u^{(m)}(y) dy. \tag{2.24}$$

On the other hand, since s_m^* is determined by OLA stopping rule, $s_m^* = \min\{i \ge 0 : u_i^{(m)} \ge \sum_{j=i+1}^{N_0} (1/(j-1)) u_j^{(m)}\} = \min\{i \ge 0 : \sum_{j=i}^{N_0} (1/j) + v_i^{(m-1)} \ge \sum_{j=i+1}^{N_0} (1/(j-1)) u_j^{(m)}\}$ $(= \min\{i \ge 0 : H_i^{(m)} \ge 0\})$. Thus $\bar{s}_m^*$ satisfies the equation

$$\int_{\bar{s}_m^*}^1 \frac{1}{y} dy + v^{(m-1)}(\bar{s}_m^*) - \int_{\bar{s}_m^*}^1 \frac{1}{y} u^{(m)}(y) dy = 0. \tag{2.25}$$

Now we know from (2.22) that $v^{(m-1)}(0+) = x v^{(m-1)}(x)$ for $x \in (0, \bar{s}_{m-1}^*]$. Hence

$$a^{(m-1)} = \bar{s}_{m-1}^* v^{(m-1)}(\bar{s}_{m-1}^*) = \cdots = \bar{s}_m^* v^{(m-1)}(\bar{s}_m^*). \tag{2.26}$$

Inserting (2.24) into (2.25) and using (2.26),

$$a^{(m)} = a^{(m-1)} - \bar{s}_m^* \log \bar{s}_m^*,$$

which yields the desired result.

From this corollary, as $N_0 \to \infty$ we see that the maximum probabilities satisfy $W_1^{(1)} \to .270670$, $W_1^{(2)} \to .472509$, and $W_1^{(3)} \to .621329$ for the problem with one, two, and three selections, respectively.

3. Poisson arrival model. Let $\tau_1, \tau_2, \cdots$ denote the arrival times of a Poisson process in chronological order and let $\{N(t)\}_{t \geq 0}$ be the corresponding counting process. For the unknown intensity λ of the process, we suppose a prior exponential distribution, $a \exp\{-a\lambda\} I(\lambda > 0)$ where a is a known nonnegative parameter. The corresponding conditional density given $\tau_i = s$ can be straightforwardly computed and yields $f(\lambda | \tau_i = s) = (\lambda^i / i!)(s + a)^{i+1} \exp\{-(s + a)\lambda\} I(\lambda > 0)$ for $s \in [0, T]$. Bruss (1987) has succeeded in showing that the optimal stopping rule, which maximizes the probability of obtaining the best object in the given time interval $[0, T]$ with single selection, is to accept (if possible) the relatively best object after time $(T + a)/e - a$. Here we consider the Bruss' problem with multiple selections. As is shown in Bruss (1987), the posterior distribution of $N(T)$ generated by $\tau_1, \cdots, \tau_i$ only depends on the values of i and τ_i and equals a negative binomial (Pascal) distribution with parameters $(i, (s + a)/(T + a))$, that is, for $0 \leq s \leq T$,

$$
\begin{aligned}
P(N(T) = n | \tau_1 = t_1, \cdots, \tau_{i-1} = t_{i-1}, \tau_i = s) &= P(N(T) = n | \tau_i = s) \\
&= \binom{n}{i} (\frac{s + a}{T + a})^{i+1} (1 - \frac{s + a}{T + a})^{n-i}.
\end{aligned} \tag{3.1}
$$

Let $W_i^{(m)}(s)$ denote the maximum probability of obtaining the best object when we confront the relatively best object which is the ith object arriving at time s $(0 < s \leq T)$, and we can select more m (≥ 1) objects thereafter. Similarly if m more selections are allowed, let $U_i^{(m)}(s)$ (resp. $V_i^{(m)}(s)$) be the corresponding probability when we accept (resp. reject) the relatively best object, which is the ith object arriving at time s. Using Bruss' result, we have

$$
\begin{aligned}
U_i^{(m)}(s) &= \sum_{n \geq i} (i/n) P(N(T) = n | \tau_i = s) + V_i^{(m-1)}(s) \\
&= \frac{s + a}{T + a} + V_i^{(m-1)}(s).
\end{aligned} \tag{3.2}
$$

Let $p_{(i,s)}^{(k,u)}$ denote the transition probability given prior exponential distribution that $(i+k)$th object arriving at time $s + u$ is the first relatively best object after ith object which is the relatively best arrived at time s, then we have

$$
V_i^{(m)}(s) = \int_0^{T-s} \sum_{k \geq 1} p_{(i,s)}^{(k,u)} W_{i+k}^{(m)}(s + u) du \tag{3.3}
$$

and for $k \geq 1,\ 0 < u < T - s$,

$$
\begin{aligned}
p_{(i,s)}^{(k,u)} &= \int_0^\infty \frac{\lambda e^{-\lambda u} (\lambda u)^{k-1}}{(k-1)!} \frac{i}{(i + k - 1)(i + k)} \frac{e^{-\lambda(s+a)} (\lambda)^i (s + a)^{i+1}}{i!} d\lambda \\
&= \frac{s + a}{(s + a + u)^2} \binom{i + k - 2}{k - 1} (\frac{s + a}{s + a + u})^i (\frac{u}{s + a + u})^{k-1},
\end{aligned} \tag{3.4}
$$

where we apply the equation $\int_0^\infty \lambda^{k+i} \exp\{-\lambda(s + a + u)\} d\lambda = \Gamma(k + i + 1)/(s + a + u)^{k+i+1}$ to the right-hand side of the first equation above. Then we have the dynamic programming equation for $i, m \geq 1, 0 < s \leq T$,

$$
W_i^{(m)}(s) = \max\{U_i^{(m)}(s), V_i^{(m)}(s)\}, \tag{3.5}
$$

with boundary conditions $W_i^{(m)}(T) = 1$ for $i, m \geq 1$ and $W_i^{(0)}(s) = 0$ for all i and s. Let $g_i^{(m)}(s)$ be the *one-stage look-ahead function*, that is,

$$\begin{aligned}
g_i^{(m)}(s) &\equiv U_i^{(m)}(s) - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} U_{i+k}^{(m)}(s+u)du \\
&= \frac{s+a}{T+a} - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} (\frac{s+a+u}{T+a})du \\
&\qquad + \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} \{W_{i+k}^{(m-1)}(s+u) - V_{i+k}^{(m-1)}(s+u)\}du \\
&= (\frac{s+a}{T+a})\{1 + \log(\frac{s+a}{T+a})\} \\
&\qquad + \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} \{W_{i+k}^{(m-1)}(s+u) - V_{i+k}^{(m-1)}(s+u)\}du, \qquad (3.6)
\end{aligned}$$

where we use $\sum_{k\geq 1} p_{(i,s)}^{(k,u)} = (s+a)/(s+a+u)^2$ (independently of i), since $p_{(i,s)}^{(k,u)} = (s+a)/(s+a+u)^2 \times \{$negative binomial distribution with parameters $(k, u/(s+a+u))\}$.

Theorem 3. *The optimal rule for the problem with random arrivals on $[0,T]$ following a Poisson process at intensity $\lambda > 0$ having an exponential distribution with rate parameter $a \geq 0$ when we can select m more objects thereafter is to accept (if possible) the first relatively best object after time $s_m^* = (T+a)/e^{C^{(m)}} - a$ ($s_0^* \equiv T$), where $C^{(m)}$ is constant. Moreover, s_m^* is non-increasing in m*

Proof. Let $h_i^{(m)}(s) = ((T+a)/(s+a))g_i^{(m)}(s)$. As induction hypotheses, we assume that $h_i^{(m)}(s)$ is independent of i and for fixed m

$$h^{(m)}(s) \geq 0 \Rightarrow h^{(m)}(s+u) \geq 0 \quad \text{for} \quad u \in [0, T-s], \qquad (AP1)$$

$h^{(m)}(s)$ for $s \in (0, s_{m-1}^*]$ has the following form,

$$h^{(m)}(s) = C^{(m)} + \log(\frac{s+a}{T+a}), \qquad (AP2)$$

where $C^{(m)}$ is constant, and for all m

$$h^{(m+1)}(s) \geq h^{(m)}(s). \qquad (AP3)$$

These are the hypotheses that $\tau_{s_m^*}^{(m)}$ is the optimal rule and $s_{m-1}^* \geq s_m^* = \inf\{0 < s \leq s_{m-1}^* : h^{(m)}(s) \geq 0\} = (T+a)/e^{C^{(m)}} - a$. They imply that

$$\begin{aligned}
W_{i+k}^{(m)}(s+u) - V_{i+k}^{(m)}(s+u) &= g^{(m)}(s+u)I(s+u \geq s_m^*), \\
&= (\frac{s+u+a}{T+a})h^{(m)}(s+u)I(s+u \geq s_m^*), \qquad (3.7)
\end{aligned}$$

which follows from (3.3) for $s+u \geq s_m^*$

$$\begin{aligned}
W_{i+k}^{(m)}(s+u) &= U_{i+k}^{(m)}(s+u), \text{ and} \\
V_{i+k}^{(m)}(s+u) &= \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} U_{i+k}^{(m)}(s+u)du
\end{aligned}$$

Inserting (3.7) into (3.6),

$$h_i^{(m+1)}(s) = h^{(1)}(s) + (\frac{T+a}{s+a}) \int_{(s_m^*-s)^+}^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} \frac{s+u+a}{T+a} h^{(m)}(s+u)du$$

$$= h^{(1)}(s) + (\frac{T+a}{s+a}) \int_{(s_m^*-s)^+}^{T-s} \frac{s+a}{(s+u+a)^2} \frac{s+u+a}{T+a} h^{(m)}(s+u)du$$

$$= h^{(1)}(s) + \int_{(s_m^*-s)^+}^{T-s} \frac{1}{s+u+a} h^{(m)}(s+u)du (\equiv h^{(m+1)}(s)), \tag{3.8}$$

being independently i, where

$$h^{(1)}(s) = 1 + \log(\frac{s+a}{T+a}), \tag{3.9}$$

which is increasing in s. Therefore $h^{(1)}(s)$ satisfies the hypotheses (AP1) and (AP2) with $C^{(1)} \equiv 1$. Because $h^{(1)}(s)$ is non-negative for $s \geq s_1^*$, by virtue of (3.8)

$$h^{(2)}(s) - h^{(1)}(s) = \int_{(s_1^*-s)^+}^{T-s} \frac{1}{s+u+a} h^{(1)}(s+u)du \geq 0.$$

Thus the hypothesis (AP3) holds for $m = 1$.

To complete the induction, we shall show that these hypotheses hold for m replaced by $m+1$. Recalling (3.8), for $s \leq s_m^* = (T+a)/e^{C^{(m)}} - a$

$$h^{(m+1)}(s) = h^{(1)}(s) + \int_{(T+a)/e^{C^{(m)}}-a-s}^{T-s} \frac{1}{s+u+a} h^{(m)}(s+u)du$$

$$= \log(\frac{s+a}{T+a}) + C^{(m+1)}, \tag{3.10}$$

where

$$C^{(m+1)} = 1 + \int_{e^{-C^{(m)}}}^{1} \frac{1}{v} h^{(m)}((T+a)v - a)dv, \tag{3.11}$$

where we change the variable from $(s+u+a)/(T+a)$ to v in the integrand in (3.10). (3.10) states that (AP2) holds with m replaced by $m+1$ and that $h^{(m+1)}(s)$ is non-decreasing in $s \in (0, s_m^*]$. On the other hand, for $s \in [s_m^*, T]$, $h^{(m+1)}(s)$ is non-negative because by the hypothesis (AP3)

$$0 \leq h^{(m)}(s) \leq h^{(m+1)}(s).$$

Hence we have

$$h^{(m+1)}(s) \geq 0 \Rightarrow h^{(m+1)}(s+u) \geq 0 \quad \text{for} \quad u \in [0, T-s], \tag{3.12}$$

which states that (AP1) holds with m replaced by $m+1$.

Now $h^{(m+2)}(s)$ can be written as

$$h^{(m+2)}(s) = h^{(1)}(s) + \int_{(s_{m+1}^*-s)^+}^{T-s} \frac{1}{s+u+a} h^{(m+1)}(s+u)du. \tag{3.13}$$

Taking the difference in the above equation from (3.8)

$$h^{(m+2)}(s) - h^{(m+1)}(s) \geq \int_{(s^*_{m+1}-s)^+}^{T-s} \frac{1}{s+u+a}\{h^{(m+1)}(s+u) - h^{(m)}(s+u)\}du \geq 0,$$

where the first inequality comes from $s^*_m \geq s^*_{m+1}$ and the second one comes from the hypothesis (AP3). Thus (AP3) holds for all m and the proof is complete.

As shown in the proof, $s^*_1 = (T+a)/e - a$. Then, from (3.11),

$$C^{(2)} = 1 + \int_{e^{-1}}^{1} \frac{1}{v} h^{(1)}((T+a)v - a)dv = 1 + \int_{e^{-1}}^{1} \frac{1}{v}\{1 + \log v\}dv = 1 + \frac{1}{2}. \tag{3.14}$$

So $s^*_2 = (T+a)/e^{3/2} - a$. By virtue of (3.8),

$$h^{(2)}(s) = \begin{cases} \frac{3}{2} + \log(\frac{s+a}{T+a}), & 0 < s \leq s^*_2, \\ 1 - \frac{1}{2}\log^2(\frac{s+a}{T+a}), & s^*_2 \leq s \leq T. \end{cases}$$

Inserting the above into (3.11), for $s \leq s^*_2$

$$C^{(3)} = 1 + \int_{e^{-3/2}}^{e^{-1}} \frac{1}{v}(\frac{3}{2} + \log v)dv + \int_{e^{-1}}^{1} \frac{1}{v}(1 - \frac{1}{2}\log^2 v)dv = 1 + \frac{23}{24}. \tag{3.15}$$

Thus $s^*_3 = (T+a)/e^{47/24} - a$.

For $a \to 0$, it is of interest to compare the values $s^*_1 = T/e \approx .367879T$, $s^*_2 = T/e^{3/2} \approx .22313T$, $s^*_3 = T/e^{47/24} \approx .141093T$, with the threshold values $n/e \approx .367879n$, $n/e^{3/2} \approx .22313n$, $n/e^{47/24} \approx .141093n$, of the well-known no-information case, where $T^{-1}s^*_k = e^{-1}, e^{-3/2}, e^{-47/24}, e^{-2761/1152}$ for $k = 1, 2, 3, 4$ respectively are the same as in the non-informative prior $\Delta_0(\lambda) \equiv 1$ case (Bruss(1987)),

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