THREE-PERSON STOPPING GAMES UNDER WINNING PROBABILITY MAXIMIZATION AND PLAYERS' UNEQUALLY WEIGHTED PRIVILEGE

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ABSTRACT. Three-person n-stage optimal stopping game where players have unequally weighted privilege and their purpose is to maximize their own winning probability (WP) is investigated and an explicit but informal solution is obtained. A distinguishing feature of this game model is the fact that players have their own weights by which at each stage player's desired decision may be taken away by an opponent as an outcome of drawing a lottery. It is shown that even in a game where players are "dictator" and "subject", there exists an equilibrium strategy-triple by adopting which the "subject" improves his disadvantage as n increases. For instance, in the $\langle \frac{1}{2}, \frac{1}{2}, 0 \rangle$-weight game 0-weight player gets $1 - 2\sqrt{2}/3$; 0.0572, for $n = 3$, and 0.122, if $n \to \infty$.

1. The Problem

We first state the three-person game we shall discuss in this paper as follows:

(1^0) There are three persons I, II, III. These players have their weights w_1, w_2 and w_3 respectively. Let $1 \geq w_1 \geq w_2 \geq w_3 \geq 0, w_1 + w_2 + w_3 = 1$, and $w_{(i,j)} \equiv w_i/(w_i + w_j), i \neq j$. (0/0 is interpreted as $\frac{1}{2}$) Players observe a sequence of n i.i.d. random variables $\{X_t\}_{t=1}^n$ sequentially one-by-one, each r.v. obeying uniform distribution on $0 \leq x \leq 1$.

(2^0) Observing each X_t players choose simultaneously and independently either to accept (A) or reject (R) the X_t. If three players' choice is A-A-A, then player I (II, III) accepts X_t with probability $w_1 (w_2, w_3)$ and drops out from the play thereafter. The two players remained continue their two-person game with their "revised" new weights. If three players' choice is R-A-A, then II (III) accepts X_t with probability $w_{(2,3)} (w_{(3,2)})$ dropping out from the game and the remaining players III (II) and I continue their two-person game with their revised new weights. If three-players' choice is R-R-A, then III accepts X_t and drops out and his opponents I and II continue the remaining two-person game. If players' choice-triple is R-R-R, then X_t is rejected and the players face the next X_{t+1}. In cases of other four choice-triples A-R-A, A-A-R, R-A-R and A-R-R, the game is played similarly as mentioned above.

(3^0) A player wins if he accepts a r.v. that is larger that those accepted by his opponents, or if all of his opponents fail to accept any r.v. The purpose of each player is to find the strategy that maximizes the probability of his winning.

Define state $(x \mid n)$ to mean that (1) three players remain in the game, and (2) there remain n r.v.s to be obverved and the players currently face the first observation $X_1 = x$. Let $W_n^i,\ i = 1, 2, 3$, be the value of the game for player i, for the n-problem.

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The statement of the problem in dynamic programming framework is as follows. Denote, by $V_n(w_{(i,j)}, x)$, the value for player i in the two-person game against j, with weights $w_{(i,j)}$ for i, and $w_{(j,i)}$ for j, under the condition that player $k(\neq i,j)$, has already dropped out from the game by accepting a past observation x. In state $(x \mid n)$, players face a trimatrix game with the payoff matrix $M_n(x)$, which is

$$M_n(x) = \begin{cases} M_{n,R}(x), & \text{if R is chosen by III} \\ M_{n,A}(x), & \text{if A is chosen by III} \end{cases} \tag{1.1}$$

where

$$M_{n,R}(x) = (I) \left\{ \begin{array}{c|c|c|} & \multicolumn{2}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{(II)}} \\ & \text{R} & \text{A} \\ \hline \text{R} & W^1, W^2, W^3 & V(w_{13}), x^{n-1}, V(w_{31}) \\ \hline \text{A} & x^{n-1}, V(w_{23}), V(w_{32}) & \begin{array}{c} w_{12}x^{n-1} + w_{21}V(w_{13}), \\ w_{12}V(w_{23}) + w_{21}x^{n-1}, \\ w_{12}V(w_{32}) + w_{21}V(w_{31}) \end{array} \\ \hline \end{array} \right.$$

and

$$M_{n,A}(x) = \begin{array}{c|c|c|} & \text{R} & \text{A} \\ \hline \text{R} & V(w_{12}), V(w_{21}), x^{n-1} & \begin{array}{c} w_{23}V(w_{13}) + w_{32}V(w_{12}), \\ w_{23}x^{n-1} + w_{32}V(w_{21}), \\ w_{23}V(w_{31}) + w_{32}x^{n-1} \end{array} \\ \hline \text{A} & \begin{array}{c} w_{13}x^{n-1} + w_{31}V(w_{21}), \\ w_{13}V(w_{23}) + w_{31}V(w_{21}), \\ w_{13}V(w_{32}) + w_{31}x^{n-1} \end{array} & \begin{array}{c} w_1x^{n-1} + w_2V(w_{13}) + w_3V(w_{12}), \\ w_1V(w_{23}) + w_2x^{n-1} + w_3V(w_{21}), \\ w_1V(w_{32}) + w_2V(w_{31}) + w_3x^{n-1} \end{array} \\ \hline \end{array}$$

In these two matrices the subscript $n-1$ of W^i and V, and x inside $V(\cdot)$ are omitted. Also $w_{(i,j)}$ are rewritten as w_{ij}. The Optimality Equation is

$$(W_n^1, W_n^2, W_n^3) = E[eq.val.M_n(X)] \quad (n \geq 1; W_0^i = V_0(w_{ij}) = 0, \forall i, j) \tag{1.2}$$

provided the eq. value of $M_n(x)$ exists uniquely.

In Sakaguchi and Hamada [9] the authors investigated the game with the equal-weight i.e. $w =< \frac{1}{3}, \frac{1}{3}, \frac{1}{3} >$ case, and an explicit solution is obtained under a somewhat convenient assumption. This assumption will be used again in the present paper.

Three-person unequal-weight games are more difficult that the equal-weight games to derive a solution even in the case with weighs $< \frac{1}{2}, \frac{1}{2}, 0 >$ and $< 1, 0, 0 >$. In Section 3 and 4 we investigate these two unequal-weight games. Denote by $G_n^{(3)}$ and $H_n^{(3)}$ n-stage sequential games with weights $< \frac{1}{2}, \frac{1}{2}, 0 >$ and $< 1, 0, 0 >$, respectively.

The problems studied in the works [8, 9, 10] belong to a class of the best-choice problems combined with sequential games. Two and three-person optimal stopping games where players have weighted privilege under full-information (FI) and expected net value (ENV) maximization are investigated in Sakaguchi [8]. The problems under WP-maximization with both of FI and NI (no-information) are studied in Sakaguchi and Hamada [9]. And those problem under WP-maximization with NI, where players' aims are selecting-best of r.v.s is treated in Ramsey and Szajowski [10].

Recent works related to these area of problems are [1] $\sim$ [5]. Also recent looks for secretary problems and optimal stopping games in various phases and their extentions can be found in Samuels [6] and Sakaguchi [7], respectively.

2. Lemmas.

In games $G_n^{(3)}$ and $H_n^{(3)}$, players drop out from the game one by one as the game goes on. If one player drops out from the game firstly by accepting an r.v. $X = x$, then the remaining two players never accept any r.v. smaller that x. If player $\left\{\begin{array}{c}\text{I or II}\\ \text{II or III}\end{array}\right\}$ drops out firstly from the game $\left\{\begin{array}{c}G_n^{(3)}\\ H_n^{(3)}\end{array}\right\}$, by accepting r.v. $X = x$, then there is left a two-person $< 1, 0 >$-weight m-stage sequential game denoted by $\left\{\begin{array}{c}G_{m,x}^{(2)}\\ H_{m,x}^{(2)}\end{array}\right\}$, $(2 < m < n)$, under the restriction that any r.v. smaller that x should be rejected by both players.

Let us denote, by $(V_{m,x}^D, V_{m,x}^S)$, the equilibrium values in these two-person $< 1, 0 >$ $-$weight games. The superscripts D and S mean “dictator” (i.e. 1-weight) and “subject” (i.e. 0-weight), respectively. The purpose of Section 2 is to discuss about the behaviors of $V_{m,x}^D$ and $V_{m,x}^S$ as functions of $x \in [0, 1]$.

Here we make an important assumption.

Assumption A. *In the two-person* $< w, \overline{w} >$ $-$*weight,* $(\frac{1}{2} \le w \le 1)$, m*-stage sequential game under the restriction that any r.v. smaller that* $x \in (0, 1)$ *shold be rejected, the players must choose A-A at the earliest r.v. that is larger than* x.

For the equal-weight case, symmetry in the players' role yields the value $V_{m,x} = \frac{1}{2}(1-x^m)$ (see[9; Section 3.2]). For the unequal-weight games $G_{m,x}^{(2)}$ and $H_{m,x}^{(2)}$, however, symmetry disappears, and the values $V_{m,x}^D$ and $V_{m,x}^S$ are not so easily expressed.

Lemma 1. *Under Assumption A we have*

$$V_{m,x}^D = x^m \beta_m(x), \tag{2.1}$$

$$V_{m,x}^S = 1 - x^m - x^m \beta_m(x), \qquad (1 \le m \le n-1;\ V_{0,x}^D \equiv 0,\ V_{1,x}^D \equiv 1 - x) \tag{2.2}$$

where $\beta_m(x) \equiv \sum_{j=1}^m j^{-1}(x^{-j} - 1)$.

Proof. Under Assumption A, we have the recursions

$$V_{m,x}^D = xV_{m-1,x}^D + \int_x^1 y^{m-1} dy, \qquad \text{with } V_{0,x}^D \equiv 0,\ V_{1,x}^D = 1 - x,$$

$$V_{m,x}^S = xV_{m-1,x}^S + \int_x^1 (1 - y^{m-1}) dy, \qquad \text{with } V_{0,x}^S \equiv 0,\ V_{1,x}^S = 0,$$

The upper equation gives

$$\begin{aligned}V^D_{m,x} =& x^2V^D_{m-2,x} + \frac{x}{m-1}(1-x^{m-1}) + \frac{1}{m}(1-x^m)\\ =& \ldots\ldots = x^{m-1}V^D_{1,x} + \sum_{j=2}^{m} x^{m-j}(1-x^j)/j\\ =& x^{m-1} + x^m\left[\sum_{j=2}^{m} j^{-1}(x^{-1}-1) - 1\right]\\ =& x^m\sum_{j=1}^{m} j^{-1}(x^{-1}-1),\end{aligned}$$

which is (2.1). The lower equation similarly gives (2.2). □

Note that $1-(V^D_{m,x}+V^S_{m,x}) = x^m$ is probability of draw in the games $G^{(2)}_{m,x}$ and $H^{(2)}_{m,x}$.

The function $\beta_m(x),\ 0<x\le 1$, ic convexly decreasing in $0<x\le 1$, with $\beta_m(0+)=\infty$, and $\beta_m(1)=0$, for all $m\ge 1$.

Lemma 2. *Let* $z = x^{-1}-1 \in [0,\infty)$, *i.e.* $x=(z+1)^{-1}\in(0,1]$. *Then*

$$\beta_m(x) = \sum_{i=1}^{m}\binom{m}{i} i^{-1}z^i, \tag{2.3}$$

Proof.

$$\begin{aligned}\beta_m(x) &= \sum_{j=1}^{m}(x^{-j}-1)/j = \sum_{j=1}^{m} j^{-1}\{(1+z)^j-1\}\\ &= \sum_{j=1}^{m} j^{-1}\sum_{i=1}^{j}\binom{j}{i}z^i = \sum_{i=1}^{m} z^i\sum_{j=i}^{m} j^{-1}\binom{j}{i} = \sum_{i=1}^{m} i^{-1}z^i\sum_{j=i}^{m}\binom{j-1}{i-1}\\ &= \sum_{i=1}^{m} i^{-1}z^i\sum_{j=i-1}^{m-1}\binom{j}{i-1} = \sum_{i=1}^{m}\binom{m}{i}i^{-1}z^i,\end{aligned}$$

since the last equality comes from the identity $\sum_{j=k}^{m}\binom{j}{k} = \binom{m+1}{k+1}$. □

Lemma 3. *The equations in* $x\in(0,1]$

$$\beta_m(x) = \frac{1}{2}(x^{-m}-1),\ x^{-m}-2, \text{ and } 1,$$

have unique roots b_m, a_m *and* γ_m, *respectively. And*

$$0 < b_m < a_m < \left(\frac{1}{3}\right)^{\frac{1}{m}} < \gamma_m < 1.$$

Proof. By wrighting $z=x^{-1}-1$, we note, from Lemma 2, that

$$\beta_m(x) - \frac{1}{2}(x^{-m}-1) = \frac{m}{2}z - \sum_{i=3}^{m}\binom{m}{i}\left(\frac{1}{2}-\frac{1}{i}\right)z^i (\equiv \phi(z),\ \text{say}),$$

and

$$\beta_m(x) - x^{-m} + 2 = 1 - \sum_{i=2}^{m} \binom{m}{i} \left(1 - \frac{1}{i}\right) z^i (\equiv \psi(z),\ \text{say}).$$

Then

$$\phi'(z) = \frac{m}{2}\left[1 - \sum_{i=2}^{m-1} \frac{i-1}{i+1}\binom{m-1}{i} z^i\right],$$

and we find that $\phi(z),\ 0 \le z < \infty$, is a unimodel concave function with $\phi(0) = 0,\ \phi'(0) = m/2, \phi(\infty) = \phi'(\infty) = -\infty$, and a sigle zero-point b_{m0}, say.

Also we have

$$\psi'(z) = -\sum_{i=1}^{m-1} \binom{m}{i+1} i z^i,$$

and hence $\psi(z),\ 0 \le z < \infty$, is a concave decreasing function with $\psi(0) = 1,\ \psi'(0) = 0,\ \psi(\infty) = \psi'(\infty) = -\infty$, and a single zero-point a_{m0}, say.

Since

$$\phi(z) - \psi(z) = \frac{1}{2}\{(1+z)^m - 3\} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0,\ \text{if } z \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 3^{1/m} - 1,$$

we find that $0 < 3^{1/m} - 1 < a_{m0} < b_{m0}$ (See Figure 1). Getting the variable z back to $x = (1+z)^{-1}$ we have

$$0 < b_m \equiv (1 + b_{m0})^{-1} < a_m \equiv (1 + a_{m0})^{-1} < (1/3)^{1/m}.$$

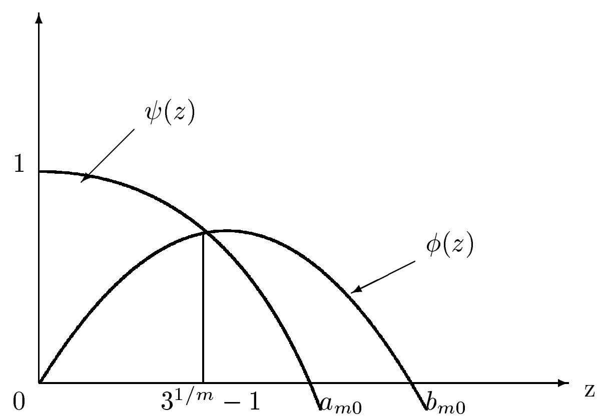

Figure 1: Graphs of $\phi(z)$ and $\psi(z)$.

Finally consider $x \in \left(a_m, \left(\frac{1}{3}\right)^{1/m}\right)$. Then $\beta_m(x) > \frac{1}{2}(x^{-m} - 1) > 1$, where the first (2nd) inequality is due to $x > a_m > b_m$ $\left(x < \left(\frac{1}{3}\right)^{1/m}\right)$. Therefore letting $x \uparrow \left(\frac{1}{3}\right)^{1/m}$, we get

$$\beta_m(3^{-1/m}) \ge 1 = \beta_m(\gamma_m),\ \text{i.e, } 3^{-1/m} \le \gamma_m.$$

Theses argaments above complete the whole proof of the lemma. □

Remark 1. $V_{m,x}^D$ is an m-th order polynomial

$$V_{m,x}^D = x^m \beta_m(x) = -\left(\sum_{j=1}^{m} j^{-1}\right) x^m + \sum_{j=0}^{m-1} (m-j)^{-1} x^j.$$

Lemma 3 and Lemma 5 (which will appear later) may be proven more easily by using this expression. We wanted to use Lemma 2 for the proofs of these lemmas.

In order to compute the numerical values of various decision numbers appeared in Lemma 3 we can use Lemma 2, and the relations below are useful. Let $z = x^{-1} - 1$. Then

$$\beta_m(x) = x^{-m} - 2 \iff \sum_{i=2}^{m} \binom{m}{i} (1 - i^{-1}) z^i = 1, \tag{2.4}$$

$$\beta_m(x) = \frac{1}{2}(x^{-m} - 1) \iff \sum_{i=3}^{m} \binom{m}{i} \left(\frac{1}{2} - i^{-1}\right) z^i = m/2, \tag{2.5}$$

$$\beta_m(x) = 1 \iff \sum_{i=1}^{m} \binom{m}{i} i^{-1} z^i = 1, \tag{2.6}$$

and solving each of these equations gives $\{a_m\}, \{b_m\}$ and $\{\gamma_m\}$, respectively. Table 1 shows the values of a_m, b_m and γ_m for $m = 2(1)10$.

Table 1. Numerical values of decision numbers in Lemma 3

	b_m	a_m	$(1/3)^{1/m}$	γ_m
$m=2$	0	$0.4142(=\sqrt{2}-1)$	$0.5774(=1/\sqrt{3})$	0.6899
3	0.25(=1/4)	0.5843	0.6934	0.7758
4	0.4163	0.6781	0.7598	0.8246
5	0.5249	0.7407	0.8027	0.8559
6	0.5999	0.7782	0.8327	0.8778
7	·	·	0.8548	0.8939
8	·	·	0.8717	0.9063
9	·	·	0.8851	0.9160
10			0.8960	0.9240
.			.	.
.			.	.
.			.	.

(The values of $\{\gamma_m\}$ are reproduced from Gilbert and Mosteller [4; Table 7])

Lemma 4. *Let* $a_m = \left(1 + \frac{c}{m}\right)^{-1}$, $b_m = \left(1 + \frac{h}{m}\right)^{-1}$ *and* $m \to \infty$. *Then* $c(\fallingdotseq 1.42)$ *is a unique root of the equation*

$$\int_0^c t^{-1}(e^t - 1)dt = e^c - 2, \ i.e. \ \sum_{j=2}^{\infty} \frac{c^j}{j^2(j-2)!} = 1, \tag{2.7}$$

$h(\fallingdotseq 2.49)$ *is a unique root of the equation*

$$\int_0^h t^{-1}(e^t-1)dt = \frac{1}{2}(e^h-1),\ i.e.\ \sum_{j=2}^{\infty} \frac{h^j}{(j+1)^2 j(j-2)!} = 1. \tag{2.8}$$

Proof. From the definition of a_m we have $\beta_m(a_m) = \left(\frac{1}{a_m}\right)^m - 2$, which is in the limit of $m \to \infty$ equal to (2.7), since

$$\sum_{j=1}^{m} j^{-1}\left\{\left(1+\frac{c}{m}\right)^j - 1\right\} \xrightarrow[(m\to\infty)]{} \int_0^c t^{-1}(e^t-1)dt,$$

and

$$\int_0^c t^{-1}(e^t-1)dt = \sum_{j=1}^{\infty} \frac{c^j}{j \cdot j!}$$

The equation (2.8) is proved similarly. □

It is well-known (See [4; Section 3b]) that the assymptotic behavior of γ_m is given by $\gamma_m = (1+r/m)^{-1}$, where $r(\fallingdotseq 0.80435)$ is a unique root of the equation

$$\int_0^r t^{-1}(e^t-1)dt = 1. \tag{2.9}$$

Note that $0 < r(\fallingdotseq 0.80435) < c(\fallingdotseq 1.42) < h(\fallingdotseq 2.49)$, reflecting the relation $0 < b_m < a_m < \gamma_m < 1$.

The final lemma is concerned with the behaviors of $V_{m,x}^D$ and $V_{m,x}^S$.

Lemma 5. *The graphs of functions $V_{m,x}^D$ and $V_{m,x}^S$ are as shown by Figure 2. We find that*

$$V_{m,x}^S \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} V_{m,x}^D, \quad if\ x \left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\} b_m$$

where $\{b_m\}$ is defined in Lemma 3.

Proof. From (2.1)-(2.2) and Lemma 3 we have

$$V_{m,x}^S < V_{m,x}^D \iff \beta_m(x) > \frac{1}{2}(x^{-m}-1) \iff x > b_m.$$

Now, for $V_{m,x}^S (= \phi(x)$, say), we obtain, after some calculus,

$$\phi'(x) = \frac{d}{dx}\{1 - x^m(1+\beta_m(x))\} = -x^{m-1}\left[m(1+\beta_m(x)) - \sum_{j=1}^{m} x^{-j}\right] \tag{2.10}$$

$$= -mx^{m-1}\left[\sum_{j=1}^{m-1}(j^{-1}-m^{-1})x^{-j} - \left(\sum_{j-1}^{m} j^{-1} - 1\right)\right] < 0,$$

(2.11)

$$\phi''(x) = -m(m-1)x^{m-2}\left[\sum_j (j^{-1}-m^{-1})x^{-j} - \sum_j j^{-1} + 1\right] + mx^{m-1}\sum_j \left(1-\frac{j}{m}\right)s^{-j-1}$$

$$= mx^{m-2}\left[-\sum_{j=1}^{m-2}(j^{-1}-m^{-1})(m-1-j)x^{j} + (m-1)\sum_{j=1}^{m-1}(j^{-1}-m^{-1})\right]$$

and hence $\phi(x)$ is a decreasing concave-convex function with $\phi(0) = 1 - m^{-1}$ (since $x^m\beta_m(x) \to m^{-1}$, as $x \to 0+$), $\phi'(0) = -(m-1)^{-1}$, $\phi(1) = \phi'(1) = 0$, $\phi''(0) = -\frac{2}{m-2}$ and $\phi''(1) = m\sum_{j=1}^{m-1}(1-j/m) = \frac{1}{2}m(m-1)$.

In the same way, for $V^D_{m,x} = 1 - x^m - V^S_{m,x} (\equiv \psi(x)$, say), we have from (2.10)-(2.11)

$$\psi'(x) = -mx^{m-1} - \phi'(x) = mx^{m-1}\left[\sum_{j=1}^{m-1}(j^{-1}-m^{-1})x^{-j} - \sum_{j=1}^{m} j^{-1}\right], \tag{2.12}$$

$$\psi''(x) = -m(m-1)x^{m-2} - \phi''(x) = mx^{m-2}\left[\sum_{j=1}^{m-2}(j^{-1}-m^{-1})(m-1-j)x^{-j} - (m-1)\sum_{j=1}^{m} j^{-1}\right],$$

with

$$\psi(0) = m^{-1},\ \psi'(0) = (m-1)^{-1},\ \psi(1) = 0,\ \psi'(1) = -m - \phi'(1) = -m,$$

$$\psi''(0) = -\phi''(0) = \frac{2}{m-2},\ \text{and } \psi''(1) = -m(m-1) - \phi''(1) = -\frac{3}{2}m(m-1),$$

and hence $\psi(x)$ is a convex-concave function with a single point of inflexion and attains its maximum at a unique $x \in (0,1)$, satisfying

$$\sum_{j=1}^{m-1}(j^{-1}-m^{-1})x^{-j} = \sum_{j=1}^{m} j^{-1}. \tag{2.13}$$

The graphs of two functions $\phi(x)$ and $\psi(x)$ are as shown by Figure 2. This completes the proof of the lemma. $\square$

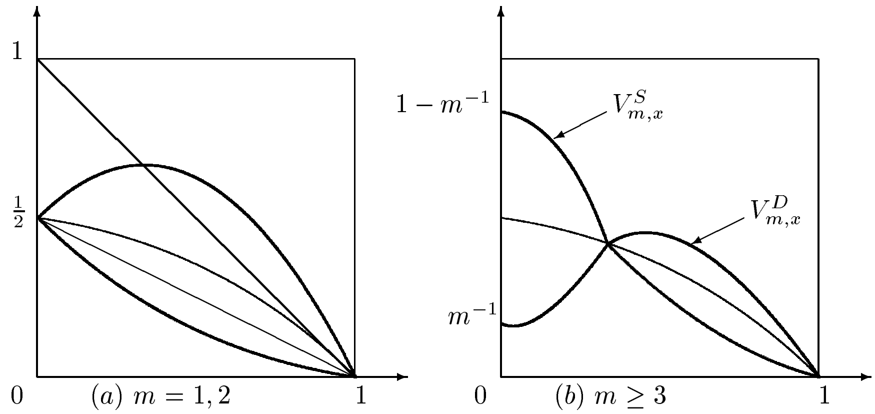

Figure 2: Graphs of $V^D_{m,x}$ and $V^S_{m,x}$.

(Thin lines mean $\frac{1}{2}\left(V^D_{m,x}+V^S_{m,x}\right) = \frac{1}{2}\left(1-x^m\right)$. Note that $V^D_{1,x} = 1-x$ and $V^D_{2,x} = \frac{1}{2}(-3x^2+2x+1)$.)

Note that $\frac{1}{2}(V^D_{m,x}+V^S_{m,x}) = \frac{1}{2}(1-x^m)$ is the common equilibrium value in the two-person equal-weight game $\Gamma^{(2)}_{m,x}$ (See the argument at the beginning of this section and [9; Section 3.2]. It will be shown in the subsequent two sections that "dictator" in games $G^{(3)}_n$ and $H^{(3)}_n$ never chooses A at any r.v. with $0<x<b_{n-1}$.

Remark 2. The two numbers $k_m \equiv \text{arg-max}_{0\leq x\leq 1} V^D_{m,x}$ and $\max_{0\leq x\leq 1} V^D_{m,x}$ are important in the next two sections. k_m is a unique root of the equation (2.13) and

$$\max_x V^D_{m,x} = (k_m)^m\beta_m(k_m) = \frac{1}{m}\sum_{j=0}^{m-1}(k_m)^j.$$

We find that $b_m < k_m < a_m$.

For the first small values of m they are :

	b_m	k_m	a_m	$\max_x V^D_{m,x}$
$m=2$	0	0.3333	0.4142	0.6667
3	0.25	0.5339	0.5843	0.6063
4	0.4163	0.6432	0.6781	0.5808
5	0.5249	0.7114	0.7407	0.5667
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Consider a full-information (but imperfect-observation) m-horizon best-choice problem where each r.v. is observed only whether it is greater than or less than some specified decision-level $x\in[0,1]$, and the objective is

$$\text{Pr.(win} \mid x) \equiv \text{Pr.}[X_1,\ldots,X_{\tau-1} < x < X_\tau \text{ and } X_\tau > X_{\tau+1},\ldots,X_m] \to \max_{0\leq x\leq 1}.$$

Winning here is meant by the event that the earliest r.v. greater that x becomes best among all r.v.s. τ is the stopping time. Then we find that

$$\text{Pr.(win} \mid x) = \sum_{s=1}^{m} x^{s-1}\int_x^1 y^{m-s}dy = x^m\sum_{j=1}^{m} j^{-1}(x^{-j}-1) = V^D_{m,x}.$$

Therefore k_m is identical to the opimal decision-level and $\frac{1}{m}\sum_{j=0}^{m-1}(k_m)^j$ is the maximal probability of winning for this best-choice problem (End of Remark 2).

3. Three-person $<\frac{1}{2},\frac{1}{2},0>$-weight Game $G_n^{(3)}$

Now we have to proceed to deriving an explicit solution to $G_n^{(3)}$. Let the equilibrium values be denoted by (W_n^E, W_n^E, W_n^S), where E means "equal-weight" and S means "subject."

In state $(x \mid n)$ of $G_n^{(3)}$, the players face a trimatrix game with payoff matrix (1.1), where

(3.1)

$$M_{n,R}(x) = (I)\left\{ \begin{array}{c|c|c|} & \overbrace{\text{R} \qquad\qquad\qquad\qquad\qquad\qquad \text{A}}^{\text{(II)}} & \\ \text{R} & W^E, \quad W^E, \quad W^S & V^D, \quad x^{n-1}, \quad V^S \\ \hline \text{A} & x^{n-1}, \quad V^D, \quad V^S & \frac{1}{2}(x^{n-1}+V^D), \quad \frac{1}{2}(x^{n-1}+V^D), \quad V^S \end{array}\right.$$

and

(3.2)

$$M_{n,A}(x) = \begin{array}{c|c|c|} & \text{R} & \text{A} \\ \hline \text{R} & \frac{1}{2}(1-x^{n-1}), \quad \frac{1}{2}(1-x^{n-1}), \quad x^{n-1} & V^D, \quad x^{n-1}, \quad V^S \\ \hline \text{A} & x^{n-1}, \quad V^D, \quad V^S & \frac{1}{2}(x^{n-1}+V^D), \quad \frac{1}{2}(x^{n-1}+V^D), \quad V^S \\ \hline \end{array}$$

In these two matrices the subscripts are omitted *i.e.*,W^E means W_{n-1}^E and $V^D(V^S)$ means $V_{n-1,x}^D(V_{n-1,x}^S)$. The Optimality Equation is

$$(3.3) \qquad (W_n^E, W_n^E, W_n^S) = E[eq.val.M_n(X)] \quad \left(n \geq 2.\ (W_1^E, W_1^E, W_1^S) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)\right)$$

provided the eq. values of $M_n(x)$ exist uniquely a.e. x.

We shall hereafter follow a decision-theoretic approach to the problem, instead of investigating along the formulation (3.1) $\sim$ (3.3). Consider the strategy-triples, in state $(x \mid n)$, such that player I (II, III) chooses $\left\{{A \atop R}\right\}$ if $x\left\{{> \atop <}\right\}e_n$ (f_n, g_n), where e_n, f_n, g_n are the decision numbers selected by I, II, III, respectively, and $0 \leq g_n \leq f_n \leq e_n \leq 1$. That is :

Players choose R-R-R, if $x \in (0, g_n)$
Players choose R-R-A, if $x \in (g_n, f_n)$
Players choose R-A-A, if $x \in (f_n, e_n)$
Players choose A-A-A, if $x \in (e_n, 1)$.

Then, under Assumption A, the expected payoffs (M_n^1, M_n^2, M_n^3) to the players when the strategy-triple (e_n, f_n, g_n) is chosen satisfy, from (3.1)-(3.2),

$$(3.4) \qquad M_n^1 = gM_{n-1}^1 + \int_g^f \frac{1}{2}(1-x^{n-1})dx + \int_f^e V_{n-1,x}^D dx + \int_e^1 \frac{1}{2}(x^{n-1} + V_{n-1,x}^D)dx,$$

$$(3.5) \qquad M_n^2 = gM_{n-1}^2 + \int_g^f \frac{1}{2}(1-x^{n-1})dx + \int_f^e x^{n-1} dx + \int_e^1 \frac{1}{2}(x^{n-1} + V_{n-1,x}^D)dx,$$

$$(3.4) \qquad M_n^3 = gM_{n-1}^3 + \int_g^f x^{n-1}dx + \int_f^e V_{n-1,x}^S dx + \int_e^1 V_{n-1,x}^S dx,$$

$$(n \geq 2; e_1 = f_1 = g_1 = 0; (M_1^1, M_1^2, M_1^3) = (\tfrac{1}{2}, \tfrac{1}{2}, 0))$$

where the subscript n for e_n, f_n and g_n are omitted in the r.h.s.

Theorem 1. *For the optimal stopping game $G_n^{(3)}$ with players' weights $w =< \frac{1}{2}, \frac{1}{2}, 0 >$ descrived by* (3.1)−(3.2), *an explicit solution under Assumption A is as follows ; Considering the strategy-triples (e_n, f_n, g_n) defined above, the equilibrium σ_n^* in state $(x \mid n)$ is given by*

$$\sigma_n^* = (e_n, f_n, g_n) = (a_{n-1}, a_{n-1}, (M_{n-1}^3)^{\frac{1}{n-1}}), \tag{3.7}$$

where $\{a_n\}$ is the sequence defined in Lemma 3 *and $\{M_n^3\}$ is determined by the recursion*

(3.8)
$$M_n^3 = \left(\frac{n-1}{n}\right)\left\{(M_{n-1}^3)^{\frac{n}{n-1}} + 1 - a_{n-1}\right\} - \frac{1}{n}\sum_{j=1}^{n-1} j^{-1}\left\{1 - (a_{n-1})^j\right\}, \quad (n \geq 2,\ M_1^3 = 0).$$

The equilibrium payoffs to the players are $(\frac{1}{2}(1 - M_n^3), \frac{1}{2}(1 - M_n^3), M_n^3)$.

Proof. Since M_{n-1}^i, $i = 1, 2, 3$, is independent of (e_n, f_n, g_n), $\frac{\partial M_n^1}{\partial e_n} = 0$ for (3.4) gives $V_{n-1,e}^D = e^{n-1}$ *i.e.*, $\beta_{n-1}(e_n) = 1$, and hence e_n is equal to γ_{n-1}, defined in Lemma 3. Similarly $\frac{\partial M_n^2}{\partial f_n} = 0$ for (3.5), gives $f_n = (1/3)^{\frac{1}{n-1}}$. And $\frac{\partial M_n^3}{\partial g_n} = 0$ for (3.6) gives $g_n = (M_{n-1}^3)^{\frac{1}{n-1}}$.

On the other hand, from (3.4)-(3.5),

$$M_n^1 - M_n^2 = g_n(M_{n-1}^1 - M_{n-1}^2) + \int_{f_n}^{e_n} (V_{n-1,x}^D - x^{n-1})dx.$$

Since

$$V_{n-1,x}^D > x^{n-1} \Longleftrightarrow \beta_{n-1}(x) > 1 \Longleftrightarrow x < \gamma_{n-1}$$

and $0 \leq f_n \leq e_n = r_{n-1}$ in equilibrium, we find by induction on n, that

$$M_n^1 - M_n^2 \geq 0, \text{ and equality holds if } f_n = e_n.$$

Therefore the stategy-triple in state $(x \mid n)$ under equilibrium must satisfy $e_n = f_n$.

Substituting this result into (3.4) and (3.6), we obtain

$$M_n^1 = gM_{n-1}^1 + \int_g^f \frac{1}{2}(1 - x^{n-1})dx + \int_f^1 \frac{1}{2}(x^{n-1} + V_{n-1,x}^D)dx, \tag{3.9}$$

$$M_n^3 = gM_{n-1}^3 + \int_g^f x^{n-1}dx + \int_f^1 V_{n-1,x}^S dx. \tag{3.10}$$

[We can easily comfirm from (3.9) − (3.10) that $2M_n^1 + M_n^3 \equiv 1, \forall n \geq 1$ by induction on n] and these equations give

$$\frac{\partial M_n^1}{\partial f} = \frac{1}{2}\left\{1 - f^{n-1}(2 + \beta_{n-1}(f))\right\} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0,\ if\ f \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} a_{n-1};$$

$$\frac{\partial M_n^3}{\partial g} = M_{n-1}^3 - g^{n-1} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0,\ if\ g \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} (M_{n-1}^3)^{\frac{1}{n-1}}.$$

Therefore, these arguments lead to the conclusion that the strategy-triple in equilibrium in state $(x \mid n)$ is $(a_{n-1}, a_{n-1}, (M^3_{n-1})^{\frac{1}{n-1}})$, *i.e.*,(3.7).

Substittuting $f = a_{n-1}$ into (3.11) and using

$$\begin{aligned}\int_{a_{n-1}}^{1} V^D_{n-1,x} dx =& \frac{1}{n}\left[x^n \beta_{n-1}(x)\right]^1_{a_{n-1}} + \frac{1}{n}\int_{a_{n-1}}^{1}\sum_{j=1}^{n-1} x^{j-1} dx \\ =& \frac{1}{n}\left[-a_{n-1} + 2(a_{n-1})^n + \sum_{j=1}^{n-1} j^{-1}\{1-(a_{n-1})^j\}\right],\end{aligned}$$

we obtain

$$M^3_n = gM^3_{n-1} + 1 - a_{n-1} - \frac{1}{n}\left[1 - a_{n-1} + g^n + \sum_{j=1}^{n-1} j^{-1}\left\{1-(a_{n-1})^j\right\}\right] \tag{3.11}$$

$$= \left(\frac{n-1}{n}\right)(M^3_{n-1})^{\frac{n}{n-1}} + \left(\frac{n-1}{n}\right)(1-a_{n-1}) - \frac{1}{n}\sum_{j=1}^{n-1} j^{-1}\left\{1-(a_{n-1})^j\right\},$$

i.e., (3.8), since $g = (M^3_{n-1})^{\frac{1}{n-1}}$. In all equations appeared between (3.9) and (3.11) the subscripts n in f and g are omitted for simplicity. This completes the proof of the theorem. □

By referring to Table 1, and computing (3.7)-(3.8), for some small n, we get Table 2, which illustates the solution described by Theorem 1. Note that $\{M^3_n\}$ (and hence $\{M^1_n\}$ also) is not monoton.

Remark 3. The solution for $n = 2$ is : Choose A-A-A in state $(x \mid 2)$, for any x. Payoffs are $(\frac{1}{2}, \frac{1}{2}, 0)$. The solution for $n = 3$ is Choose R-R-A(A-A-A) if $x < (>)\ a_2 = \sqrt{2} - 1 \risingdotseq 0.4142$, in state $(x \mid 3)$. Payoffs are $(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, 1 - \frac{2\sqrt{2}}{3})$. And so on for $n \geq 4$.

Table 2. An explicit solution to $G^{(3)}_n$ based on Theorem 1.

	$\sigma^*_n =$ $(a_{n-1},$	$a_{n-1},$	$(M^3_{n-1})^{\frac{1}{n-1}})$	$(M^1_n,$	$M^2_n,$	$M^3_n)$
$n = 2$	0	0	0	1/2	1/2	0
3	0.4142	0.4142	0.1716	0.4714	0.4714	0.0572
4	0.5843	0.5843	0.3853	0.4402	0.4402	0.1196
5	0.6781	0.6781	0.5881	0.4450	0.4450	0.1101
6	0.7407	0.7407	0.6432	0.4469	0.4469	0.1063
7	0.7782	0.7782	0.6883	0.4469	0.4469	0.1062
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Corollary 1.1. *If M^3_n given by* (3.8) *converges when $n \to \infty$, the limit $\alpha \risingdotseq 0.122$ is equal to a unique root of the equation*

$$-\alpha \log \alpha = K - \alpha, \tag{3.12}$$

where

$$K = \int_0^c t^{-1}(e^{-1} - 1 + t)dt = \sum_{j=2}^{\infty} \frac{(-c)^j}{j \cdot j!} (\risingdotseq 0.379) \tag{3.13}$$

and $c(\risingdotseq 1.42)$ *is a constant given by* (2.7) *in* Lemma 4.

Proof. Substituting $a_{n-1} = (1 + \frac{c}{n-1})^{-1}$ into(3.8), we obtain

$$n(M_n^3 - M_{n-1}^3) = (n-1)(M_{n-1}^3)^{\frac{n}{n-1}} - nM_{n-1}^3 + (n-1)(1 - a_{n-1}) - \sum_{j=1}^{n-1} j^{-1}\{1 - (a_{n-1})^j\}$$

$$= M_{n-1}^3 \left\{(M_{n-1}^3)^{\frac{1}{n-1}} - 1\right\} / (n-1)^{-1} - M_{n-1}^3 + \frac{c}{1 + c/(n-1)} - \sum_{j=1}^{n-1} j^{-1} \left\{1 - \left(1 + \frac{c}{n-1}\right)^{-j}\right\},$$

and therefore we get when $n \to \infty$

$$0 = \alpha \log \alpha - \alpha + c - \int_0^1 t^{-1}(1 - e^{-ct})dt = \alpha \log \alpha - \alpha + \int_0^c t^{-1}(e^{-t} - 1 + t)dt.$$

This proves the corollary. $\square$

Remark 4. Theorem 1 combined with Corollary 1.1 implies that if M_n^3 converges when $n \to \infty$, the opimal stopping game $G_n^{(3)}$ has the limitting equilibrium payoffs

$$\lim_{n \to \infty} (M_n^1, M_n^2, M_n^3) = (0.439, 0, 439, 0.122).$$

It is interesting to note that in the two-person $< 1, 0 >$-weight game under WP-maximization the limtting equilibrium payoffs are $(1 - \beta, \beta)$, where $\beta(\risingdotseq 0.3276)$ is a unique root of the equation $-\beta \log \beta = \log 2 - \beta$ (see Sakaguchi and Hamada [9]).

Remark 5. From Corollary 1.1 we find that the decision threshold $(M_n^3)^{\frac{1}{n-1}}$ for player III in state $(x \mid n)$ is asymptotically

$$\exp\left(\frac{1}{n-1} \log M_{n-1}^3\right) = \exp\left(\frac{\log 0.122}{n-1}\right) \risingdotseq 1 - \frac{2.1037}{n-1} + o(n^{-1}).$$

On the other hand, from Lemma 4, for the decision threshold a_{n-1} for players I and II, is

$$a_{n-1} = \left(1 + \frac{1.42}{n-1}\right)^{-1} = 1 - \frac{1.42}{n-1} + o(n^{-1}).$$

Therefore we observe that $a_{n-1} > (M_{n-1}^3)^{\frac{1}{n-1}}$ asymptotically.

4. Three-person $< 1, 0, 0 >$- wight Game $H_n^{(3)}$.

Next we discuss about the game $H_n^{(3)}$. Denote, by (W_n^D, W_n^S, W_n^S), the eq. values, where the superscripts D and S mean "dictator" and "subject", repectively.

In state $(x \mid n)$ players face a trimatrix game with the payoff matrix $M_n(x)$, which is

$$M_n(x) = \begin{array}{l} \nearrow^{\text{R by I}} \; M_{n,R}(x) \\ \searrow_{\text{A by I}} \; M_{n,A}(x) \end{array}$$

where

(4.1)

$$M_{n,R}(x) = \text{(II)} \left\{ \begin{array}{c|ccc|ccc} & & & & \multicolumn{3}{c}{\overbrace{\text{R} \qquad\qquad\qquad\qquad \text{A}}^{\text{(III)}}} \\ \text{R} & W^D, & W^S, & W^S & V^D, & V^S, & x^{n-1} \\ \text{A} & V^D, & x^{n-1}, & V^S & V^D, & \frac{1}{2}(x^{n-1}+V^S), & \frac{1}{2}(x^{n-1}+V^S) \end{array} \right.$$

and $M_{n,A}(x) = [\left(x^{n-1}, \frac{1}{2}(1-x^{n-1}), \frac{1}{2}(1-x^{n-1})\right)$, for the four choice-triples].
In the matrix $M_{n,R}(x)$ the subscripts are again omitted *i.e.*, $W^D(W^S)$ means $W^D_{n-1}(W^S_{n-1})$ and $V^D(V^S)$ means $V^D_{n-1,x}(V^S_{n-1,x})$. The same four triples in $M_{n,A}(x)$ are due to the fact that, for any choice-triple with A by I the game is left as in the two-person equal-weight game thereafter. The Optimality Equation is

$$(W^D_n, W^S_n, W^S_n) = E[\text{eq.val.} M_n(X)] \tag{4.2}$$

$$(n \geq 2,\ (W^D_1, W^S_1, W^S_1) = (1,0,0))$$

provided that eq. values of $M_n(x)$ exist uniquely a.e.x.

Similarly as in the previous analysis made for $G^{(3)}_n$, we follow the decision-theoretic approach. Then, under Assumption A, the expected payoffs to the players, when the strategy-triple (e_n, f_n, g_n) is chosen in state $(x \mid n)$, satisfy, from (4.1), the recursion

$$M^1_n = gM^1_{n-1} + \int_g^f V^D_{n-1,x}dx + \int_f^e V^D_{n-1,x}dx + \int_e^1 x^{n-1}dx, \tag{4.3}$$

$$M^2_n = gM^2_{n-1} + \int_g^f V^S_{n-1,x}dx + \int_f^e \frac{1}{2}(x^{n-1} + V^S_{n-1,x})dx + \int_e^1 \frac{1}{2}(1-x^{n-1})dx, \tag{4.4}$$

$$M^3_n = gM^3_{n-1} + \int_g^f x^{n-1}dx + \int_f^e \frac{1}{2}(x^{n-1} + V^S_{n-1,x})dx + \int_e^1 \frac{1}{2}(1-x^{n-1})dx, \tag{4.5}$$

$$(n > 2;\ e_1 = f_1 = g_1 = 0,\ (M^1_1, M^2_1, M^3_1) = (1,0,0))$$

where the subscript n for e_n, f_n, g_n in the r.h.s. are omitted. Note that

$$M^1_n + M^2_n + M^3_n = 1,\ \forall n \geq 1,$$

since $V^S_{n-1,x} + V^S_{n-1,x} + x^{n-1} = 1$.

Now we arrive at the following statement.

Theorem 2. *For the optimal stopping game $H^{(3)}_n$ with palyers' weights $w =< 1,0,0 >$, descrived by (4.1)−(4.2), an explicit solution under Assumption A is as follows: Considering strategy-triples (e_n, f_n, g_n) in state $(x \mid n)$, the equilibrium is given by*

$$(e_n, f_n, g_n) = (\gamma_{n-1}, f_n^*, f_n^*), \tag{4.6}$$

where $\{\gamma_n\}$ *is given in* Lemma 3, *and* f_n^* *is given by*

$$f_n^* = \text{arg–max}_{0 \le f \le \gamma_{n-1}} \left[f M_{n-1}^2 + \int_f^{\gamma_{n-1}} \frac{1}{2}(1 - V_{n-1,x}^D) dx \right]. \tag{4.7}$$

The equilibrium payoffs to the players are given by the recursion

$$M_n^1 = 1 - 2M_n^2 = \left[f M_{n-1}^1 + \frac{1}{n} \left\{ 1 - f^n \beta_{n-1}(f) + \sum_{j=1}^{n-1} j^{-1} \{ (\gamma_{n-1})^j - f^j \} \right\} \right]_{f = f_n^*} \tag{4.8}$$

$$(n \ge 2,\ (M_1^1, M_1^2, M_1^3) = (1, 0, 0)).$$

Proof. Similarly as in the proof of Theorem 1, the conditions $\frac{\partial M_n^1}{\partial e_n} = \frac{\partial M_n^2}{\partial f_n} = \frac{\partial M_n^3}{\partial g_n} = 0$ on the equations (4.3) $\sim$ (4.5) yield that the strategy-triple in equilibrium in state $(x \mid n)$ is $(\gamma_{n-1}, a_{n-1}, (M_{n-1}^3)^{\frac{1}{n-1}})$.

On the other hand, we have from (4.4) -(4.5)

$$M_n^2 - M_n^3 = g_n(M_{n-1}^2 - M_{n-1}^3) + \int_{g_n}^{f_n} (V_{n-1,x}^S - x^{n-1}) dx.$$

Since, by Lemmas 3 and 4,

$$V_{n-1,x}^S > x^{n-1} \Longleftrightarrow \beta_{n-1}(x) < x^{-n+1} - 2 \Longleftrightarrow x < a_{n-1},$$

and since $0 \le g_n \le f_n = a_{n-1}$ in equilibrium, we find that, for all $n \ge 1$,

$$M_n^2 - M_n^3 \ge 0, \text{ and equality holds if } g_n = f_n.$$

Therefore the strategy-triple in state $(x \mid n)$ under equilibrium must satisfy $g_n = f_n$.

Substituting this result into (4.3)-(4.4) we obtain

$$M_n^1 = f M_{n-1}^1 + \int_f^e V_{n-1,x}^D dx + \int_e^1 x^{n-1} dx, \tag{4.9}$$

$$M_n^2 = f M_{n-1}^2 + \int_f^e \frac{1}{2}(x^{n-1} + V_{n-1,x}^S) dx + \int_e^1 \frac{1}{2}(1 - x^{n-1}) dx, \tag{4.10}$$

and the equation (4.9) gives

$$\frac{\partial M_n^1}{\partial e} = e^{n-1}(\beta_{n-1}(e) - 1) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0, \quad if\ e \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \gamma_{n-1}.$$

Hence from (4.10) we get

$$M_n^2 = f M_{n-1}^2 + \int_f^{\gamma_{n-1}} \frac{1}{2}(1 - V_{n-1,x}^D) dx + \int_{\gamma_{n-1}}^1 \frac{1}{2}(1 - x^{n-1}) dx,$$

and

$$\frac{\partial M_n^2}{\partial f} = M_{n-1}^2 - \frac{1}{2}(1 - V_{n-1,f}^D) = \frac{1}{2}(V_{n-1,f}^D - M_{n-1}^1),$$

which is increasing in $f \in (0, k_{n-1})$ and decreasing in $f \in (k_{n-1}, \gamma_{n-1})$, (See Lemma 5 and Remark 1).

Moreover we find , after integration by parts, that

$$\begin{aligned}\int_f^{\gamma_{n-1}} V_{n-1,x}^D dx =& \frac{1}{n}\left[x^n \beta_{n-1}(x)\right]_f^{\gamma_{n-1}} + \frac{1}{n}\int_f^{\gamma_{n-1}} \sum_{j=1}^{n-1} x^{j-1} dx \\ =& \frac{1}{n}\left\{(\gamma_{n-1})^n - f^n \beta_{n-1}(f)\right\} + \frac{1}{n}\sum_{j=1}^{n-1} j^{-1}\left\{(\gamma_{n-1})^j - f^j\right\},\end{aligned}$$

which implies, from (4.9), that the recursion (4.8) holds true.

This completes the proof of the theorem. □

Remark 6. The solution for $n = 2$ is : We find $f_2^* = 0$. Choose R-A-A(A-A-A) if $x < (>)$ $\gamma_1 = \frac{1}{2}$, in state $(x \mid 2)$. Payoffs are $(M_2^1, M_2^2, M_2^3) = (\frac{3}{4}, \frac{1}{8}, \frac{1}{8})$, since $M_2^1 = \int_0^{1/2}(1-x)dx + \int_{1/2}^1 x dx = \frac{3}{4}$. The solution for $n = 3$ is : We find $f_3^* = \text{arg-max}_f \left[f M_2^2 + \int_f^{\gamma_2} \frac{1}{2}(1 - V_{2,x}^D)dx\right] = \text{arg-max}_f \left[\frac{1}{8}f + \int_f^{\gamma_2} \frac{1}{4}(3x^2 - 2x + 1)dx\right] = 0$. Choose R-A-A (A-A-A) if $x < (>)$ $\gamma_2 = \frac{1}{5}(1+\sqrt{6}) \fallingdotseq 0.6899$, in state $(x \mid 3)$. Payoffs are $(M_3^1, M_3^2, M_3^3) = (0.6426, 0.1787, 0.1787)$, since $M_3^1 = \frac{1}{3}(1 + \gamma_2 + \frac{1}{2}\gamma_2^2) = 0.6426$. Similarly the solution for $n = 4$ is : Since $f_4^* = 0$, choose R-A-A (A-A-A), if $x < (>)$ $\gamma_3 = 0.7758$ in state $(x \mid 4)$. Payoffs are $(M_4^1, M_4^2, M_4^3) = (0.5581, 0.2210, 0.2210)$, since $M_4^1 = \frac{1}{4}(1 + \gamma_3 + \frac{1}{2}\gamma_3^2 + \frac{1}{3}\gamma_3^3) = 0.5581$. And so on for $n \geq 5$.

Remark 7. The Assumption A brings our game into the easier from too much extent. In the "subgames" $G_{m,x}^{(2)}$ and $H_{m,x}^{(2)}$ here, each player fixes his strategy. The equilibrium stategies in $G_n^{(3)}$ and $H_n^{(3)}$ without assuming Assumption A are as yet unknown. We must think about the fact that the optimal play in the $< 1, 0 >$-weight game $G_{m,x}^{(2)} \left[H_{m,x}^{(2)}\right]$ is different from one in $G_m^{(2)} \left[H_m^{(2)}\right]$ (See Lemmas 1 and 5, Theorems 1 and 2, and the author's previous work [9; Section 2]).

Moreover, as to the expected payoffs (M_n^1, M_n^2, M_n^3) from the equilibrium play in the game $H_n^{(3)}$, when $n \to \infty$. are as yet unknown. See Remark 4.

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