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ON LINEAR OPERATORS FROM ORLICZ SPACES INTO LOCALLY CONVEX LINEAR-TOPOLOGICAL SPACES

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ABSTRACT. Let Y be a sequentially complete,locally convex linear-topological space,(E,Σ,μ) a non-atomic measure space and φ a real nondecreasing and continous for $u \geq 0$ function,equal to 0 for $u = 0$.We prove the identity of the class of linear and pseudomodular continous operators from the Orlicz space $L^{*\varphi}_{\rho}(E,\Sigma,\mu)$ into Y with the class of similar operators from the Orlicz space $L^{*\overline{\varphi}}_{\rho}(E,\Sigma,\mu)$ into Y,where

$$\overline{\varphi}(u) = \int_0^u p(t)dt \quad \text{for} \quad u \geq 0 \quad \text{and} \quad p(t) = \inf_{t<s} \frac{\varphi(s)}{s} \quad \text{for} \quad t \geq 0.$$

Also,we show that the class of linear and pseudonorm continous operators from the space of finite elements $L^{o\varphi}(E,\Sigma,\mu)$ into Y is the same as from the space $L^{o\overline{\varphi}}(E,\Sigma,\mu)$ into Y.The other proof of this fact, using Rademacher functions one can find in [1](th.2.2 b).Our result is a little bit more general than the one mentioned above.But first of all,our proof is simple and esentially different from that in [1].

1. THE ORLICZ SPACE

Definition 1.1 *A $\tilde{\varphi}$-function we call a real,nondecreasing and continous for $u \geq 0$ function,equal to 0 for $u = 0$.A $\tilde{\varphi}$-function φ is called convex,if it satisties the Jensen inequality*

$$\varphi(\alpha u + \beta v) \leq \alpha\varphi(u) + \beta\varphi(v) \quad \text{for } u,v,\alpha,\beta \geq 0 \text{ , where } \alpha + \beta = 1.$$

As usual in the theory of the Orlicz spaces ([2],[3],[4]),a φ-function we call a real,nondecreasing and continous for $u \geq 0$ function,equal to 0 only for $u = 0$ and tending to ∞ as $u \to \infty$.In this paper the use of the $\tilde{\varphi}$-functions instead of the φ-functions simplifies our considerations.

1.2. Let (E,Σ,μ) denote a measure space.Obviously,we assume that the measure μ is σ-finite on E.By $S = S(E,\Sigma,\mu)$ we denote the space of μ-measurable,real or complex-valued functions defined on E.The measure space (E,Σ,μ) we call non-atomic,if for every set $G \in \Sigma$ there exists a set $F \subset G$, $F \in \Sigma$ with $\mu(F) = \frac{1}{2}\mu(G)$.Then we say also that the measure μ and the space S are non-atomic.

1.3. Let φ be a $\tilde{\varphi}$-function and (E,Σ,μ) a measure space.For $f \in S$ we write

$$\rho_\varphi(f) = \int_E \varphi(|f(x)|)d\mu.$$

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In the space S the functional $\rho_\varphi(\cdot)$ is a pseudomodular in the Musielak-Orlicz sense [6].

By $L^{*\varphi}_\rho = L^{*\varphi}_\rho(E,\Sigma,\mu)$ we denote the class of those functions $f \in S$ for which $\rho_\varphi(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$ and by $L^{o\varphi} = L^{o\varphi}(E,\Sigma,\mu)$ the class of those functions $f \in S$ for which $\rho_\varphi(\lambda f) < \infty$ for all $\lambda > 0$.The classes $L^{*\varphi}_\rho$ and $L^{o\varphi}$ are linear subspaces of S and $L^{o\varphi} \subset L^{*\varphi}_\rho$.
The class $L^{*\varphi}_\rho$ we call the Orlicz space and the class $L^{o\varphi}$ the space of finite elements ([2],[3]).

The spaces $L^{*\varphi}_\rho$ and $L^{o\varphi}$ we call non-atomic if the measure space (E,Σ,μ) is non atomic.

1.4. In the Orlicz space $L^{*\varphi}_\rho$ the functional $||\cdot||_\varphi$ defined by the formula

$$||f||_\varphi = \inf\{\varepsilon > 0 : \rho_\varphi(\frac{f}{\varepsilon}) \le \varepsilon\} \quad (f \in L^{*\varphi}_\rho),$$

is an F-pseudonorm.In the case,when φ is a convex $\tilde{\varphi}$-function,the formula

$$||f||^*_\varphi = \inf\{\varepsilon > 0 : \rho_\varphi(\frac{f}{\varepsilon}) \le 1\} \quad (f \in L^{*\varphi}_\rho),$$

determines a B-pseudonorm in $L^{*\varphi}_\rho$ equivalent to $||\cdot||_\varphi$,([2],[3],[6]).

Definition 1.5 *We say that the sequence (f_n) in $L^{*\varphi}_\rho$ is pseudonorm convergent to f in $L^{*\varphi}_\rho$,if $||f_n - f||_\varphi \to 0$ as $n \to \infty$.*

It is easily verified that the sequence (f_n) in $L^{*\varphi}_\rho$ is pseudonorm convergent to f in $L^{*\varphi}_\rho$ if and only if for any $\lambda > 0$ there holds $\rho_\varphi(\lambda(f_n - f)) \to 0$ as $n \to \infty$. The space $L^{o\varphi}$ is a closed subspace of the Orlicz space $L^{*\varphi}_\rho$ with respect on the pseudonorm convergence.
Also,we say that the sequence (f_n) in $L^{*\varphi}_\rho$ is pseudomodular convergent or φ-convergent to $f \in L^{*\varphi}_\rho$,and write $f_n \xrightarrow{\varphi} f$, if $\rho_\varphi(\lambda(f_n - f)) \to 0$ as $n \to \infty$ for some $\lambda > 0$,dependent on (f_n),([3]).

1.6. In the case,when φ is a φ-function,the suffix "pseudo" in 1.3,1.4 and 1.5 we omit,because then we have the classical Orlicz spaces ([2],[3]).

2. Convex $\tilde{\varphi}$-functions $\overline{\varphi}$ and $\overline{\overline{\varphi}}$ generated by φ.

2.1. Let φ be a $\tilde{\varphi}$-function.We define

$$\overline{\varphi}(u) = \int_0^u p(t)dt \quad \text{for} \quad u \ge 0 \ \text{, where} \quad p(t) = \inf_{t<s} \frac{\varphi(s)}{s} \quad \text{for} \quad t \ge 0.$$

Since the function p is non-negative and nondecreasing for $t \ge 0$, so the function $\overline{\varphi}$ is a convex $\tilde{\varphi}$-function,([2]).Also,we observe that if φ is such that $\lim_{u\to\infty} \inf \frac{\varphi(u)}{u} = 0$,then $\overline{\varphi}(u) = 0$ for $u \ge 0$,and if φ is a φ-function with the property $\lim_{u\to\infty} \inf \frac{\varphi(u)}{u} > 0$, then $\overline{\varphi}$ is a convex φ-function.

Further,by Ψ_φ we denote the class of those convex $\tilde{\varphi}$-function ψ for which the inequality $\psi(u) \le \varphi(u)$ for $u \ge 0$ holds.The class Ψ_φ is not empty,because the function $\psi_0(u) = 0$ for $u \ge 0$ belongs to Ψ_φ.We define

$$\overline{\overline{\varphi}}(u) = \sup\{\psi(u) : \psi \in \Psi_\varphi\} \quad \text{for} \quad u \geq 0.$$

Obviously,$\overline{\overline{\varphi}}$ is a function satisfying the inequality $0 \leq \overline{\overline{\varphi}}(u) \leq \varphi(u)$ for $u \geq 0$.The function $\overline{\overline{\varphi}}$ is nondecreasing and convex for $u \geq 0$,because it is a supremum of nondecreasing and convex functions for $u \geq 0$.The Jensen inequality for $\overline{\overline{\varphi}}$ guarantees the continuity of $\overline{\overline{\varphi}}$ for $u > 0$ and the inequality $0 \leq \overline{\overline{\varphi}} \leq \varphi$ the continuity of $\overline{\overline{\varphi}}$ for $u = 0$.Hence $\overline{\overline{\varphi}}$ is the greatest convex $\tilde{\varphi}$-function satisfying the inequality $\overline{\overline{\varphi}}(u) \leq \varphi(u)$ for $u \geq 0$.

Theorem 2.2 *For any $\tilde{\varphi}$-function φ the following inequality holds*

$$\overline{\varphi}(u) \leq \overline{\overline{\varphi}}(u) \leq \overline{\varphi}(2u) \quad \text{for} \quad u \geq 0.$$

Proof. From the definition of $\overline{\varphi}$ we get for $u > 0$

$$\overline{\varphi}(u) \leq up(u) = u \inf_{u<s} \frac{\varphi(s)}{s} \leq u\frac{\varphi(u)}{u} = \varphi(u).$$

Since $\overline{\varphi}$ is a convex $\tilde{\varphi}$-function,so it must be $\overline{\varphi}(u) \leq \overline{\overline{\varphi}}(u)$ for $u \geq 0$.
On the other hand,from the Jensen inequality for $\overline{\overline{\varphi}}$ and the fact that $\overline{\overline{\varphi}}(0) = 0$ it follows that the quotient $\dfrac{\overline{\overline{\varphi}}(u)}{u}$ is a nondecreasing function for $u > 0$.Therefore by virtue of the inequality $\overline{\overline{\varphi}} \leq \varphi$ we have for $u > 0$

$$\overline{\overline{\varphi}}(u) = u\frac{\overline{\overline{\varphi}}(u)}{u} = u \inf_{u<s} \frac{\overline{\overline{\varphi}}(s)}{s} \leq u \inf_{u<s} \frac{\varphi(s)}{s} = up(u) \leq \int_u^{2u} p(t)dt \leq \overline{\varphi}(2u).$$

Hence the inequality $\overline{\overline{\varphi}}(u) \leq \overline{\varphi}(2u)$ for $u \geq 0$ is also true. □

2.3 *If the $\tilde{\varphi}$-function φ satisfies the condition $\lim\limits_{u\to\infty} \dfrac{\varphi(u)}{u} = \infty$ then the following equality holds $\overline{\overline{\varphi}} = (\varphi^*)^*$, where*

$$\varphi^*(v) = \sup\{uv - \varphi(u) : u \geq 0\} \quad \text{for} \quad v \geq 0.$$

This theorem for φ-functions one can find in [4].Since the proof of our theorem is analogical,we omit it.

2.4 *For arbitrary $\tilde{\varphi}$-function φ the following equality holds*

$$(*) \qquad \overline{\overline{\varphi}}(u) = \inf \sum_{k=1}^{m} \alpha_k \varphi(u_k) \qquad (u \geq 0),$$

where infimum is taken over all convex combinations

$$u = \sum_{k=1}^{m} \alpha_k u_k, \text{ where } \alpha_k, u_k \geq 0 \text{ for } k = 1, \ldots, m \text{ and } \sum_{k=1}^{m} \alpha_k = 1.$$

Proof. By $\tilde{\varphi}$ we denote the function defined by the right side of the equality (*).Let us take

$$u = \sum_{k=1}^{m} \alpha_k u_k, \text{ where } \alpha_k, u_k \geq 0 \text{ for } k = 1, \ldots, m \text{ and } \sum_{k=1}^{m} \alpha_k = 1.$$

Then by virtue of the inequality $\overline{\overline{\varphi}} \leq \varphi$ and the convexity of $\overline{\overline{\varphi}}$ we have

$$\overline{\overline{\varphi}}(u) \leq \sum_{k=1}^{m} \alpha_k \overline{\overline{\varphi}}(u_k) \leq \sum_{k=1}^{m} \alpha_k \varphi(u_k).$$

From this we deduce that $\overline{\overline{\varphi}}(u) \leq \tilde{\varphi}(u)$ dla $u \geq 0$.

On the other hand,from the definition of $\tilde{\varphi}$ we get immediately the inequality $\tilde{\varphi}(u) \leq \varphi(u)$ for $u \geq 0$.We shall show that $\tilde{\varphi}$ is a convex $\tilde{\varphi}$-function.Let us take arbitrary $\varepsilon > 0$, $u', u'', \alpha, \beta \geq 0, \alpha + \beta = 1$. We observe that there exist $\alpha'_k, u'_k \geq 0$,where $k = 1, \ldots, m'$,such that

$$\sum_{k=1}^{m'} \alpha'_k = 1\ ,\ u' = \sum_{k=1}^{m'} \alpha'_k u'_k, \ \text{ and } \ \sum_{k=1}^{m'} \alpha'_k \varphi(u'_k) \leq \tilde{\varphi}(u') + \varepsilon,$$

and $\alpha''_k, u''_k \geq 0$,where $k = 1, \ldots, m''$ such that

$$\sum_{k=1}^{m''} \alpha''_k = 1\ ,\ u'' = \sum_{k=1}^{m''} \alpha''_k u''_k, \ \text{ and } \ \sum_{k=1}^{m''} \alpha''_k \varphi(u''_k) \leq \tilde{\varphi}(u'') + \varepsilon.$$

Let us denote $m = m' + m'', \alpha_k = \alpha\alpha'_k, u_k = u'_k$ for $k = 1, \ldots, m'$
and $\alpha_{m'+k} = \beta\alpha''_k, u_{m'+k} = u''_k$ for $k = 1, \ldots, m''$. Since $\alpha_k, u_k \geq 0$ for $k = 1, \ldots, m$,

$$\sum_{k=1}^{m} \alpha_k = \alpha \sum_{k=1}^{m'} \alpha'_k + \beta \sum_{k=1}^{m''} \alpha''_k = 1 \quad \text{and}$$

$$\sum_{k=1}^{m} \alpha_k u_k = \alpha \sum_{k=1}^{m'} \alpha'_k u'_k + \beta \sum_{k=1}^{m''} \alpha''_k u''_k = \alpha u' + \beta u'',$$

so we get

$$\begin{aligned}\tilde{\varphi}(\alpha u' + \beta u'') &\leq \sum_{k=1}^{m} \alpha_k \varphi(u_k) = \alpha \sum_{k=1}^{m'} \alpha'_k \varphi(u'_k) + \beta \sum_{k=1}^{m''} \alpha''_k \varphi(u''_k) \\ &\leq \alpha(\tilde{\varphi}(u') + \varepsilon) + \beta(\tilde{\varphi}(u'') + \varepsilon) = \alpha\tilde{\varphi}(u') + \beta\tilde{\varphi}(u'') + \varepsilon.\end{aligned}$$

From this we obtain the Jensen inequality for $\tilde{\varphi}$ and thus the function $\tilde{\varphi}$ is convex for $u \geq 0$.Now,the convexity of $\tilde{\varphi}$ implies the continuity of $\tilde{\varphi}$ for $u > 0$ and the inequality $\overline{\overline{\varphi}} \leq \tilde{\varphi} \leq \varphi$ the continuity of $\tilde{\varphi}$ for $u = 0$ and $\tilde{\varphi}(0) = 0$.The function $\tilde{\varphi}$ is also nondecreasing for $u \geq 0$,because for $0 \leq u_1 < u_2$ we have

$$\tilde{\varphi}(u_1) = \tilde{\varphi}(\frac{u_1}{u_2}u_2 + (1 - \frac{u_1}{u_2})0) \leq \frac{u_1}{u_2}\tilde{\varphi}(u_2) \leq \tilde{\varphi}(u_2).$$

Hence $\tilde{\varphi}$ is a convex $\tilde{\varphi}$-function.This fact and the inequality $\overline{\overline{\varphi}} \leq \tilde{\varphi} \leq \varphi$ imply finally the equality $\overline{\overline{\varphi}} = \tilde{\varphi}$. □

2.5.Remark *In 2.4 we may assume that the numbers α_k are of finite binary representation.*

Proof. According to the formula 2.4 (*) for arbitrary $u \geq 0$ and $\varepsilon > 0$ there exist such $\alpha_k, u_k \geq 0$,where $k = 1, \ldots, m$ that

$$\sum_{k=1}^{m} \alpha_k = 1\ , \quad u = \sum_{k=1}^{m} \alpha_k u_k \ \text{ i } \ \overline{\overline{\varphi}}(u) \leq \sum_{k=1}^{m} \alpha_k \varphi(u_k) \leq \overline{\overline{\varphi}}(u) + \frac{\varepsilon}{2}.$$

If $m = 1$,then $\alpha_1 = 1$.Therefore let us suppose $m > 1$.Since $\alpha_k \geq 0$ for $k = 1, \dots, m$ and $\alpha_1 + \dots + \alpha_m = 1$,so it must be $\alpha_{k_0} \geq \frac{1}{m}$ for some index k_0.We may assume that $k_0 = m$.Let M be a positive number such that $u_k \leq M$ and $\varphi(u_k) \leq M$ for $k = 1, \dots, m$.Since the function φ is continous in the point u_m, so there exists δ satisfying $0 < \delta \leq \varepsilon$ and such that $|\varphi(u_m) - \varphi(v)| \leq \frac{\varepsilon}{4}$ if $v \geq 0$ and $|u_m - v| \leq \delta$.We choose numbers β_k, $k = 1, \dots, m-1$,of finite binary representation such that $0 \leq \beta_k \leq \alpha_k$ and $\alpha_k - \beta_k \leq \frac{\delta}{2m^2 M}$ for $k = 1, \dots, m$. We observe that the number $\beta_m = 1 - (\beta_1 + \dots + \beta_{m-1})$ is of finite binary representation and satisfies the inequalities

$$\frac{1}{m} \leq \alpha_m \leq \beta_m \leq 1 \quad \text{and} \quad \beta_m - \alpha_m = \sum_{k=1}^{m-1} (\alpha_k - \beta_k) \leq \frac{\delta}{2mM}.$$

Let us put $v_k = u_k$ for $k = 1, \dots, m-1$,

$$v_m = \frac{1}{\beta_m}\Big(\sum_{k=1}^{m-1} (\alpha_k - \beta_k) u_k + \alpha_m u_m \Big).$$

We see that $v_k \geq 0$ for $k = 1, \dots, m$, $\sum_{k=1}^{m} \beta_k v_k = \sum_{k=1}^{m} \alpha_k u_k = u$,

$$|u_m - v_m| \leq \frac{1}{\beta_m}\Big(\sum_{k=1}^{m-1} (\alpha_k - \beta_k) u_k + (\beta_m - \alpha_m) u_m \Big) \leq$$
$$\leq m\Big(m \frac{\delta}{2m^2 M} M + \frac{\delta}{2mM} M\Big) \leq \delta.$$

Therefore

$$\sum_{k=1}^{m} \beta_k \varphi(v_k) \leq \sum_{k=1}^{m} \beta_k \varphi(v_k) - \sum_{k=1}^{m} \alpha_k \varphi(u_k) + \overline{\overline{\varphi}}(u) + \frac{\varepsilon}{2}$$
$$\leq \beta_m \varphi(v_m) - \alpha_m \varphi(u_m) + \overline{\overline{\varphi}}(u) + \frac{\varepsilon}{2} \leq \beta_m(\varphi(v_m) - \varphi(u_m))$$
$$+ (\beta_m - \alpha_m)\varphi(u_m) + \overline{\overline{\varphi}}(u) + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2mM} M + \overline{\overline{\varphi}}(u) + \frac{\varepsilon}{2} \leq \overline{\overline{\varphi}}(u) + \varepsilon$$

and we see that the remark is true. □

3. Lemmas on simple functions.

Definition 3.1 *A simple function we call a function $g \in S$ of the form*

$$g = \sum_{i=1}^{n} a_i \chi_{E_i}, \tag{+}$$

where a_i denote numbers and χ_{E_i} characteristic functions of sets $E_i \in \Sigma$ with finite measures.We will assume that the sets E_i are pairwise disjoint.

It is well known that for every function $f \in S$ there exists a sequence of simple functions (g_n) convergent to f everywhere on E and such that

$|g_n(x)| \le |f(x)|$ and $|f(x) - g_n(x)| \le |f(x)|$ for $x \in E$ and $n = 1, 2, \ldots$. From this fact we get immediately the following lemma.

Lemma 3.2 *For every function $f \in L^{*\varphi}_{\rho}$ with $\rho_\varphi(f) < \infty$ there exists a sequence of simple functions (g_n) such that $\rho_\varphi(g_n) \le \rho_\varphi(f)$ for $n = 1, 2, \ldots$ and $\rho_\varphi(f - g_n) \to 0$ as $n \to \infty$. Moreover,for every function $f \in L^{o\varphi}$ there exists a sequence of simple functions (g_n) such that $||g_n||_\varphi \le ||f||_\varphi$ for $n = 1, 2, \ldots$ and $||f - g_n||_\varphi \to 0$ as $n \to \infty$.*

Now we prove our fundamental lemma.

Lemma 3.3 *Let (E, Σ, μ) be a non-atomic measure space, φ a $\tilde{\varphi}$-function, $\delta > 0$ and g a simple function such that $\rho_{\overline{\overline{\varphi}}}(g) \le \delta$.Then there exist simple functions $g_1, \ldots, g_{2^k}$ such that $\rho_\varphi(g_l) \le 2\delta$ for $l = 1, \ldots, 2^k$ and*

$$g = \frac{1}{2^k}(g_1 + \cdots + g_{2^k}).$$

Proof. Let the simple functions g be of the form 3.1 (+).We denote $u_i = |a_i|$ for $i = 1, \ldots, n$. By virtue of 2.4 for every i there exist $u_{i,j} \ge 0$ and $\alpha_{i,j} > 0$, where $j = 1, \ldots, m_i$, such that

$$\sum_{j=1}^{m_i} \alpha_{i,j} = 1 \qquad\qquad u_i = \sum_{j=1}^{m_i} \alpha_{i,j} u_{i,j}$$

and

$$\sum_{j=1}^{m_i} \alpha_{i,j} \varphi(u_{i,j}) \le \overline{\overline{\varphi}}(u_i) + \delta\big(\sum_{i=1}^{n} \mu(E_i)\big)^{-1}.$$

On virtue of 2.5 we may assume that the the numbers $\alpha_{i,j}$ are of finite binary representation.Let k be a non-negative integer such that all numbers

$$k_{i,j} = 2^k \alpha_{i,j}, \quad (j = 1, \ldots, m_i, i = 1, \ldots, n),$$

are positive integers.Next,we denote by $K_{i,j,l}$, where the indices may assume the values $j = 1, \ldots, m_i, i = 1, \ldots, n$ and $l = 1, \ldots, 2^k$,the set of those positive integers $r \le 2^k$ for which there holds

$$k_{i,j-1,l} < r \le k_{i,j,l} \quad \text{or} \quad k_{i,j-1,l} < r + 2^k \le k_{i,j,l},$$

$$\text{where} \quad k_{i,0,l} = l \quad \text{and} \quad k_{i,j,l} = l + \sum_{v=1}^{j} k_{i,v}.$$

The measure space (E, Σ, μ) is non-atomic,therefore every set E_i may be divided on 2^k pairwise disjoint sets $E_{i,r}$ with measures $\mu(E_{i,r}) = \frac{1}{2^k}\mu(E_i)$ for $r = 1, \ldots, 2^k$.

We set

$$g_l = \sum_{i=1}^{n} \sum_{j=1}^{m_i} u_{i,j} \operatorname{sign} a_i \sum_{r \in K_{i,j,l}} \chi_{E_{i,r}} \quad \text{for} \quad l = 1, \ldots, 2^k.$$

The set $K_{i,j,l}$ posesses $k_{i,j}$ elements and therefore we have

$$\rho_\varphi(g_l) = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\varphi(u_{i,j})\sum_{r\in K_{i,j,l}}\mu(E_{i,r}) = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\varphi(u_{i,j})k_{i,j}\frac{1}{2^k}\mu(E_i) =$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m_i}\alpha_{i,j}\varphi(u_{i,j})\mu(E_i) \le \sum_{i=1}^{n}\overline{\overline{\varphi}}(u_i)\mu(E_i) + \delta = \rho_{\overline{\overline{\varphi}}}(g) + \delta \le 2\delta.$$

Finally,let us observe that

$$\frac{1}{2^k}\sum_{l=1}^{2^k} g_l = \frac{1}{2^k}\sum_{i=1}^{n}\sum_{j=1}^{m_i} u_{i,j}\operatorname{sign} a_i \sum_{l=1}^{2^k}\sum_{r\in K_{i,j,l}}\chi_{E_{i,r}} =$$
$$= \frac{1}{2^k}\sum_{i=1}^{n}\sum_{j=1}^{m_i} k_{i,j}u_{i,j}\chi_{E_i}\operatorname{sign} a_i = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\alpha_{i,j}u_{i,j}\chi_{E_i}\operatorname{sign} a_i =$$
$$= \sum_{i=1}^{n} u_i\chi_{E_i}\operatorname{sign} a_i = g.$$

□

3.4 *Let (E,Σ,μ),φ,δ be as in 3.3 and let g be a simple function such that $||g||_{\overline{\overline{\varphi}}} \le \delta$.Then there exist simple functions $g_1,\dots,g_{2^k}$ such that $||g_l||_\varphi \le 2\delta$ for $l = 1,\dots,2^k$ and $g = \frac{1}{2^k}(g_1 + \dots + g_{2^k})$.*

Proof. Let us observe that the inequality $||g||_{\overline{\overline{\varphi}}} \le \delta$ implies $\rho_{\overline{\overline{\varphi}}}(\frac{g}{\delta}) \le \delta$.
We apply 3.3 to the simple function $h = \frac{g}{\delta}$.So,there exist simple functions $h_1,\dots,h_{2^k}$ such that

$$\rho_\varphi(h_l) \le 2\delta \quad \text{for} \quad l = 1,\dots,2^k \quad \text{and} \quad h = 2^{-k}(h_1 + \dots + h_{2^k}).$$

We set $g_l = \delta h_l$ for $l = 1,\dots,2^k$ and observe that g_l are simple functions such that $g = 2^{-k}(g_1 + \dots + g_{2^k})$ and $||g_l||_\varphi \le 2\delta$, because $\rho_\varphi(\frac{g_l}{2\delta}) = \rho_\varphi(\frac{h_l}{2}) \le \rho_\varphi(h_l) \le 2\delta$. □

4. The locally convex spaces

Definition 4.1 *Let X be a linear space over field K real or complex numbers and let τ be a topology in the set X.A space $\langle X,\tau\rangle$ is called a linear-topological space,when the operations $+ : X\times X \to X$ and $\cdot : K\times X \to X$ are continous functions from $X\times X$ into X and $K\times X$ into X respectively.*

The linear-topological spaces with a topology defined by the metric in linear spaces are normed spaces,countable-normed spaces and F^*-spaces, because we have

$$|(x+y)-(x_0+y_0)| \le |x-x_0| + |y-y_0| \quad \text{and} \quad |ax - a_0x_0| \le$$
$$\le |a(x-x_0)| + |(a-a_0)x_0|,$$

and it implies the continuity of the operators $+$ and $\cdot$.

4.2.A neighbourhood base of the point 0 of linear-topological space has the following properties:

a) for every $V \in \mathcal{V}_0$ there exists $U \in \mathcal{V}_0$ such that $U + U \subset V$

b) if $V \in \mathcal{V}_0$, $\alpha \in K$ and $|\alpha| \leq 1$, then $\alpha V \subset V$

c) if $V \in \mathcal{V}_0$ and $x \in X$, then there exists a number $\alpha_0 > 0$ such that $\alpha_0 x \in V$.

4.3. Let $\langle X, \tau \rangle$ be a linear-topological space and let $x_0 \in X$.If $V \in \mathcal{V}_0$ is a neighbourhood base of the point 0 in the linear-topological space $\langle X, \tau \rangle$ and $x_0 \in X$, then a family $\mathcal{V}_{x_0}$ of the sets of the form $V_{x_0} = V_0 + x_0$,when $V \in \mathcal{V}_0$ is the neighbourhood base of the point x_0 in $\langle X, \tau \rangle$.

4.4. For any subset A of linear-topological space X and any base of the neighbourhood $\mathcal{V}_0$ of $0 \in X$ the formula $\overline{A} = \bigcap_{V \in \mathcal{V}_0} (A + V)$ defines the closure of the set A.

The proof of this fact one can find in [8].

From 4.4 and the property (1) of base of the neighbourhood of 0 it follows that for every $V \in \mathcal{V}_0$ there exists $U \in \mathcal{V}_0$ such that

$$\overline{U} = \bigcap_{W \in \mathcal{V}_0} (U + W) \subset U + U \subset V.$$

From this it follows that there exists a neighbourhood base of 0 which consists of closed sets.

Definition 4.5 *The set A of the linear space X is called a convex set if from the conditions $x, y \in A$,$\alpha, \beta \geq 0, \alpha + \beta = 1$ it follows that $\alpha x + \beta y \in A$.*

4.6. If the sets A and B are convex in a linear space over the field K of the real or complex numbers and $a \in K$,then the sets aA,$A + B$,$A - B$ are convex in X.

Definition 4.7 *The linear-topological space $\langle X, \tau \rangle$ is called a locally convex if there exists a neighbourhood base $\mathcal{V}_0$ in X of the point 0 which consists of convex sets.*

5. Linear operators from the Orlicz spaces into locally convex linear-topological spaces

5.1. In the Orlicz space $L^{*\varphi}_{\rho}$ and also in its subspace $L^{o\varphi}$ we have two types of convergences,one is the pseudonorm convergence and the other the pseudomodular convergence.Therefore let us denote by $\alpha(L^{o\varphi}, Y)$ the class of pseudonorm continous linear operators from $L^{o\varphi}$ into Y,where Y is a locally convex linear-topological space and by $\alpha(L^{*\varphi}_{\rho}, Y)$ the class of pseudomodular continous linear operators from $L^{*\varphi}_{\rho}$ into Y.Obviously,the linear operator A from $L^{o\varphi}$ into Y is pseudonorm continous,i.e. belongs to $\alpha(L^{o\varphi}, Y)$, if and only if for every $V \in \mathcal{V}_0$ there exists $\delta > 0$ such that the conditions $f \in L^{o\varphi}$ and $||f||_{\varphi} \leq \delta$ imply $A(f) \in V$.Let us observe that the linear operator A from $L^{*\varphi}_{\rho}$ into Y is pseudomodular continous,i.e. belongs to $\alpha(L^{*\varphi}_{\rho}, Y)$, if and only if for every $V \in \mathcal{V}_0$ there exists $\delta > 0$ such that the conditions $f \in S$ and $\rho_{\varphi}(f) \leq \delta$ imply $A(f) \in V$.

5.2. Let φ be a $\tilde{\varphi}$-function and $\overline{\overline{\varphi}}$ its convex $\tilde{\varphi}$-function as in 2.1.For these functions the inequality $\overline{\overline{\varphi}}(u) \leq \varphi(u)$ for $u \geq 0$ holds.Hence we have $\rho_{\overline{\overline{\varphi}}}(f) \leq \rho_\varphi(f)$ for $f \in S$ and next the inclusions $L^{*\varphi}_\rho \subset L^{*\overline{\overline{\varphi}}}_\rho, L^{o\varphi} \subset L^{o\overline{\overline{\varphi}}}$ and the inequality $||f||_{\overline{\overline{\varphi}}} \leq ||f||_\varphi$ for $f \in L^{*\varphi}_\rho$.

From this follows immediately

5.3. If $A \in \alpha(L^{*\overline{\overline{\varphi}}}_\rho, Y)$, then the restriction $A|_{L^{*\varphi}_\rho}$ belongs to $\alpha(L^{*\varphi}_\rho, Y)$ and if $A \in \alpha(L^{o\overline{\overline{\varphi}}}, Y)$,then $A|_{L^{o\varphi}} \in \alpha(L^{o\varphi}, Y)$.

Now we present two main theorems

Theorem 5.4 *If the Orlicz space $L^{*\varphi}_\rho$ is non-atomic,then for every linear operator $A \in \alpha(L^{*\varphi}_\rho, Y)$ there exists a linear operator $B \in \alpha(L^{*\overline{\overline{\varphi}}}_\rho, Y)$ such that $A = B|_{L^{*\varphi}_\rho}$.*

Proof. On virtue of 4.4 one can assume that in the space Y there exists a neighbourhood base $\mathcal{V}_0$ of the point 0 which consists of convex and closed sets.Let $A \in \alpha(L^{*\varphi}_\rho, Y)$ and let us take $V \in \mathcal{V}_0$.Then there exists $\delta > 0$ such that the conditions $f \in S$ and $\rho_\varphi(f) \leq 2\delta$ imply $A(f) \in V$.Let us take an arbitrary simple function g such that $\rho_{\overline{\overline{\varphi}}}(g) \leq \delta$. On virtue of 3.3 there exist simple functions $g_1, \ldots, g_{2^k}$ such that $\rho_\varphi(g_l) \leq 2\delta$ for $l = 1, \ldots, 2^k$ and $g = 2^{-k}(g_1 + \cdots + g_{2^k})$.Therefore one can write $A(g_l) \in V$ for $l = 1, \ldots, 2^k$ and next on virtue of convexity of the set V there is $A(g) \in V$.

Thus we have proved the following remark:
(R) If $A \in \alpha(L^{*\varphi}_\rho, Y)$, then for every $V \in \mathcal{V}_0$ there exists $\delta > 0$ such that the inequality $\rho_{\overline{\overline{\varphi}}}(g) \leq \delta$ for any simple function g implies $A(g) \in V$.

Now we shall prove the theorem.Let $A \in \alpha(L^{*\varphi}_\rho, Y)$.We take an arbitrary $f \in L^{*\overline{\overline{\varphi}}}_\rho$.Then $\rho_{\overline{\overline{\varphi}}}(\lambda f) < \infty$ for some $\lambda > 0$ and on virtue of 3.2 there exists a sequence of simple functions (g_n) such that $\rho_{\overline{\overline{\varphi}}}(\lambda(f - g_n)) \to 0$ as $n \to \infty$.Since

$$\rho_{\overline{\overline{\varphi}}}(\frac{1}{2}\lambda(g_n - g_m)) \leq \rho_{\overline{\overline{\varphi}}}(\lambda(f - g_n)) + \rho_{\overline{\overline{\varphi}}}(\lambda(f - g_m)),$$

so we have $\rho_{\overline{\overline{\varphi}}}(\frac{1}{2}\lambda(g_n - g_m)) \to 0$ as $n, m \to \infty$.Whence,by virtue of the remark (R),we get

$$A(\frac{1}{2}\lambda(g_n - g_m)) = \frac{1}{2}\lambda(A(g_n) - A(g_m)) \to \theta \ \ , \ \text{as} \ \ n, m \to \infty.$$

This shows that the sequence $(A(g_n))$ satisfies the Cauchy condition.Since Y is a sequentially complete space,then the sequence $(A(g_n))$ is convergent;we denote $B(f) = \lim_{n\to\infty} A(g_n)$. Further,let (h_n) be an arbitrary sequence of simple functions such that $h_n \overset{\overline{\overline{\varphi}}}{\to}\to f$.Then $\rho_{\overline{\overline{\varphi}}}(\lambda_1(f - h_n)) \to 0$ as $n \to \infty$ for some $\lambda_1 > 0$.From this,on virtue of the inequality

$$\rho_{\overline{\overline{\varphi}}}(\frac{1}{2}\lambda_2(g_n - h_n)) \leq \rho_{\overline{\overline{\varphi}}}(\lambda(f - g_n)) + \rho_{\overline{\overline{\varphi}}}(\lambda_1(f - h_n)),$$

where $\lambda_2 = \inf\{\lambda, \lambda_1\}$, we get $\rho_{\overline{\overline{\varphi}}}(\frac{1}{2}\lambda_2(g_n - h_n)) \to 0$ as $n \to \infty$.Whence,in view of the remark (R),it follows that

$$A(\frac{1}{2}\lambda_2(g_n - h_n)) = \frac{1}{2}\lambda_2(A(g_n) - A(h_n)) \to \theta \ \text{as} \ n \to \infty.$$

Hence the sequence $(A(h_n))$ is convergent and

$$B(f) = \lim_{n\to\infty} A(g_n) = \lim_{n\to\infty} A(h_n).$$

This means that the value $B(f)$ is independent on the choice of a sequence of simple functions (h_n) satisfying $h_n \overset{\overline{\overline{\varphi}}}{\to} f$.

Let us take $f_1, f_2 \in L_\rho^{*\overline{\overline{\varphi}}}$ and numbers a and b.Then from 3.2 it follows the existence of sequences of simple functions $(g_{1,n})$ and $(g_{2,n})$ such that $g_{1,n} \overset{\overline{\overline{\varphi}}}{\to} f_1$ and $g_{2,n} \overset{\overline{\overline{\varphi}}}{\to} f_2$.This implies $ag_{1,n} + bg_{2,n} \overset{\overline{\overline{\varphi}}}{\to} af_1 + bf_2$ and therefore

$$B(af_1 + bf_2) = \lim_{n\to\infty} A(ag_{1,n} + bg_{2,n}) = a \lim_{n\to\infty} A(g_{1,n}) + b \lim_{n\to\infty} A(g_{2,n}) =$$
$$= aB(f_1) + bB(f_2).$$

Next,let us take $f \in L_\rho^{*\varphi}$.On virtue of 3.2 there exists a sequence of simple functions (g_n) such that $g_n \overset{\varphi}{\to} f$.From 5.2 it follows that the convergence $g_n \overset{\overline{\overline{\varphi}}}{\to} f$ holds too. Therefore,in this case,we have

$$B(f) = \lim_{n\to\infty} A(g_n) = A(f).$$

Further,let $V \in \mathcal{V}_0, \delta > 0$ be such that the condition $\rho_{\overline{\overline{\varphi}}}(g) \leq \delta$ for simple function g implies $A(g) \in V$ and let $f \in L_\rho^{*\overline{\overline{\varphi}}}$ be such that $\rho_{\overline{\overline{\varphi}}}(f) \leq \delta$.Then,from 3.2 it follows the existence of a sequence of simple functions (g_n) such that $\rho_{\overline{\overline{\varphi}}}(g_n) \leq \rho_{\overline{\overline{\varphi}}}(f)$ for $n = 1, 2, \ldots$ and $\rho_{\overline{\overline{\varphi}}}(f - g_n) \to 0$ as $n \to \infty$. Hence there holds $B(f) = \lim_{n\to\infty} A(g_n) \in \overline{V} = V$.

Thus we have proved the existence of a operator $B \in \alpha(L_\rho^{*\overline{\overline{\varphi}}}, Y)$ with the property $B|_{L_\rho^{*\varphi}} = A$.Such operator is only one.Namely if B and B_1 are operators with the required properties,then taking an arbitrary $f \in L_\rho^{*\overline{\overline{\varphi}}}$ and more,on virtue of 3.2,a sequence of simple functions (g_n) such that $g_n \overset{\overline{\overline{\varphi}}}{\to} f$ we see that

$$B(f) = \lim_{n\to\infty} B(g_n) = \lim_{n\to\infty} A(g_n) = \lim_{n\to\infty} B_1(g_n) = B_1(f).$$

□

5.5 *If the space $L^{o\varphi}$ is non-atomic,then for every operator $A \in \alpha(L^{o\varphi}, Y)$ there exists a unique operator $B \in \alpha(L^{o\overline{\overline{\varphi}}}, Y)$ such that $A = B|_{L^{o\varphi}}$.*

Since the proof is similar to the previous one,we omit it.

5.6. On virtue of 2.2 for the convex $\tilde{\varphi}$-functions $\overline{\varphi}$ and $\overline{\overline{\varphi}}$ generated by φ the inequality $\overline{\varphi}(u) \leq \overline{\overline{\varphi}}(u) \leq \overline{\varphi}(2u)$ for $u \geq 0$ holds,therefore we may replace in the statements 5.2,5.3,5.4 and 5.5 the function $\overline{\overline{\varphi}}$ by the function $\overline{\varphi}$.Moreover,we observe that the results 5.3,5.4 and 5.5 may be formulated in the form announced in the abstract.Namely,

If (E, Σ, μ) is a non-atomic measure space,then

$$\alpha(L_\rho^{*\varphi}, Y) = \alpha(L_\rho^{*\overline{\varphi}}, Y) \quad \text{and} \quad \alpha(L^{o\varphi}, Y) = \alpha(L^{o\overline{\varphi}}, Y).$$

The first equality is closely connected with the theorem of W.Orlicz in [7].The other proof of the second equality one can find in [1](th.2.2 b).Our result is a little bit more general than the one mentioned above.But first of all,our proof is esentially different from that in [7].

In the particularity,from our results one can obtain the following theorem with the first part known from [7].

Theorem 5.7 *If the $\tilde{\varphi}$-function φ is such that $\lim_{u\to\infty}\inf \frac{\varphi(u)}{u} = 0$, then only trivial linear operator from the non-atomic Orlicz space $L^{*\varphi}_{\rho}$ into locally convex space Y is pseudomodular continous and only trivial linear operator from the non-atomic space of finite elements $L^{o\varphi}$ into Y is pseudonorm continous.*

Proof. In this case we have $\overline{\varphi}(u) = 0$ for $u \geq 0$.This implies $\rho_{\overline{\varphi}}(f) = 0$ for all $f \in S$, and next $L^{*\overline{\varphi}}_{\rho} = L^{o\overline{\varphi}} = S$ and $||f||_{\overline{\varphi}} = 0$ for all $f \in S$.Whence we get $\alpha(L^{*\overline{\varphi}}_{\rho}, Y) = \alpha(L^{o\overline{\varphi}}, Y) = \{0\}$, because only the trivial operator satisfies the inequality $A(f) \in V$ for all $f \in S$ and all $V \in \mathcal{V}_0$.Thus,by virtue of 5.6 we obtain $\alpha(L^{*\varphi}_{\rho}, Y) = \alpha(L^{o\varphi}, Y) = \{0\}$.

The other proof of this theorem one can find in [1](th. 2.2 a). □

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