THE CONTINUITY ON PATTERN RECOGNITIONS

Gao Zhi min* and Yoshikazu Yasui**

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Abstract. In pattern analysis and image management, the information of an objective image can be recovered from a sequence approximate images. In mathematical form of expression we need to consider some types of continuity. In [3], W.Gong defined the upper limit and lower limit of a sequence, and the concepts are used to characterize the convergency and continuity in the space consisting of images. In the present paper, we shall give first some examples to show that there are some theoretical shortcomings in [3], and furthermore we shall prove some corresponding correct results.

1 Preliminaries Let R^n be the n-dimensional Euclidean space, and for $x, y \in R^n$, $\rho(x,y)$ is the distance between the points x and y. If ϵ is a positive real number and $x \in R^n$, then we denote $U(x,\epsilon) = \{y \in R^n : \rho(x,y) < \epsilon\}$ which is called to be open ball centred at x of radius ϵ. If $x = O$ is the origin of R^n, we denote $U(x,\epsilon)$ by ϵD.

In general, for the subset A of a space X, $\overline{A}$ means the closure of A in X. We denotes the set of natural numbers by letter Ω. For a topological space $(X, \mathcal{T})$, if a sequence $\{x_i : i \in \Omega\}$ converges to x^* in X, then we write it by $x_i \xrightarrow{X} x^*$. When (X,ρ) is a metric space, and a sequence $\{x_i : i \in \Omega\}$ converges to x^* in X, then $x_i \xrightarrow{X} x^*$ will be denoted for simplicity by the symbol $x_i \to x^*$ or $\rho(x_i, x^*) \to 0$.

If y is a point in R^n and A, B are subsets of R^n, we let $A[y]$ be its translation by the point y, i.e., $A[y] = \{a + y : a \in A\}$, and $\check{A}$ be the symmetric set of A with respect to the origin, i.e., $\check{A} = \{-a : a \in A\}$. $A \oplus B = \{a + b : a \in A, b \in B\}$ is called the dilation of set A by set B, and $A \ominus \check{B} = \cap_{b \in \check{B}} A[b]$ is the erosion of set A by set B. It is clear that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$, and $U(A,\epsilon) = A \oplus \epsilon \mathrm{D}$, and $A \ominus \check{B} = \{x : B[x] \subset A\}$.

Let $Y \subset R^n$ be a 'very big' bounded closed set which contains the origin as its interior point. We shall construct four families as follows.

$\mathcal{F} = \{F \subset R^n : F \neq \emptyset, F \text{ is closed}\}$, $\mathcal{K} = \{K \subset R^n : K \neq \emptyset, K \text{ is compact}\}$ and $\mathcal{P} = \{K \subset Y : K \neq \emptyset, K \text{ is compact}\}$, and $\mathcal{P}^* = \{K \subset Y \oplus Y : K \neq \emptyset, K \text{ is compact}\}$.

It is clear that $\mathcal{K} \subset \mathcal{F}$ and $\mathcal{P} = \{K \cap Y : K \in \mathcal{K} \text{ and } K \cap Y \neq \emptyset\}$.

Definition 1.1 ([3]). *Let $G_1, G_2, \cdots, G_m$ be finite non-empty open sets, and $K_1, K_2, \cdots, K_p$ finite compact sets of R^n (K_j can be empty). We set*

$N(\{G_i\},\{K_j\}) = \{F \in \mathcal{F} : F \cap G_i \neq \emptyset$ *for each* $i = 1, 2, \cdots, m; F \cap K_j = \emptyset$ *for each* $j = 1, 2, \cdots, p\}$

$\mathcal{B}(\mathcal{F}) = \{N(\{G_i\},\{K_j\}) : \{G_i\}$ is a finite family of non-empty open sets in R^n, and $\{K_j\}$ is a finite family of compact sets in $R^n\}$, *and* $\mathcal{T}(\mathcal{F}) = \{\cup \mathcal{B}^* : \mathcal{B}^* \subset \mathcal{B}(\mathcal{F})\}$.

It can be easily proved that $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$ is a topological space. Similarly we know that $(\mathcal{K}, \mathcal{T}(\mathcal{K}))$, $(\mathcal{P}, \mathcal{T}(\mathcal{P}))$ and $(\mathcal{P}^*, \mathcal{T}(\mathcal{P}^*))$ are topological spaces.

2. Some Lemmas

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Lemma 2.1 ([3,4]). *Topological spaces $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$ $((\mathcal{K}, \mathcal{T}(\mathcal{K}))$, $(\mathcal{P}, \mathcal{T}(\mathcal{P}))$ and $(\mathcal{P}^*, \mathcal{T}(\mathcal{P}^*)))$ are Hausdorff spaces with countable bases.*

From Lemma 2.1 we know that the convergency can be characterized by sequence in above spaces.

Lemma 2.2 ([3]). *Let $\{F_i : i \in \Omega\}$ be a sequence in $\mathcal{F}$. Then $F_i \xrightarrow{\mathcal{F}} F$ if and only if the following two conditions are satisfied:*

1). *If G is an open set of R^n and $G \cap F \neq \emptyset$, then G intersects eventually with $\{F_i : i \in \Omega\}$ (that is, there is an $N \in \Omega$, such that $G \cap F_i \neq \emptyset$ for each $i > N$),*

2). *If K is a compact set of R^n with $K \cap F = \emptyset$, then K does not cofinally intersect with $\{F_i : i \in \Omega\}$ (that is, there is an $N \in \Omega$, such that $K \cap F_i = \emptyset$ for each $i > N$).*

Lemma 2.3 ([4]). *If $F_i \xrightarrow{\mathcal{F}} F$ in $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$, $y_i \in F_i (i \in \Omega)$ and $y_i \to y$ in R^n, then $y \in F$.*

Lemma 2.4 ([4]). *If $F_i \xrightarrow{\mathcal{F}} F$ in $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$ and $y \in F$, then we can pick $y_i \in F_i (i \in \Omega)$ with $y_i \to y$.*

It is clear that the arguments of Lemma 2.3 and Lemma 2.4 are also true for spaces $(\mathcal{K}, \mathcal{T}(\mathcal{K}))$, $(\mathcal{P}, \mathcal{T}(\mathcal{P}))$ and $(\mathcal{P}^*, \mathcal{T}(\mathcal{P}^*))$.

Lemma 2.5 ([4]). *The spaces $(\mathcal{P}, \mathcal{T}(\mathcal{P}))$ and $(\mathcal{P}^*, \mathcal{T}(\mathcal{P}^*))$are compact spaces.*

3. Two counterexamples

In the section we shall give some examples which imply that some results are not true in [3].

Definition 3.1 ([3,6]). *Let $\{F_i : i \in \Omega\}$ be a sequence in the space $\mathcal{F}$. The upper limit $\overline{\lim} F_i$ and lower limit $\underline{\lim} F_i$ of $\{F_i : i \in \Omega\}$ are defined respectively as follows:*

$$\overline{\lim} F_i = \cap_{i>0} \overline{\cup_{j>i} F_j} \qquad \underline{\lim} F_i = \overline{\cup_{i>0} \cap_{j>i} F_j}$$

As concerns the upper limit and the lower limit, there are the following results in [3]:

a). Let $\{F_i : i \in \Omega\}$ be a sequence in the space $\mathcal{F}$. Then $F_i \xrightarrow{\mathcal{F}} F$ iff $\overline{\lim} F_i = \underline{\lim} F_i = F$ ([3,p.18]).

b). The set $\underline{\lim} F_i$ is the intersection of the limits of all converging subsequence of $\{F_i : i \in \Omega\}$. That means $\underline{\lim} F_i = \cap \{F^* : \exists \{F_{n_i} : i \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $F_{n_i} \xrightarrow{\mathcal{F}} F^*\}$. ([3,p.18]).

c). Let $\{F_i : i \in \Omega\}$ be a sequence in $\mathcal{F}$. If $F \in \mathcal{F}$ satisfies condition *) as follows:

*) When an open set U of R^n intersects with F, then U intersects eventually with $\{F_i : i \in \Omega\}$,

then $F \subset \underline{\lim} F_i$. That is $\underline{\lim} F_i$ is the biggest closed set satisfying condition *) ([3,p.18. Th.1.2.4]).

The following example 3.2 shows that a), b) and c) are not true.

Example 3.2. Let $F_i = \{(\frac{1}{i}, 0, \cdots, 0)\}$ for each $i \in \Omega$ and $F = \{(0, 0, \cdots, 0)\}$. Then it is clear that $F_i \xrightarrow{\mathcal{K}} F$. But $\overline{\lim} F_i = \cap_{i>0} \overline{\cup_{j>i} F_j} = F$, and $\underline{\lim} F_i = \overline{\cup_{i>0} \cap_{j>i} F_j} = \emptyset \neq F$. Hence a) is not true. It is not difficult to prove that b) and c) are not true.

Definition 3.3 ([3]). *Suppose that E is a separable space, and $\Psi : E \longrightarrow \mathcal{F}$ is a map. We call Ψ to be upper semi-continuous (resp. lower semi-continuous), if for any convergent sequence $\{x_i : i \in \Omega\}$ of E with $x_i \xrightarrow{E} x$, $\Psi(x) \supset \overline{\lim}\Psi(x_i)$ (resp. $\Psi(x) \subset \underline{\lim}\Psi(x_i)$) holds.*

There are some results concerning with upper semicontinuity and lower semicontinuity in [3]. They are following:

d). A map $\Psi : E \longrightarrow \mathcal{F}$ is continuous iff Ψ is upper semicontinuous and lower semicontinuous ([3,p.19]).

e). The map $\Psi : \mathcal{F} \times \mathcal{K} \longrightarrow \mathcal{F}$ with $\Psi(F, K) = F \oplus K$ is continuous, and the map $\Phi : \mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K}$ with $\Phi(B, K) = B \oplus K$ is continuous, where $F \in \mathcal{F}$ and, K and B are in $\mathcal{K}$ ([3,p.17, Th.1.2.3]).

Unfortunately we find that the argument d) is not true according as Definition 3.1. In fact, if we let Ψ be the identity map from $\mathcal{K}$ to $\mathcal{K}$, $F_i = \{(\frac{1}{i}, 0, \cdots, 0)\}$ for each $i \in \Omega$ and $F = \{(0, \cdots, 0)\} \in \mathcal{K}$, then Ψ is continuous, and $\Psi(F_i) = F_i \xrightarrow{\mathcal{K}} F = \Psi(F)$, but $\underline{\lim}\Psi(F_i) = \underline{\lim} F_i = \emptyset \neq F = \overline{\lim} F_i = \overline{\lim}\Psi(F_i)$. (see Example 3.2).

The following example implies that the argument e) is not true too.

Example 3.5. Let $n = 1$, that is $R^n = R$, set $F_i = \{0, i\}$, $K_i = \{0, -i+1\}$, and $F = K = \{0\}$. Then we have that $F_i \xrightarrow{\mathcal{K}} F$ and $K_i \xrightarrow{\mathcal{K}} K$ by Lemma 2.2. In this case, $F_i \oplus K_i = \{0, 1, -i+1, i\}$ and $F \oplus K = \{0\}$. Also we get that $F_i \oplus K_i \xrightarrow{\mathcal{K}} \{0, 1\} \in \mathcal{K}$ by Lemma 2.2. But $F \oplus K = \{0\} \neq \{0, 1\}$. Hence the dilation is not a continuous map from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$.

4. The continuity on the erosion and dilation.

Firstly we shall give the following definition which is important for the main results of this section.

Definition 4.1. *For a sequence $\{F_i : i \in \Omega\}$ of $\mathcal{F}$, let*

$\widetilde{F} = \{x$: *For each neighborhood U_x of x, there is a subsequence $\{F_{i_j} : j \in \Omega\}$ such that $U_x \cap F_{i_j} \neq \emptyset$.$\}$ and*

$\underline{F} = \{x$: *For each neighborhood U_x of x, U_x intersects eventually with $\{F_i : i \in \Omega\}\}$, where U_x is the neighborhood of x in R^n. We call $\widetilde{F}$ upper closed limit and $\underline{F}$ lower closed limit.*

By use of $\widetilde{F}$ and $\underline{F}$, we have already proved the following:

Theorem 4.2 ([4]). *Let $\{F_i : i \in \Omega\}$ be a sequence in $\mathcal{F}$. Then $F_i \xrightarrow{\mathcal{F}} F$ iff $F = \widetilde{F} = \underline{F}$.*

Next we shall have the following characterizations of $\underline{F}$.

Theorem 4.3. *Let $\{F_i : i \in \Omega\}$ be a sequence in $\mathcal{P}$. Then $\underline{F}$ is the intersection of the* *limits of all converging subsequence of* $\{F_i : i \in \Omega\}$*, that is,* $\underline{F} = \cap\{F^* : \exists\{F_{i_j} : j \in \Omega\} \subset \{F_i : i \in \Omega\}$ *with* $F_{i_j} \xrightarrow{\mathcal{P}} F^*\}$.

Proof. Suppose that $x \in \underline{F}$ and $\{F_{i_j} : j \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $F_{i_j} \xrightarrow{\mathcal{P}} F^*$. We shall prove $x \in F^*$. Let $\{U_i(x) : i \in \Omega\}$ be a decreasing neighborhood base of x in Y.

For $U_1(x)$, we have a sequence $\{x_j^1 \in U_1(x) \cap F_{i_j} : j \in \Omega$ and $j > k_1\}$ for some k_1. Similarly for $U_2(x)$, we have a sequence $\{x_j^2 \in U_2(x) \cap F_{i_j} : j \in \Omega$ and $j > k_2\}$ for some $k_2 > k_1$. By induction, we have a sequence $\{x_j^l \in U_l(x) \cap F_{i_j} : j \in \Omega$ and $j > k_l\}$ for some $k_l > k_{l-1}$. Hence $\{x_{k_l}^l : l \in \Omega\} \to x$, that is $x \in F^*$ by Lemma 2.3.

On the other hand, let $x \in \cap\{F^* : \exists\{F_{i_j} : j \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $F_{i_j} \xrightarrow{\mathcal{P}} F^*\}$, we shall prove that $x \in \underline{F}$. That is, we must prove that U_x intersects eventually with $\{F_i : i \in \Omega\}$ for each neighborhood U_x of x in Y.

Suppose not, there were a neighborhood U of x, and a subsequence $\{F_i^1 : i \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $U \cap F_i^1 = \emptyset$ (for each $i \in \Omega$). Since $\mathcal{P}$ is a compact space (see Lemma 2.5), we have a subsequence $\{F_{i_j}^1 : j \in \Omega\} \subset \{F_i^1 : i \in \Omega\}$ with $F_{i_j}^1 \xrightarrow{\mathcal{P}} F^*$. We know that $U \cap F^* = \emptyset$ by Lemma 2.2, i.e., $x \notin F^*$. This contradicts to that $x \in \cap\{F^* : \exists\{F_{i_j} : j \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $F_{i_j} \xrightarrow{\mathcal{P}} F^*\}$. Hence $x \in \underline{F}$.

Theorem 4.4. *Let* $\{F_i : i \in \Omega\}$ *be a sequence in* $\mathcal{F}$*. If* $F \in \mathcal{F}$ *satisfies condition *) as follows:*

**). When an open set* U *of* R^n *intersects with* F*, then* U *intersects eventually with* $\{F_i : i \in \Omega\}$*,*

then $F \subset \underline{F}$*. That is* $\underline{F}$ *is the biggest closed set satisfying condition *).*

Proof. Suppose $x \in F$ and F satisfies condition *). Then U_x intersects eventually with $\{F_i : i \in \Omega\}$ for each neighborhood U_x of x in R^n. By Definition 4.1, $x \in \underline{F}$, i.e., $F \subset \underline{F}$. It is clear that $\underline{F}$ is closed.

Theorems 4.2, 4.3 and 4.4 show that a), b) and c) in section 3 are true by Definition 4.1.

Definition 4.5. *For* A *and* $B \subset R^n$*, let* $\rho(A, B) = inf\{\epsilon : B \subset U(A, \epsilon)\}$*, and* $H(A, B) = max\{\rho(A, B), \rho(B, A)\}$.

Lemma 4.6 ([4]). *The topological spaces* $\mathcal{P}$ *and* $\mathcal{P}^*$ *are metrizable, and* $H(A, B)$ *is their metric.*

The following theorem is interesting in comparison with Example 3.5.

Theorem 4.7. *The map* $\Psi : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}^*$ *with* $\Psi(K, B) = K \oplus B$ *is a continuous map.*

Proof. We prove first that, if K and B are compact, then $K \oplus B$ is compact. In fact, if $z_i \in K \oplus B = \{x + y : x \in K, y \in B\}$, then $\exists x_i \in K$ and $y_i \in B$ with $z_i = x_i + y_i$. Since K and B are compact, we can get a converging subsequences $\{x_{i_j} : j \in \Omega\} \subset \{x_i : i \in \Omega\}$ and $\{y_{i_{j_k}} : k \in \Omega\} \subset \{y_{i_j} : j \in \Omega\}$, hence $\{z_{i_{j_k}} : k \in \Omega\} \subset \{z_i : i \in \Omega\}$ is converging. That is $K \oplus B$ is compact.

Suppose that $B_i \xrightarrow{\mathcal{P}} B$, and $K_i \xrightarrow{\mathcal{P}} K$. Then we have that $H(B_i, B) \to 0$ and $H(K_i, K) \to 0$ by Lemma 4.6. Therefore for each $\epsilon > 0$, there is an $N \in \Omega$, such that $B_i \subset B \oplus \epsilon D$, $B \subset B_i \oplus \epsilon D$, $K_i \subset K \oplus \epsilon D$ and $K \subset K_i \oplus \epsilon D$ for each $i > N$. Then

$$B_i \oplus K_i \subset (B \oplus \epsilon D) \oplus (K \oplus \epsilon D) = B \oplus K \oplus 2\epsilon D$$

and

$$B \oplus K \subset (B_i \oplus \epsilon D) \oplus (K_i \oplus \epsilon D) = B_i \oplus K_i \oplus 2\epsilon D.$$

Hence $H(B_i \oplus K_i, B \oplus K) \to 0$, i.e.,$B_i \oplus K_i \xrightarrow{\mathcal{P}^*} B \oplus K$, and hence Ψ is continuous.

Lemma 4.8. *If $\{F_n : n \in \Omega\}$ is a sequence in $\mathcal{F}$, then* $\widetilde{F} = \overline{\lim} F_n$.

Proof. It is clear by Definitions 3.1 and 3.4.

Theorem 4.9. *The map $\Phi : \mathcal{P} \times \mathcal{P} \to \mathcal{P}^*$ with $\Phi(K, F) = K \ominus F$ is upper semicontinuous.*

Proof. If F, K and $\check{K}$ are in $\mathcal{P}$, then $F \ominus \check{K} = \cap_{k \in \check{K}} F[k]$ is compact, and it is clear that $F \ominus \check{K} = \{x : K[x] \subset F\}$. For $F_i \xrightarrow{\mathcal{P}} F$, and $K_i \xrightarrow{\mathcal{P}} K$, we have $F_i \ominus \check{K}_i = \{x : K_i[x] \subset F_i\}$ and $F \ominus \check{K} = \{x : K[x] \subset F\}$. We shall prove that $F \ominus \check{K} \supset \overline{\lim}(F_i \ominus \check{K}_i)$ (see Lemma 4.8).

For $z \in \overline{\lim}(F_i \ominus \check{K}_i)$, let $\{U_j(z) : j \in \Omega\}$ be a decreasing neighborhood base of z in $Y \oplus Y$. For each $j \in \Omega$, $U_j(z)$ intersects infinitely many elements of $\{F_i \ominus \check{K}_i : i \in \Omega\}$. Then we pick out $z_j \in U_j(z) \cap (F_{i_j} \ominus \check{K}_{i_j})$, where $i_j < i_{j+1}$, then $K_{i_j}[z_j] \subset F_{i_j}$, i.e., for any $k_{i_j} \in K_{i_j}$, $k_{i_j} + z_j = f_{i_j}$ for some $f_{i_j} \in F_{i_j}$. Furthermore it is seen that $z_j \to z$.

We need to prove that $z \in F \ominus \check{K}$, that is, $K[z] \subset F$. Equivalently we shall prove that for each $k \in K$, there is some $f \in F$ with $k + z = f$. For each $k \in K$, since $K_i \xrightarrow{\mathcal{P}} K$, we can get a sequence $\{k_i : i \in \Omega\}$ with $k_i \to k$ by Lemma 2.4, where $k_i \in K_i$. We know that if $k_{i_j} \in \{k_i : i \in \Omega\}$, then $k_{i_j} + z_j = f_{i_j}$ for some $f_{i_j} \in F_{i_j}$. Because $z_j \to z$ and $k_{i_j} \to k$, we have $f_{i_j} = k_{i_j} + z_j \to k + z$. Hence $k + z = f \in F$ by Lemma 2.3. Therefore we have proved that for each $k \in K$, and $k + z = f \in F$, that is $K[z] \subset F$. i.e., $F \ominus \check{K} \supset \overline{\lim}(F_i \ominus \check{K}_i)$. So Φ is upper semicontinuous.

5. The conclusions

In [4] we proved that the topological space $\mathcal{P}$ is a compact metric space with a countable base, and $H(A, B)$ is a metric on $\mathcal{P}$. The metric $H(A, B)$ coincides with our physical intuition. In this paper we proved the dilation operation is a continuous map and the erosion operation is a upper semicontinuous map on the space $\mathcal{P}$. From the erosion and dilation we can get all other operations of mathematical morphology. In actual pattern recognition, the treated pattern is always limited in a large scope, and a pattern can be considered as a compact set in mathematics. Hence we deem that the space $\mathcal{P}$ is a good mathematical space for pattern recognition.

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*Dept. of Math. Shantou University,
Shantou 515063, Guang-Dong, China.
e-mail address: zmgao@mailserv.stu.edu.cn

**Dept. of Math. Osaka kyoiku University,
4-698 Asahigaoka, Kashiwara, Osaka 582-8582, Japan
e-mail adress: yasui@cc.osaka-kyoiku.ac.jp