POSITIVE IMPLICATIVE HYPERK-IDEALS

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Abstract. In this manuscript we define the notions of positive implicative hyper$K$-ideals of types 1,2,3 and 4. Then by given many examples we show that these notions are different. After that we state and prove some theorems which determine the relation between these notions. Also by defining the concept of scalar element and additive condition we obtain another results. Finally we give a theorem which states that where the image and the inverse image of a positive implicative hyper$K$-ideals are also positive implicative hyper$K$-ideals under a homomorphism of hyper$K$-algebras.

1. Introduction

The hyper algebraic structure theory was introduced by F. Marty in 1934 [7]. Imai and Iseki in 1966 [3] introduced the notion of a BCK-algebra. Recently [1,6,9] Borzoei, Jun and Zahedi et al applied the hyper structures to BCK-algebras and introduced the concept of hyper$K$-algebra which is a generalization of BCK-algebra. Now we follow [1,9] and obtain some results, which are mentioned in the abstract.

2. Preliminaries

Definition 2.1. [1]. Let H be a nonempty set and "$\circ$" be a hyper operation on H, that is $\circ$ is a function from $H \times H$ to $P^*(H) = P(H) - \{\emptyset\}$. Then H is called a hyper$K$-algebra iff it contains a constant "0" and satisfies the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$
(HK3) $x < x$
(HK4) $x < y, y < x \implies x = y$
(HK5) $0 < x$,
for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

If H is a hyper$K$-algebra with the hyperopration "$\circ$" and constant "0", then we show it by $(H, \circ, 0)$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{a \in A, b \in B} a \circ b$ of H.

Example 2.2. (1) Let $(H, *, 0)$ be a BCK-algebra and define a hyperoperation "$\circ$" on H by $x \circ y = \{x * y\}$ for all $x, y \in H$. Then $(H, \circ, 0)$ is a hyper$K$-algebra.

(2) Let $n \in N$. Define the hyper operation "$\circ$" on $H = [n, +\infty)$ as follows:

$$xoy = \begin{cases} [n, x] & \text{if } x \leq y \\ (n, y] & \text{if } x > y \neq n \\ \{x\} & \text{if } y = n \end{cases}$$

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for all $x, y \in H$. Then $(H, \circ, n)$ is a hyper$K$-algebra.

Theorem 2.3[1]. Let $(H, \circ, 0)$ be a hyper$K$-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H the following statements hold:
(i) $(A \circ B) \circ C = (A \circ C) \circ B$,
(ii) $x \circ y < z \Leftrightarrow x \circ z < y$,
(iii) $A \circ B < C \Leftrightarrow A \circ C < B$,
(iv) $(x \circ z) \circ (x \circ y) < y \circ z$,
(v) $(A \circ C) \circ (B \circ C) < A \circ B$,
(vi) $A \subseteq B$ implies $A < B$,
(vii) $x \circ y < x$,
(viii) $A \circ B < A$,
(ix) $x \in x \circ 0$,
(x) $0 \in x \circ y \Leftrightarrow 0 \in (x \circ y) \circ 0$

Definition 2.4.[1]. Let I be a nonempty subset of a hyper$K$-algebra $(H, \circ, 0)$. Then I is called a *weak hyperK-ideal* of H if
(WHKI1) $0 \in I$
(WHKI2) $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.5.[1]. Let I be a nonempty subset of a hyper$K$-algebra $(H, \circ, 0)$. Then I is said to be a *hyperK-ideal* of H if
(HKI1) $0 \in I$,
(HKI2) $x \circ y < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Theorem 2.6[1]. Let $(H, \circ, 0)$ be a hyper$K$-algebra and let I be a hyper$K$-ideal of H. Then I is a weak hyper$K$-ideal of H.

Theorem 2.7[9]. If $\{I_i | i \in I\}$ is a family of hyper$K$-ideals of a hyper$K$-algebra H, then $\bigcap_{i \in I} I_i$ is a hyper$K$-ideal of H.

Definition 2.8. Let H be a hyper$K$-algebra. An element $a \in H$ is called to be a *left(resp. right) scalar* if $|a \circ x| = 1$(resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar,we say that a is an *scalar element.*

Definition 2.9.[9]. Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be two hyper$K$-algebras and $f : H_1 \longrightarrow H_2$ be a function. Then f is said to be a homomorphism iff
(i) $f(0_1) = 0_2$
(ii) $f(x \circ_1 y) = f(x) \circ_2 f(y), \quad \forall x, y \in H_1$.
If f is 1-1 (onto) we say that f is a monomorphism (epimorphism) and if f is both 1-1 and onto, we say that f is an isomorphism. Also we get $\ker f = f^{-1}(0_2)$.

Definition 2.10. Let A be a nonempty subset of H. By the hyper$K$-ideal *generated* by A, which is written by $< A >$, we mean the intersection of all hyper$K$-ideals of H containing A. If $A = \{a\}$, then we write $< a >$ instead of $< A >$.

Definition 2.11. A nonempty subset I of H is called *proper*, if $I \neq \{0\}$ and $I \neq H$.
Note: From now on in this paper we let H is a hyper$K$-algebra.

3. positive implicative hyper$K$-ideals

Definition 3.1. Let I be a nonempty subset of H such that $0 \in I$. Then I is said to be a *positive implicative hyperK-ideal* of
(i) type 1, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply that $x \circ z \subseteq I$,

(ii) type 2, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ imply that $x \circ z \subseteq I$,
(iii) type 3, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ imply that $x \circ z \subseteq I$,
(iv) type 4, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ imply that $x \circ z < I$.

Example 3.2.(i) Let H be the hyper$K$-algebra of Example 2.2(1). If I is a positive implicative ideal of BCK-algebra $(H, *, 0)$, then I is a positive implicative hyper$K$-ideal of type 1,2,3 and 4 of hyper$K$-algebra H.

(ii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

Clearly $I_2 = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 1. But $I_1 = \{0, 1\}$ is not, since $(2 \circ 1) \circ 0 = \{0, 1\} \subseteq I_1$, $1 \circ 0 = \{1\} \subseteq I_1$ and $2 \circ 0 = \{2\} \not\subseteq I_1$

(iii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

It can be checked that $I_1 = \{0, 1\}$ is a positive implicative hyper$K$-ideal of type 2. But $I_2 = \{0, 2\}$ is not, since $(1 \circ 2) \circ 0 < I_2, 2 \circ 0 \subseteq I_2$ and $1 \circ 0 \not\subseteq I_2$.

(iv) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

Now we can check that $I_2 = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 3. But $I_1 = \{0, 1\}$ is not, since $(2 \circ 1) \circ 0 < I_1, 1 \circ 0 < I_1$ and $2 \circ 0 \not\subseteq I_1$.

(v) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

It is easy to see that $I_2 = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 4. But $I_1 = \{0, 1\}$ is not, since $(2 \circ 1) \circ 0 = \{1, 2\} < I_1$, $1 \circ 0 = \{1\} < I_1$ and $2 \circ 0 = \{2\} \not< I_1$.

Theorem 3.3. Let $0 \in H$ be a right scalar element. If I is a positive implicative hyper$K$-ideal of type 1, then I is a weak hyper$K$-ideal of H.

Proof. Let $x, y \in H$, $x \circ y \subseteq I$ and $y \in I$. Since $0 \in H$ is a right scalar element, then $(x \circ y) \circ 0 = x \circ y \subseteq I$ and $y \circ 0 = \{y\} \subseteq I$. Thus $x \in \{x\} = x \circ 0 \subseteq I$. Therefore I is a weak hyper$K$-ideal of H.

Example 3.4.(i) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

We see that the $0 \in H$ is not a right scalar element and $I = \{0,2\}$ is a positive implicative hyper$K$-ideal of type 1. But I is not a weak hyper$K$-ideal, since $1 \circ 2 \subseteq I, 2 \in I$ and $1 \notin I$.

(ii) Let $H = \{0,1,2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

Clearly that $0 \in H$ is a right scalar element. Moreover $I = \{0,2\}$ is a weak hyper$K$-ideal of H, but it is not a positive implicative hyper$K$-ideal of type 1.

Definition 3.5. H is called to be a positive implicative hyper$K$-algebra, if it satisfies the following condition,

$$(x \circ z) \circ (y \circ z) = (x \circ y) \circ z$$

for all $x, y, z \in H$.

Example 3.6.(i) Let $H = \{0,1,2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

Then $(H, \circ, 0)$ is a positive implicative hyper$K$-algebra.

(ii) Consider Example 3.4(i). Since,

$$(2 \circ 0) \circ (1 \circ 0) = \{0,1,2\} \neq \{1,2\} = (2 \circ 1) \circ 0$$

then H is not a positive implicative hyper$K$-algebra.

Theorem 3.7. Let H be a positive implicative hyper$K$-algebra. Then any weak hyper$K$-ideal of H is a positive implicative hyper$K$-ideal of type 1.

Proof. Let I be a weak hyper$K$-ideal of H and let $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ for $x, y, z \in H$. Since $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z \subseteq I, y \circ z \subseteq I$ and I is a weak hyper$K$-ideal of H, then we get that $x \circ z \subseteq I$. Therefore I is a positive implicative hyper$K$-ideal of type 1.

Corollary 3.8. Let H be a positive implicative hyper$K$-algebra, such that $0 \in H$ is a right scalar element and I is a nonempty subset of H. Then I is a positive implicative hyper$K$-ideal of type 1 iff I is a weak hyper$K$- ideal of H.

Theorem 3.9. Let I be a nonempty subset of H. Then the following statements hold:

(i) If I is a positive implicative hyper$K$-ideal of type 2, then I is a positive implicative hyper$K$-ideal of type 1.

(ii) If I is a positive implicative hyper$K$-ideal of type 3, then I is a positive implicative hyper$K$-ideal of type 2 and 4.

Proof.(i) Let $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ for $x, y, z \in I$. Then $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$. So by hypothesis we get that $x \circ z \subseteq I$. Therefore I is a positive implicative hyper$K$-ideal of type 1.

(ii) The proof is similar to the proof of (i).

Example 3.10.(i) Consider Example 3.4(i). Then $I = \{0,1\}$ is a positive implicative hyper$K$-ideal of type 1. But it is not of type 2, since $(1 \circ 0) \circ 0 = \{1,2\} < \{0,1\} = I$, $0 \circ 0 = \{0,1\} \subseteq I$ and $1 \circ 0 = \{1,2\} \not\subseteq I$.

(ii) The following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Then $I = \{0,2\}$ is a positive implicative hyper$K$-ideal of type 2. But it is not a positive implicative hyper$K$-ideal of type 3, since $(2 \circ 2) \circ 0 = \{0,1,2\} < I$, $2 \circ 0 = \{1,2\} < I$ and $2 \circ 0 = \{1,2\} \not\subseteq I$.

(iii) Consider Example 3.2(v). Then $I = \{0,2\}$ is a positive implicative hyper$K$-ideal of type 4, but it is not a positive implicative hyper$K$-ideal of type 3, since $(2 \circ 1) \circ 1 < I$, $1 \circ 1 < I$ and $2 \circ 1 \not\subseteq I$

Theorem 3.11. Let I be a nonempty subset of H and $0 \in H$ is a right scalar element. If I is a positive implicative hyper$K$-ideal of type 2 or 3, then I is a hyper$K$-ideal of H.

Proof. The proof is similar to the proof of Theorem 3.3.

Example 3.12.(i) Let $H = \{0,1,2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

We see that $0 \in H$ is a right scalar element and $I = \{0,2\}$ is a hyper$K$-ideal of H. But I is not a positive implicative hyper$K$-ideal of type 2, since $(2 \circ 0) \circ 2 = \{0,1,2\} < I$, $0 \circ 2 = \{0\} \subseteq I$ and $2 \circ 2 = \{0,1,2\} \not\subseteq I$.

(ii) The following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Clearly $0 \in H$ is not a right scalar element and $I = \{0,2\}$ is a positive implicative hyper$K$-ideal of type 2. But I is not a hyper$K$-ideal of H, since $1 \circ 2 = \{1,2\} < I$, $2 \in I$ and $1 \notin I$.

(iii) Consider the following table which shows that a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

Then $0 \in H$ is a right scalar element and $I = \{0,1\}$ is a hyper$K$-ideal of H. But it is not a positive implicative hyper$K$-ideal of type 3.

(iv) Consider the following table which shows that a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Then $0 \in H$ is not a right scalar element and $I = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 2. But it is not a hyper$K$-ideal, since $1 \circ 2 < I$, $2 \in I$ and $1 \notin I$.

Definition 3.13. Let I be a nonempty subset of H. Then we say that I satisfies the *additive condition*, if $x < y$ and $y \in I$ implies that $x \in I$, for all $x, y \in H$.

Example 3.14. Consider Example 3.2(iii). Then $I_1 = \{0, 1\}$ satisfies the additive condition. But $I_2 = \{0, 2\}$ does not satisfy the additive condition, since $1 < 2$, $2 \in I_2$ and $1 \notin I_2$.

Theorem 3.15. Let I be a positive implicative hyper$K$-ideal of type 4 and satisfies the additive condition. Then I is a hyper$K$-ideal of H.

Proof. Let $x \circ y < I$ and $y \in I$ for $x, y \in H$. By Theorem 2.3(ix), $(x \circ y) \circ 0 < I$ and $y \circ 0 < I$. Since I is a positive implicative hyper$K$-ideal of type 4, then $x \circ 0 < I$. Thus there is $b \in I$ such that $x \circ 0 < b$. By Theorem 2.3(ii), $x \circ b < 0$ and so there is $a \in x \circ b$ such that $a < 0$. By (HK5) and (HK4) we have $a = 0$. Therefore $0 \in x \circ b$ and hence $x < b$. Since I satisfies the additive condition and $b \in I$, we get that $x \in I$. So I is a hyper$K$-ideal of H.

Example 3.16.(i) Consider Example 3.2(v). Then $I = \{0\}$ is a hyper$K$-ideal of H and it satisfies the additive condition. But I is not a positive implicative hyper$K$-ideal of type 4, since $(2 \circ 1) \circ 1 < \{0\}, 1 \circ 1 < \{0\}$ and $2 \circ 1 \not< \{0\}$.

(ii) Consider Example 3.6(i). Then $I = \{0, 1\}$ is a positive implicative hyper$K$-ideal of type 4 and dose not satisfies the additive condition and it is not a hyper$K$-ideal of H, since $1 \circ 2 = \{0\} < I$, $2 \in I$ and $1 \notin I$. Therefore the additive condition in Theorem 3.15 is necessary.

Corollary 3.17. Let H be a positive implicative hyper$K$-algebra, which $0 \in H$ is a right scalar element. If I is a positive implicative hyper$K$-ideal of type 4 such that it satisfies the additive condition then, I is a positive implicative hyper$K$-ideal of type 1.

Proof. By Theorem 3.15, I is a hyper$K$-ideal of H. By Theorem 2.6, I is a weak hyper$K$-ideal of H. Thus by Theorem 3.7, I is positive implicative hyper$K$-ideal of type 1.

Theorem 3.18. Let $f : H_1 \to H_2$ be a homomorphism of hyper$K$-algebras. Then

(i) If J is a positive implicative hyper$K$-ideal of type 1(resp. 2,3,4) of H_2, then $f^{-1}(J)$ is also a positive implicative hyper$K$-ideal of type 1(resp. 2,3,4) of H_1.

(ii) Let f be onto and $ker f \subseteq I$. Then

(a) If I is a positive implicative hyper$K$-ideal of type 1 of H_1 and I be a hyper$K$-ideal of H_1, then $f(I)$ is a positive implicative hyper$K$-ideal of type 1 of H_2.

(b) If $0 \in H_1$ is a right scalar element and I is a positive implicative hyper$K$-ideal of type 2(type 3) of H_1, then $f(I)$ is a positive implicative hyper$K$-ideal of type 2(type 3) of H_2.

(c) If I is a positive implicative hyper$K$-ideal of type 4 of H_1 and I satisfies the additive condition, then $f(I)$ is a positive implicative hyper$K$-ideal of type 4 of H_2.

Proof.(i) The proof is straightforward.

(ii)(a) Let $(x \circ y) \circ z \subseteq f(I)$ and $y \circ z \subseteq f(I)$. Since f is onto, then there are $x, y, z \in H_1$ such that $f(x_1) = x, f(y_1) = y$ and $f(z_1) = z$. Thus

$$f((x_1 \circ y_1) \circ z_1) = (f(x_1) \circ f(y_1)) \circ f(z_1) = (x \circ y) \circ z \subseteq f(I)$$

Let $a \in (x_1 \circ y_1) \circ z_1$. Then $f(a) \in f(I)$. Hence there is $b \in I$ such that $f(a) = f(b)$. Since $0 \in f(a) \circ f(b) = f(a \circ b)$, then there is $t \in a \circ b$ such that $0 = f(t)$. Thus $t \in kerf \subseteq I$, hence $a \circ b < I$. Now $b \in I$ and I is a hyper$K$-ideal of H_1, imply that $a \in I$. Therefore $(x_1 \circ y_1) \circ z_1 \subseteq I$. Similarly , we can get $y_1 \circ z_1 \subseteq I$. Since I is a positive implicative hyper$K$-ideal of type 1 of H_1 then $x_1 \circ z_1 \subseteq I$. So,

$$x \circ z = f(x_1) \circ f(z_1) = f(x_1 \circ z_1) \subseteq f(I)$$

Therefore $f(I)$ is a positive implicative hyper$K$-ideal of type 1 of H_2

The proof of (ii)(b) and (ii)(c) is nearly similar to the proof of (ii)(1) by imposing the suitable modifications.

Example 3.19(i) Let $H_1 = \{0, 1, 2\}$ and $H_2 = H_3 = \{0, 1, 2, 3\}$. Consider the following tables:

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

$\circ_2$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{0, 1\}$	$\{3\}$	$\{0, 1, 3\}$

$\circ_3$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{1, 2\}$	$\{0, 1\}$	$\{0, 2\}$

Then $(H_1, \circ_1, 0)$, $(H_2, \circ_2, 0)$ and $(H_3, \circ_3, 0)$ are hyper$K$-algebras. Let $f_1 : H_1 \to H_2$ and $f_2 : H_1 \to H_3$ are defined as follows:

$$f_1(x) = \begin{cases} 1 & \text{if } x = 2 \\ 2 & \text{if } x = 1 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f_2(x) = x \quad , \quad \forall x \in H_1$$

Then f_1 and f_2 are homomorphism, but are not onto. Moreover, $I = \{0, 1\}$ is a positive implicative hyper$K$-ideal of type 1,2,3,4 of H_1 and $kerf_1 = kerf_2 \subseteq I$. But $f_1(I)$ is not a positive implicative hyper$K$-ideal of type 1,2,3 and $f_2(I)$ is not a positive implicative hyper$K$-ideal of type 4.

Theorem 3.20. If $\{I_i | i \in I\}$ is a family of positive implicative hyper$K$-ideals of type 1,2,3 or 4 , then $\bigcap_{i \in I} I_i$ is also a positive implicative hyper$K$-ideals of type 1,2,3 or 4, respectively.

Theorem 3.21. Let I be a nonempty subset of H. Then

(i) I is a positive implicative hyper$K$-ideal of type 1 if and only if, for all $a \in H, I_a = \{x \in H : x \circ a \subseteq I\}$ is a weak hyper$K$-ideal of H.

(ii) Let I be a positive implicative hyper$K$-ideal of type 2, then for all $a \in H, I_a = \{x \in H : x \circ a \subseteq I\}$ is a hyper$K$-ideal of H.

Proof. (i) Let $x, y, a \in H, x \circ y \subseteq I_a$ and $y \in I_a$. Thus $(x \circ y) \circ a \subseteq I$ and $y \circ a \subseteq I$. Since I is of type 1, then $x \circ a \subseteq I$ and so $x \in I_a$. Therefore I_a is a weak hyper$K$-ideal of H.
Conversely, let $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ for $x, y, z \in H$. Then $x \circ y \subseteq I_z$ and $y \in I_z$.

Since I_z is a weak hyper$K$-ideal of H, then $x \in I_z$ and so $x \circ z \subseteq I$. Thus I is a positive implicative hyper$K$-ideal of type 1.

(ii) Let $x, y, z \in H, x \circ y < I_a$ and $y \in I_a$. Then, there are $z \in x \circ y$ and $w \in I_a$ such that $0 \in z \circ w$. Since $w \circ a \subseteq I$, then

$$0 \in 0 \circ a \subseteq (z \circ w) \circ a \subseteq ((x \circ y) \circ w) \circ a$$

This implies that $((x \circ y) \circ w) \circ a < I$. Thus there is $d \in x \circ y$ such that $(d \circ w) \circ a < I$. Since $w \circ a \subseteq I$ and I is a positive implicative hyper$K$-ideal of type 2, then $d \circ a \subseteq I$. Thus $(x \circ y) \circ a < I$. Now since $y \circ a \subseteq I$ we get that $x \circ a \subseteq I$ and so $x \in I_a$. Therefore I_a is a hyper$K$-ideal of H.

Theorem 3.22. Let I be a nonempty subset of H. Then I is a positive implicative hyper$K$-ideal of type 4 if and only if, for all $a \in H$, $I_a^< = \{x \in H : x \circ a < I\}$ is a least hyper$K$-ideal of H containing $I \cup \{a\}$, that is $I_a^< =< I \cup \{a\} >$.

Proof.($\Rightarrow$) Let $a \in H$. Then if we do similar to the proof of Theorem 3.21, by considering the suitable changes, we see that $I_a^<$ is a hyper$K$-ideal of H. Since $a \circ a < I$ then $a \in I_a^<$. If $x \in I$, then by Theorem 2.3(vii), we get that $x \circ a < x$ and this implies that $x \circ a < I$. So $x \in I_a^<$ and hence $I \subseteq I_a^<$. Now, let J be a hyper$K$-ideal of H containing $I \cup \{a\}$. Let $x \in I_a^<$. Then $x \circ a < I$. Since $I \subseteq J$, we have $x \circ a < J$. Thus $a \in J$ implies that $x \in J$, that is $I_a^< \subseteq J$.

($\Leftarrow$) Let $(x \circ y) \circ z < I$ and $y \circ z < I$, for $x, y, z \in H$. Then $x \circ y < I_z^<$ and $y \in I_z^<$. Since $I_z^<$ is a hyper$K$-ideal of H and $y \in I_z^<$, then $x \in I_z^<$. Hence $x \circ z < I$ and so I is a positive implicative hyper$K$-ideal of type 4.

Definition 3.23. Let $a \in H$. We define the subset $(a]$ of H as follows:

$$(a] = \{x \in H \ : \ x < a\}$$

Note that it is clear that $\{0, a\} \subseteq (a]$.

Theorem 3.24. The following conditions on H are equivalent:

(i) $\{0\}$ is a positive implicative hyper$K$-ideal of type 4,

(ii) $(a]$ is a hyper$K$-ideal of H, for all $a \in H$.

Proof. $(i) \Rightarrow (ii)$ Let $\{0\}$ be a positive implicative hyper$K$-ideal of type 4. Then by Theorem 3.22, for all $a \in H$, $\{0\}_a^<$ is a hyper$K$-ideal of H . But,

$$\{0\}_a^< = \{x \ : \ x \circ a < \{0\}\} = \{x \ : \ x < a\} = (a] \quad , \quad (1)$$

Therefore for all $a \in H, (a]$ is a hyper$K$-ideal of H.

$(ii) \Rightarrow (i)$ Let for all $a \in H, (a]$ is a hyper$K$-ideal of H. By (1), $\{0\}_a^< = (a]$. Then for all $a \in H, \{0\}_a^<$ is a hyper$K$-ideal of H containing $\{a\}$. So by the proof of ($\Leftarrow$) in Theorem 3.22, $\{0\}$ is a positive implicative hyper$K$-ideal of type 4.

Theorem 3.25. Let A be a nonempty subset of H and let $x \in H$ be such that $(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n < \{0\}$, for some $a_1, a_2, \cdots, a_n \in A$. Then $x \in < A >$, i.e.,

$$< A > \supseteq \{x \in H : \ (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n < \{0\} \text{ for some } a_1, a_2, \cdots, a_n \in A\}.$$

In particular, if $\{0\}$ is a positive implicative hyper$K$-ideal of type 4 and $a \in H$, then

$$< a > = \{x \in H : \ (\cdots((x \circ \underbrace{a) \circ a) \circ \cdots) \circ a}_{\text{n times}} < \{0\} \text{ for some } n \in \mathbb{N}\}.$$

Proof. Assume that $x \in H$ satisfies the following inequality

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n < \{0\}$$

for some $a_1, a_2, \cdots, a_n \in A$. Thus there is

$$a \in (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_{n-1},$$

such that $a \circ a_n < \{0\}$ and hence $a \circ a_n < < A >$, so that we have $a \in < A >$. Therefore

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_{n-1} < < A >,$$

By continuing this process, we can conclude that $x \in < A >$.

Now,let $\{0\}$ be a positive implicative hyper$K$-ideal of type 4 and let

$$B = \{x \in H : \quad (\cdots((x \circ \underbrace{a) \circ a) \circ \cdots) \circ a}_{\text{n times}} < \{0\} \text{ for some } n \in \mathbb{N}\}.$$

Since by the above satement we have $B \subseteq < a >$, it is enough to show that $< a > \subseteq B$. To show this, first we show that $< a > = (a]$. By Theorem 3.24, $(a]$ is a hyper$K$-ideal of H. Thus we show that $(a]$ is a least hyper$K$-ideal of H containing a. Let I be a hyper$K$-ideal of H and $a \in I$. Let $x \in (a]$, then $x < a$. Since $0 \in x \circ a$ and $0 \in I$, then $x \circ a < I$ and so $x \in I$. Hence $(a] \subseteq I$.Thus $(a] = < a >$. Now, let $x \in < a >$. Then $x \in (a]$ and so $x < a$. Thus $x \circ a < 0$ and this implies that $x \in B$.

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