ON TWO CLASSES OF BCI-ALGEBRAS

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Abstract. In this paper we introduce two new classes of BCI-algebras, namely the class of branchwise positive implicative BCI-algebras and the class of branchwise implicative BCI-algebras. The class of branchwise positive implicative BCI-algebras contains the class of positive implicative BCK-algebras [10], the class of medial BCI-algebras, the class of positive implicative BCI-algebras [1, 4, 13] and the class of branchwise implicative BCI-algebras contains the class of implicative BCK-algebras [10], the class of weakly implicative BCI-algebras [1] and the class of quasi-implicative BCI-algebras [17]. Necessary and sufficient condition for a BCI-algebra to be a branchwise implicative BCI-algebra have been investigated.

1 Introduction

K. Iseki and S. Tanaka [10] introduced the notions of positive implicative, implicative and commutative BCK-algebras. Further, K. Iseki [8, 11] introduced the notion of a BCI-algebra, which is a generalization of the concept of a BCK-algebra. K. Iseki [11] and K. Iseki and A.B. Thaheem [12] have shown that if the definitions of the above-mentioned classes of BCK-algebras are adopted for the corresponding classes of BCI-algebras, then no proper classes of such BCI-algebras exist, that is, such BCI-algebras are BCK-algebras of the corresponding type. Thus a natural question arises whether it is possible to introduce such generalizations of these notions for BCI-algebras which not only give proper classes of such BCI-algebras but also contain the corresponding classes of BCK-algebras. During the past ten years, M.A. Chaudhry [1, 3, 4], J. Meng and X.L. Lin [13, 14], C.S. Hoo [6, 7] and S.M. Wei, et. al. [17] have discussed this problem.

In this paper we introduce two new classes of BCI-algebras, the class of branchwise positive implicative BCI-algebras and the class of branchwise implicative BCI-algebras . The class of branchwise positive implicative BCI-algebras contains positive implicative BCK-algebras [9, 10], medial BCI-algebras [6, 7, 16], weakly positive implicative BCI-algebras [1] as well as positive implicative BCI-algebras [13]. The class of branchwise implicative BCI-algebras contains implicative BCK-algebras [9, 10], weakly implicative BCI-algebras [1] and quasi-implicative BCI-algebras [17]. Our classes of BCI-algebras are so general that they contain all the corresponding BCI-algebras introduced so far. We also establish necessary and sufficient conditions for a BCI-algebra to be a branchwise implicative BCI-algebra, which give a generalization of the following well-known result of K. Iseki [10].

Theorem A. *A BCK-algebras is implicative if and only if it is positive implicative and commutative.*

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Key words and phrases. Brachwise positive implicative BCI-algebras, branchwise implicative BCI-algebras, branchwise commutative BCI-algebras, branch.

2 Preliminaries In this section we describe certain definitions and known results which will be used in the sequel.

Deffinition 1 [11]. A. BCI-algebras is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms for $x, y, z \in X$:

(1) $\big((x * y) * (x * z)\big) * (z * y) = 0$,

(2) $\big(x(x * y)\big) * y = 0$,

(3) $x * x = 0$,

(4) $x * x = 0$ and $y * x = 0$ imply $x = y$,

(5) $x * 0 = 0$ implies $x = 0$.

A partial ordering $\leq$ on X is defined by "$x \leq y$ if and only if $x * y = 0$".

If (5) is replaced by $0 * x = 0$, then the algebra is called a BCK-algebra [10]. It is known that every BCK-algebra is a BCI-algebra but the converse is not true [11].

In a BCI-algebra the following hold [11]:

(6) $(x * y) * z = (x * z) * y$,

(7) $x * 0 = x$,

(8) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,

(9) $x * \big(x * (x * y)\big) = x * y$,

(10) $0 * (x * y) = (0 * x) * (0 * y)$,

(11) $(x * z) * (y * z) \leq x * y$.

Definition 2 [10]. A BCK-algebra is called positive implicative if it satisfies

(12) $(x * y) * z = (x * z) * (y * z)$.

Definirion 3 [10]. A BCK-algebra is called commutative if it satisfies

(13) $x * (x * y) = y * (y * x)$.

Definition 4 [9, 10]. A BCK-algebra is called implicative if it satisfies

(14) $x * (y * y) = x$.

Theorem B [10]. *A BCK-algebra X is positive implicative if and only if it satisfies*

(15) $x * y = (x * y) * y$ for all $x, y \in X$.

It has been shown in [11, 12] that no proper classes of positive implicative BCI-algebras, commutative BCI-algebras and implicative BCI-algebras exist. This has led to the following generalizations of these notions. Each generalization contains the corresponding class of BCK-algebras.

Definition 5 [1]. A BCI-algebra X satisfying

(16) $(x * y) * z = \big((x * z) * z\big) * (y * z)$ for all $x, y, z \in X$,

is called a weakly positive implicative BCI-algebra.

Theorem C [1]. *A BCI-algebra X is weakly positive implicative if and only if*

(17) $x * y = \big((x * y) * y\big)*(0 * y)$ for all $x, y \in X$.

Definition 6 [13]. A BCI-algebra X is said to be positive implicative if it satisfies

(18) $\big(x * (x * y)\big)*(y * x) = x * \Big(x * \big(y * (y * x)\big)\Big)$ for all $x, y \in X$.

The following theorem shows that the notions of weak positive implicativeness and positive implicativeness for BCI-algebras are equivalent.

Theorem D [4,Theorems 1 and 2]. *A BCI-algebra X is weakly positive implicative if and only if it is positive implicative.*

A BCI-algebra satisfying $(x*y)*(z*u) = (x*z)*(y*u)$ is called a medial BCI-algebra.

Let X be a BCI-algebra and $M = \{x : x \in X \text{ and } 0 * x = 0\}$. Then M is called its BCK-part. If $M = \{0\}$, then X is called p-semisimple.

It has been shown in [5], [6], [7] and [16] that in a BCI-algebra X, the following are equivalent:

(19) X is medial,

(20) $x * (x * y) = y$ for all $x, y \in X$,

(21) $0 * (0 * x) = x$ for all $x \in X$,

(22) X is p-semisimple.

We now describe the notions of branches of a BCI-algebra and branchwise commutative BCI-algebras defined and investigated in [2] and [3].

Deffinition 7 [2]. Let X be a BCI-algebra, then the set $\text{Med}(X) = \{x : x \in X \text{ and } 0 * (0 * x) = x\}$ is called the medial part of X.

Obviously, $0 \in \text{Med}(X)$. It is known that $\text{Med}(X)$ is a medial subalgebra of X [2]. Further, for each $x \in X$, there is a unique $x_0 = 0 * (0 * x) \in \text{Med}(X)$ such that $x_0 \leq x$ [2]. This is because $0 * (0 * x_0) = 0 * (0 * (0 * (0 * x))) = 0 * (0 * x) = x_0$. Obviously, for a BCK-algebra X, $\text{Med}(X) = \{0\}$. In the sequel the elements of $\text{Med}(X)$ will be denoted by $x_0, y_0, \ldots$

Definition 8 [2]. Let X be a BCI-algebra and $x_0 \in \text{Med}(X)$, then the set

$$B(x_0) = \{x : x \in X \text{ and } x_0 * x = 0\}$$

is called a branch of X determined by the element x_0.

Remark 1. A BCK-algebra X is a one-branch BCI-algebra and in this case $X = B(0)$.

The following theorem proved in [2] and [3] shows that the branches of a BCI-algebra X are pairwise disjoint and form a partition of X. So the study of branches of a BCI-algebra X plays an important role in the investigation of the properties of X.

Theorem E [2, 3]. *Let X be a BCI-algebra with medial part* $\mathrm{Med}(X)$, *then*

(*i*) $\cup\Big\{B(x_0) : x_0 \in \mathrm{Med}(X)\Big\} = X$,

(*ii*) $B(x_0) \cap B(y_0) = \emptyset$ *for* $x_0, y_0 \in \mathrm{Med}(X)$ *and* $x_0 \neq y_0$,

(*iii*) *if* $x, y \in B(x_0)$, *then* $0 * x = 0 * y = 0 * x_0 = 0 * y_0$ *and* $x * y \in M$, $y * x \in M$, *that is,*

$$0 * (x * y) = 0 = 0 * (y * x).$$

Definition 9 [3]. A BCI-algebra X is said to be branchwise commutative if and only if for $x_0 \in \mathrm{Med}(X)$ and $x, y \in B(x_0)$ the equality

(23) $x * (x * y) = y * (y * x)$ holds.

Since a BCK-algebra is a one-branch BCI-algebra, therefore it is commutative if and only if it is branchwise commutative.

Theorem F [3]. *A BCI-algebra X is branchwise commutative if and only if*

(24) $x * (x * y) = y * \Big(y * \big(x * (x * y)\big)\Big)$ for all $x, y \in X$.

3 Branchwise Positive implicative BCI-algebras.

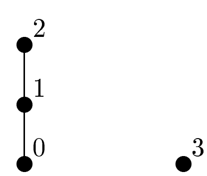
In this section we define branchwise positive implicative BCI-algebras and show that this proper class of BCI-algebras contains the class of positive implicative BCK-algebras, the class of medial BCI-algebras and the class of weakly positive implicative BCI-algebras.

Definition 10. A BCI-algebra X is called a branchwise positive implicative BCI-algebra if, for all $x_0 \in \mathrm{Med}(X)$ and x, y beloging to the same branch $B(x_0)$, it satisfies

(25) $x * y = (x * y) * \Big(y * \big(0 * (0 * y)\big)\Big)$.

Example 1. Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0



It is easy to verify that X is branchwise positive implicative. This shows that proper branchwise positive implicative BCI-algebras exist.

Remark 2. (*i*) A BCK-algebra X is a one-branch BCI-algebra and $X = B(0)$. If $x, y \in X = B(0)$, then $0 * y = 0$ gives $0 * (0 * y) = 0$. Further, if X is positive implicative, then $x * y = (x * y) * y$ or $x * y = (x * y) * \Big(y * \big(0 * (0 * y)\big)\Big)$. Thus X is a branchwise positive implicative BCI-algebra.

(*ii*) It is known that every branch of a medial BCI-algebra is singleton. Let X be a medial BCI-algebra and $x_0 \in \mathrm{Med}(X)$, then $B(x_0) = \{x_0\}$. Hence $x_0 * x_0 = 0 =$

$(x_0 * x_0) * (x_0 * x_0) = (x_0 * x_0) * \Big(x_0 * \big(0 * (0 * x_0)\big)\Big)$. Thus X is branchwise positive imlplicative.

We now show that every weakly positive implicative BCI-algebra [1] is branchwise positive implicative.

Theorem 1. *If X is a weakly positive implicative BCI-algebra, then it is a branchwise positive implicative BCI-algebra.*

Proof. Let X be a weakly positive implicative BCI-algebra, then (17) gives

$$(a) \qquad p * q = \Big((p * q) * q\Big) * (0 * q) \quad \text{for all} \quad p, q \in X.$$

Let $x_0 \in \mathrm{Med}(X)$ and $x, y \in B(x_0)$. Then

$$\begin{aligned}
&\Big[(x * y) * \Big(y * \big(0 * (0 * y)\big)\Big)\Big] * (x * y) \\
&= \Big((x * y) * (x * y)\Big) * \Big(y * \big(0 * (0 * y)\big)\Big) \qquad \text{(by (6))} \\
&= 0 * \Big(y * \big(0 * (0 * y)\big)\Big) = (0 * y) * \Big(0 * \big(0 * (0 * y)\big)\Big) \qquad \text{(by (10))} \\
&= (0 * y) * (0 * y) \qquad \text{(by (9))} \\
&= 0
\end{aligned}$$

Thus

$$(b) \qquad (x * y) * \Big(y * \big(0 * (0 * y)\big)\Big) \leq x * y.$$

Since X is weakly positive implicative, therefore using (a), we get

$$\begin{aligned}
&(x * y) * \Big[(x * y) * \Big(y * \big(0 * (0 * y)\big)\Big)\Big] \\
&= \Big[\Big((x * y) * y\Big) * (0 * y)\Big] * \Big[(x * y) * \Big(y * \big(0 * (0 * y)\big)\Big)\Big] \\
&= \Big[[(x * y) * [(x * y) * \Big(y * \big(0 * (0 * y)\big)\Big)]] * y\Big] * (0 * y) \qquad \text{(using (6) twice)} \\
&\leq \Big(\big(y * \big(0 * (0 * y)\big)\big) * y\Big) * (0 * y) \qquad \text{(by (2))} \\
&= \Big((y * y) * \big(0 * (0 * y)\big)\Big) * (0 * y) = \Big(0 * \big(0 * (0 * y)\big)\Big) * (0 * y) \\
&= (0 * y) * (0 * y) \qquad \text{(by (9))} \\
&= 0
\end{aligned}$$

Hence

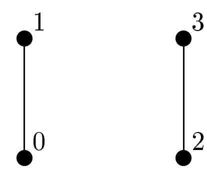
$$(c) \qquad x * y \leq (x * y) * \Big(y * \big(0 * (0 * y)\big)\Big).$$

Using (b) and (c), we get $x * y = (x * y) * \Big(y * \big(0 * (0 * y)\big)\Big)$ for all $x, y \in B(x_0)$. Hence X is a branchwise positive implicative BCI-algebras. This completes the proof.

The following example shows that the converse of the above theorem is not true.

Example 2 [15]. Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by

$*$	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	2	1	0



Routine calculations give that X is branchwise positive implicative but not weakly positive implicative because $3 * 2 = 1$ and $\big((3 * 2) * 2\big)*(0 * 2) = (1 * 2) * (0 * 2) = 2 * 2 = 0$. Thus $3 * 2 \neq \big((3 * 2) * 2\big)*(0 * 2)$.

4 Branchwise Implicative BCI-algebras In this section we define branchwise implicative BCI-algebras and show that this proper class of BCI-algebras contains the class of implicative BCK-algebras [10], the class of medial BCI-algebras [7] and the class of quasi-implicative BCI-algebras [17]. We also investigate necessary and sufficient conditions for a BCI-algebra to be a branchwise positive implicative BCI-algebra.

Since no proper class of implicative BCK-algebras exists, therefore the following generalizations of this notion have been made during the past ten years.

Definition 11 [1]. A BCI-algebra X is called weakly implicative if and only if
(26) $x = \big(x * (y * x)\big)*\big(0 * (y * x)\big)$ for all $x, y \in X$.

Definition 12 [17]. A BCI-algebra X is called quasi-implicative if and only if
(27) $y * \Big(y * \big(x * (x * y)\big)\Big) = \Big(\big(x * (x * y)\big)*(x * y)\Big)*\big(0 * (x * y)\big)$ for all $x, y \in X$.

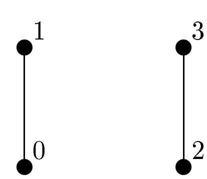
We further generalize this concept and prove a generalization of Theorem A, a well-known result of K. Iseki [10].

Definition 13. A BCI-algebra X is said to be a branchwise implicative BCI-algebra if and only if

$$x * (y * x) = x, \text{ for all } x_0 \in \mathrm{Med}(X) \text{ and for all } x, y \in B(x_0). \tag{28}$$

Example 3 [15]. The set X with the binary operation $*$ defined as

$*$	0	1	2	3
0	0	0	2	2
1	1	0	3	2
2	2	2	0	0
3	3	2	1	0



is a branchwise implicative BCI-algebra. Thus there exist proper branchwise implicative BCI-algebras .

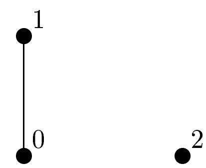
Remark 2. (i) A BCK-algebra X is one-branch BCI-algebra and $X = B(0)$. If $x, y \in X = B(0)$ and X is an implicative BCK-algebra, then $x * (y * x) = x$ for $x, y \in X = B(0)$. Hence X is a branchwise implicative BCI-algebra.

(ii) Let X be a medial BCI-algebra, then each beanch is a singleton. Thus $B(x_0) = \{x_0\}$. Further, $x_0 = x_0 * 0 = x_0 * (x_0 * x_0)$. Hence X is branchwise implicative.

(iii) Let X be a weakly implicative BCI-algebra, then $x = \big(x * (y * x)\big) * \big(0 * (y * x)\big)$. Let $x_0 \in \mathrm{Med}(X)$ and $x, y \in B(x_0)$, then Theorem E part (iii) implies $0 * (y * x) = 0$. Hence $x = \big(x * (y * x)\big) \; * \; 0 = x * (y * x)$ for all $x, y \in B(x_0)$. Thus X is branchwise implicative. But the following example shows that the converse is not true.

Example 4. Let $X = \{0, 1, 2\}$ in which $*$ is defined by

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0



Then X is branchwise implicative but not weakly implicative because $\big(1*(2*1)\big)*\big(0*(2*1)\big) = (1 * 2) * (0 * 2) = 2 * 2 = 0 \neq 1$.

We now investigate necessary and sufficient conditions for a BCI-algebra to be a branchwise implicative BCI-algebra.

Theorem 2. *If a BCI-algebra X is a branchwise positive implicative and branchwise commutative BCI-algebra, then it is a branchwise implicative BCI-algebra.*

Proof. Let X be a branchwise positive implicative as well as a branchwise commutative BCI-algebra. Let $x_0 \in \mathrm{Med}(X)$ and $x, y \in B(x_0)$. Then (24) gives

$$x * \big(x * (y * x)\big) = (y * x) * \Big((y * x) * \Big(x * \big(x * (y * x)\big)\Big)\Big).$$

Using (25) we get

$$\begin{aligned}
x * \big(x * (y * x)\big) &= \Big((y * x) * \Big(x * \big(0 * (0 * x)\big)\Big)\Big) * \Big((y * x) * \Big(x * \big(x * (y * x)\big)\Big)\Big) \\
&= \Big[(y * x) * \Big((y * x) * \Big(x * \big(x * (y * x)\big)\Big)\Big)\Big] * \Big(x * \big(0 * (0 * x)\big)\Big) \\
&= \Big(x * \big(x * (y * x)\big)\Big) * \Big(x * \big(0 * (0 * x)\big)\Big) \qquad \text{(by (24))} \\
&\leq \big(0 * (0 * x)\big) * \big(x * (y * x)\big) \qquad \text{(by (1))} \\
&= \Big(0 * \big(x * (y * x)\big)\Big) * (0 * x) \qquad \text{(by (6))} \\
&= \Big((0 * x) * \big(0 * (y * x)\big)\Big) * (0 * x) \qquad \text{(by (10))} \\
&= \big((0 * x) * 0\big) * (0 * x) \qquad \text{(by Th. E (part iii))} \\
&= (0 * x) * (0 * x) = 0.
\end{aligned}$$

Hence

(d) $$x \leq x * (y * x).$$

Further, $\big(x * (y * x)\big) \ * \ x = (x * x) * (y * x) = 0 * (y * x) = 0$. Thus

$$x * (y * x) \leq x \tag{e}$$

which along with (d) implies $x = x * (y * x)$ for all $x, y \in B(x_0)$. Hence X is branchwise implicative. This completes the proof.

We now state and use the following theorem.

Theorem G [17, Theorem 1]. *If X is a quasi-implicative BCI-algebra, then it is both weakly positive implicative and branchwise commutative.*

Remark 3. Theorem 1 gives that every weakly positive implicative BCI-algebra X is branchwise positive implicative. Thus Theorem G implies that every quasi-implicative BCI-algebra is both branchwise positive implicative and branchwise commutative. Using Theorem 2 we get that every quasi-implicative BCI-algebra is branchwise implicative. But its converse is not true because that BCI-algebra of Example 2 is branchwise positive implicative as well as branchwise implicative but it is not quasi-implicative. This is because

$$1 * \Big(1 * \big(3 * (3 * 1)\big)\Big) = 1 * \big(1 * (3 * 2)\big) = 1 * (1 * 1) = 1 * 0 = 1$$

and

$$\begin{aligned} \Big(\big(3 * (3 * 1)\big) * (3 * 1)\Big) * \big(0 * (3 * 1)\big) &= \big((3 * 2) * 2\big) * (0 * 2) \\ &= (1 * 2) * 2 = 2 * 2 = 0, \end{aligned}$$

which implies

$$y * \Big(y * \big(x * (x * y)\big)\Big) \neq \Big(\big(x * (x * y)\big) * (x * y)\Big)\big(0 * (x * y)\big).$$

Theorem 3. *If X is a branchwise implicative BCI-algebra, then it is both branchwise positive implicative and branchwise commutative.*

Proof. Let X be branchwise implicative. Let $x_0 \in \mathrm{Med}(X)$ and $x, y \in B(x_0)$. Then $x = x * (y * x)$, which implies

$$\begin{aligned} x * (x * y) &= \big(x * (y * x)\big) * (x * y) \\ &= \big(x * (x * y)\big) * (y * x) \leq y * (y * x). \end{aligned}$$

Interchanging x and y we get

$$y * (y * x) \leq x * (x * y).$$

Thus $x * (x * y) = y * (y * x)$ for all $x, y \in B(x_0)$. Hence X is branchwise commutative. Further

$$\begin{aligned}&\Big[(x*y)*\Big(y*\big(0*(0*y)\big)\Big)\Big]*(x*y)\\&=\Big((x*y)*(x*y)\Big)*\Big(y*\big(0*(0*y)\big)\Big)\\&=0*\Big(y*\big(0*(0*y)\big)\Big)=(0*y)*\Big(0*\big(0*(0*y)\big)\Big)\\&=(0*y)*(0*y)=0.\end{aligned}$$

Thus

$$\text{(f)}\qquad (x*y)*\Big(y*\big(0*(0*y)\big)\Big)\leq x*y.$$

Since $x,y\in B(x_0)$, therefore $x*y\in M=B(0)$. Further $0*(0*y)\leq y$ gives that $0*(0*y)$ and y belong to the same branch $B(y_0)$. Thus $y*\big(0*(0*y)\big)\in M=B(0)$. Since X is branchwise commutative, therefore

$$\begin{aligned}&(x*y)*\Big((x*y)*\Big(y*\big(0*(0*y)\big)\Big)\Big)\\&\quad=\Big(y*\big(0*(0*y)\big)\Big)*\Big(\Big(y*\big(0*(0*y)\big)\Big)*(x*y)\Big)\\&\quad=\Big(y*\big(0*(0*y)\big)\Big)*\Big(\big(y*(x*y)\big)*\big(0*(0*y)\big)\Big)\\&\quad=\Big(y*\big(0*(0*y)\big)\Big)*\Big(y*\big(0*(0*y)\big)\Big),\qquad\text{(by (28))}\end{aligned}$$

because X is branchwise implicative. Hence

$$\begin{aligned}&(x*y)*\Big((x*y)*\Big(y*\big(0*(0*y)\big)\Big)\Big)\\&\quad=\Big(y*\big(0*(0*y)\big)\Big)*\Big(y*\big(0*(0*y)\big)\Big)=0.\end{aligned}$$

Thus

$$\text{(g)}\qquad x*y\leq(x*y)*\Big(y*\big(0*(0*y)\big)\Big),$$

which along with (f) implies $x*y=(x*y)*\Big(y*\big(0*(0*y)\big)\Big)$. Thus X is branchwise positive implicative. This complete the proof.

Combining Theorems 2 and 3, we get the following theorem.

Theorem 4. *A BCI-algebra X is branchwise implicative if and only if it is both branchwise positive implicative and branchwise commutative.*

Remark 4. Since in a BCK-algebra X branchwise implicativeness, branchwise positive implicativeness and branchwise commutativeness coincide with implicativeness, positive implicativeness and commutativeness, respectively, therefore the following well-known result, Theorem A, of K. Iseki [10] follows as a corollary from Theorem 4.

Corollary 1. *A BCK-algebras X is implicative if and only if it is both positive implicative and commutative.*

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