CHAOTIC ORDER AND FURUTA INEQUALITY

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Abstract. We show a satellite theorem of chaotic Furuta inequality. For positive invertible operators A and B, if $\log A \geq \log B$, then

$$(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \leq A^{\frac{r}{2}}BA^{\frac{r}{2}} \quad and \quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{\frac{r}{2}}AB^{\frac{r}{2}}$$

for $p \geq 1$ and $r \geq 0$.

1. Introduction. Throughout this note, we use a capital letter as an operator on a Hilbert space H. An operator A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$, and also an operator A is strictly positive (in symbol: $A > 0$) if A is positive and invertible.

For positive opretors A and B, Kubo-Ando [13] have established operator meantheory. Especialy, the α-power mean $\sharp_\alpha$ which is defined as follows:

$$A \ \sharp_\alpha \ B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, \quad for \ \alpha \in [0,1].$$

In [2], we have defined the relative operator entropy $S(A|B)$ for positive invertible operators A, B as follows:

$$\lim_{\alpha \to 0} \frac{A \ \sharp_\alpha \ B - A}{\alpha} = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S(A|B).$$

The case where A and B commute, this coincides with the relative entropy introduced by Umegaki [17]. Moreover $S(A|I) = -A\log A$ is the operator entropy which is introduced by Nakamura and Umegaki [14]. On the other hand, we interprete $S(I|A) = \log A$ is the "chaos" of A itself. So we have called $\log A \geq \log B$ the chaotic order and denoted it by $A \gg B$ ([5],[6]). Ando's exponential inequality of [1] inspired us a practical tool of the chaotic order as the following Theorem A [4].

Theorem A (Chaotic Furuta inequality). *Let A and B be positive invertible operators. Then the followings are equivalent.*

(i) $$A \gg B$$

(ii) $$(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \leq A^r \quad for \quad p \geq 0 \quad and \quad r \geq 0$$

(iii). $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \quad for \quad p \geq 0 \quad and \quad r \geq 0$$

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Very recently, Uchiyama [16] has given an excelent proof of (i) $\Rightarrow$ (ii) of this theorem only using the Furuta inequality, whose tool is just the fact that

$$\lim_{n\to\infty}(1+\frac{X}{n})^n = e^X \quad \text{for selfadjoint operator } X.$$

In this formula, if we put $\log A \geq \log B$, Theorem A is easily obtained.

It seems that Furuta's assertion in [9] says inferiority of the chaotic order to the usual operator order. The purpose of this note says some clarify the difference between the chaotic order and usual one.

2. Furuta inequality. The original form of the Furuta inequality is the following [7](cf.[8]):

Furuta inequality:
If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for p and q such that $p \geq 0$ and $q \geq 1$ with

$$(1+r)q \geq p+r.$$

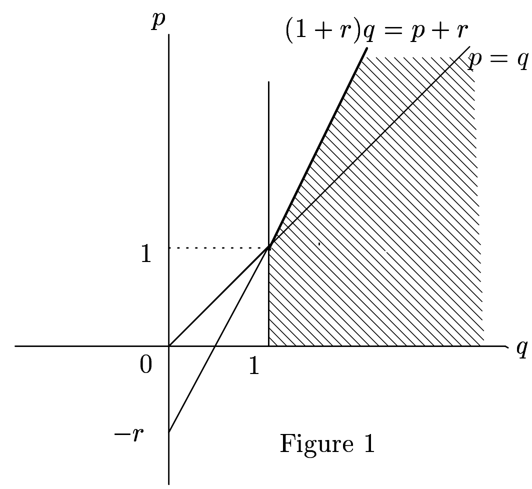

Figure 1

In this inequality, the case where $r = 0$ is the Löwner-Heinz inequality. The best possibility of the domain for p, q and r in the Figure 1 is proved by Tanahashi [15].

From our view point of operator mean, we can rewrite the Furuta inequality as follows (cf. [3],[10],[11],[12] etc.):

$$(F) \qquad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq A \quad \textit{and} \quad B \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$$

for $p \geq 1$ and $r \geq 0$.

Moreover, we have given a proof of these inequalities by using α-power mean $\sharp_\alpha$ and succeeded to arrange these inequalities in one line as follows [10]:

Satellite theorem of the Furuta inequality: *If $A \geq B \geq 0$, then*

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$$

for all $p \geq 1$ and $r \geq 0$.

The satellite theorem can be written without use of operator means as follows:

$$(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \leq A^{\frac{r}{2}}BA^{\frac{r}{2}} \leq A^{1+r} \quad \text{for} \quad p \geq 1 \quad \text{and} \quad r \geq 0$$

and

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{\frac{r}{2}}AB^{\frac{r}{2}} \geq B^{1+r} \quad \text{for} \quad p \geq 1 \quad \text{and} \quad r \geq 0$$

3. Results. Under the chaotic order $A \gg B$, we can transform the Furuta inequality (F) as follows:

Theorem 1 (Satellite theorem of chaotic Furuta inequality). *Let A and B be positive invertible operators. If $A \gg B$, then*

$$(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \leq A^{\frac{r}{2}}BA^{\frac{r}{2}} \quad \textit{and} \quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{\frac{r}{2}}AB^{\frac{r}{2}}$$

for $p \geq 1$ and $r \geq 0$.

To prove this theorem, we need the following Furuta's lemma (cf.[4]).

Lemma(Furuta). *Let A and B be positive invertible operators. Then for any real number r,*

$$(BAB)^r = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^2A^{\frac{1}{2}})^{r-1}A^{\frac{1}{2}}B.$$

Proof of Theorem 1. Using Theorem A, we have

$$\begin{aligned} A^{-\frac{r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{-\frac{r}{2}} &= B^{\frac{p}{2}}(B^{\frac{p}{2}}A^rB^{\frac{p}{2}})^{\frac{1-p}{p+r}}B^{\frac{p}{2}} \\ &= B^{\frac{p}{2}}((B^{\frac{p}{2}}A^rB^{\frac{p}{2}})^{-\frac{p}{p+r}})^{\frac{p-1}{p}}B^{\frac{p}{2}} \\ &\leq B^{\frac{p}{2}}(B^{-p})^{\frac{p-1}{p}}B^{\frac{p}{2}} = B. \end{aligned}$$

The first equality is obtained by Furuta's lemma.

From this theorem, we can obtain the Furuta inequality since $A \geq B$ implies $A \gg B$. As a generalization of this theorem, we can show the next characterizatins of the chaotic order which will be useful in our preparing paper.

Theorem 2. *Let A and B be positive invertible operators. Then the followings are equivalent:*

(1) $$A \gg B \text{ (i.e.,} \log A \geq \log B).$$

(2) $$(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} \leq A^{\frac{r}{2}}B^{\delta}A^{\frac{r}{2}} \quad \textit{for} \quad 0 \leq r \quad \textit{and} \quad 0 \leq \delta \leq p.$$

(3) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} \geq B^{\frac{r}{2}}A^{\delta}B^{\frac{r}{2}} \quad \textit{for} \quad 0 \leq r \quad \textit{and} \quad 0 \leq \delta \leq p.$$

(4) $$(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{-\delta+r}{p+r}} \le A^{r-\delta} \quad for \ \ 0 \le \delta \le r \ \ and \ \ 0 \le p.$$

(5) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{-\delta+r}{p+r}} \ge B^{r-\delta} \quad for \ \ 0 \le \delta \le r \ \ and \ \ 0 \le p.$$

Proof. We first prove the equivalence of the formulas from (2) to (5). (2) is represented as follows:

$$\begin{aligned} A^{\frac{r}{2}}B^{\delta}A^{\frac{r}{2}} &\ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} \\ &= A^{\frac{r}{2}}B^{\frac{p}{2}}(B^{\frac{p}{2}}A^rB^{\frac{p}{2}})^{\frac{\delta+r}{p+r}-1}B^{\frac{p}{2}}A^{\frac{r}{2}} \\ &= A^{\frac{r}{2}}B^{\frac{p}{2}}(B^{-\frac{p}{2}}A^{-r}B^{-\frac{p}{2}})^{\frac{p-\delta}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}}, \end{aligned}$$

the first equality is led by Furuta's lemma. So we have $B^{-\delta} \le B^{-\frac{p}{2}}(B^{\frac{p}{2}}A^rB^{\frac{p}{2}})^{\frac{p-\delta}{p+r}}B^{-\frac{p}{2}}$, that is, $B^{p-\delta} \le (B^{\frac{p}{2}}A^rB^{\frac{p}{2}})^{\frac{p-\delta}{p+r}}$. Since p and r are exchanged, we obtain (5). The equivalences of (2) $\Longleftrightarrow$ (3) and (4) $\Longleftrightarrow$ (5) are easily seen by exchanging A and B for B^{-1} and A^{-1} respectively. By Theorem A, since $(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \le A^r$, (1) implies (4) is given as follows:

$$(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{-\delta+r}{p+r}} = ((A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}})^{\frac{-\delta+r}{r}} \le (A^r)^{\frac{-\delta+r}{r}} = A^{r-\delta}.$$

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