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BIOPERATIONS AND SOME SEPARATION AXIOMS

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Abstract. We generalize the notion of operation-open sets in the sense of H.Ogata [16] to bioperations and define bioperation-closures and bioperation-generalized closed sets. We obtain properties of bioperation-continuities, bioperation-separation axioms and bioperation-normal spaces.

1. Introduction. A topological space X with topology τ will be denoted by (X, τ). The interior and the closure of a subset S of (X, τ) will be denoted by τ-Int(S) (or Int(S)) and τ-Cl(S) (or Cl(S)), respectively. The power set of X will be denoted by $P(X)$. An operation γ on τ is a function from τ into $P(X)$ such that $U \subset U^\gamma$ for every $U \in \tau$, where U^γ denotes the value of γ at U. The study of this concept was initiated by S.Kasahara [9] and the operation γ is denoted by α in his paper [9]. S. Kasahara [9] unified several characterizations of compact spaces [8], nearly compact spaces [22] and H-closed spaces [5] [7] by generalizing the notion of compactness with the help of operation. After the work of S. Kasahara, D.S. Janković [6] defined the concept of operation-closures and investigated some properties of functions with operation-closed graphs. Moreover, H.Ogata [16][17] investigated the notion of operation-open sets, i.e.,γ-open sets, and used it to investigate some new separation axioms. Using these notions of operation-open sets [16] and operation-closures [6], some operator-approaches to topological properties were studied, cf.[18], [19], [24]. For two operations on τ some bioperation-open sets and bioperation-separation axioms were defined [25][20][23].

In this paper we shall introduce an alternative bioperation-open sets and investigate more bioperator-approaches to properties of topological spaces. In section 2 we introduce a different type of bioperation-open sets and investigate relations between it and that of [16], [25] and [20]. We define two different types of bioperation-closures in section 3 and by using basic properties of them we study bioperation-generalized closed sets in section 4. The notion of new bioperation-separation axioms is introduced in section 5. We compare their separation-axioms with the separation-axioms in [16], [25] and the ordinary T_i-separation axioms (i=0,1/2,1,2). The notions of bioperation-continuous functions and bioperation-closed functions are introduced in section 7. We show that the set of all bioperation-homeomorphisms from (X, τ) onto itself has a group structure. Finally, in section 8, we obtain some relations of bioperation-continuous functions, bioperation-separation axioms introduced in section 5 and bioperation-normal spaces.

Throughout this paper, let γ and γ' be given two operations on τ in the sense of [9] and [16]. That is, $\gamma : \tau \to P(X)$ and $\gamma' : \tau \to P(X)$ are functions such that $U \subset U^\gamma$ and $V \subset V^{\gamma'}$ for every $U \in \tau$ and every $V \in \tau$ where $U^\gamma = \gamma(U)$ and $V^{\gamma'} = \gamma'(V)$. We recall the following:

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(1.1) [9] γ is said to be *regular* if for every open neighbourhoods U and V of each $x \in X$, there exists an open neighbourhood W of x such that $W^\gamma \subset U^\gamma \cap V^\gamma$.

(1.2) [16] A non-empty subset A of (X,τ) is γ-*open* if for each $x \in A$, there exists an open neighbourhood U of x such that $U^\gamma \subset A$. We suppose that the empty set is γ-open for any operation γ.

(1.3) [16] γ is said to be *open* if for every open neighbourhood U of each $x \in X$, there exists a γ-open set S such that $x \in S$ and $S \subset U^\gamma$.

2. $[\gamma,\gamma']$-open sets. In this section the notion of $[\gamma,\gamma']$-open sets is defined and the relations among $[\gamma,\gamma']$-open sets, (γ,γ')-open sets [25] and γ-open sets due to Ogata [16] are investigated.

Definition 2.1. A non-empty subset A of (X,τ) is said to be $[\gamma,\gamma']$-*open* (resp. (γ,γ')-*open* [25]) if for each $x \in A$ there exist open neighbourhoods U and V of x such that $U^\gamma \cap V^{\gamma'} \subset A$ (resp. $U^\gamma \cup V^{\gamma'} \subset A$). We suppose that the empty set $\emptyset$ is (γ,γ')-open and also $[\gamma,\gamma']$-open for any operations γ and γ'.

Proposition 2.2. *Let A and B be subsets of (X,τ).*

(i) *If A is γ-open and B is γ'-open, then $A \cap B$ is $[\gamma,\gamma']$-open.*

(ii) *If A is $[\gamma,\gamma']$-open, then A is open.*

(iii) *If A_i is $[\gamma,\gamma']$-open for every $i \in \Gamma$, then $\cup\{A_i \mid i \in \Gamma\}$ is $[\gamma,\gamma']$-open.*

(iv) *If A is γ-open, then A is $[\gamma,\gamma']$-open for any operation γ'.*

(v) *If (X,τ) is a γ-regular space* [9] *and A is $[\gamma,\gamma']$-open for an operation γ', then A is γ-open.*

(vi) *A is γ-open if and only if A is $[\gamma,X]$-open, where $X : \tau \to P(X)$ is the operation defined by $U^X = X$ for every $U \in \tau$.* □

Definition 2.3. The set of all $[\gamma,\gamma']$-open (resp.(γ,γ')-open) sets of (X,τ) is denoted by $\tau_{[\gamma,\gamma']}$ (resp. $\tau_{(\gamma,\gamma')}$).

Remark 2.4. (i) The following relation (2.5) (resp.(2.6)) is shown by Proposition 2.2 (i),(ii), (iv) and [25;(2.4)] (resp. Proposition 2.2(vi)):

(2.5) $\tau_\gamma \cap \tau_{\gamma'} = \tau_{(\gamma,\gamma')} \subset \tau_\gamma \subset \tau_\gamma \cup \tau_{\gamma'} \subset \tau_{[\gamma,\gamma']} \subset \tau$.

(2.6) $\tau_{[\gamma,X]} = \tau_\gamma$.

Remark 2.7. In (2.5) the set $\tau_{(\gamma,\gamma')}$ is a proper subset of $\tau_{[\gamma,\gamma']}$ and $\tau_{[\gamma,\gamma']}$ is a proper subset of τ as shown by the following example.

Example 2.8. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Let γ and γ'' be the closure operation, i.e. $U^\gamma = U^{\gamma''} = \mathrm{Cl}(U)$. Moreover, let $\gamma' : \tau \to P(X)$ be the interior-closure operation, i.e. $U^{\gamma'} = \mathrm{Int}(\mathrm{Cl}(U))$ for every $U \in \tau$. It is shown that $\tau_\gamma = \tau_{\gamma''} = \{\emptyset, X\}, \tau_{\gamma'} = \tau$ and $\tau_{[\gamma,\gamma']} = \tau$. Therefore, $\tau_{(\gamma,\gamma')} = \{\emptyset, X\} \subsetneqq \tau_{[\gamma,\gamma']}$ and $\tau_{[\gamma,\gamma'']} \subsetneqq \tau$.

Proposition 2.9. *Let γ and γ' be regular operations.*

(i) *If A and B are $[\gamma,\gamma']$-open, then $A \cap B$ is $[\gamma,\gamma']$-open.*

(ii) *$\tau_{[\gamma,\gamma']}$ is a topology on X.* □

Remark 2.10. The regularity on γ and γ' of Proposition 2.9 can not be removed as shown by the following example.

Example 2.11. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ be a topology on X. For each $A \in \tau$ we define two operations γ and γ', respectively, by

$$A^\gamma = \begin{cases} A & \text{if } b \in A, \\ \mathrm{Cl}(A) & \text{if } b \notin A, \end{cases} \quad \text{and } A^{\gamma'} = \begin{cases} A & \text{if } A \neq \{a\}, \\ A \cup \{c\} & \text{if } A = \{a\}. \end{cases}$$

Then γ is not regular [16;Example 2.8]. The sets $\{a, b\}$ and $\{a, c\}$ are $[\gamma, \gamma']$-open, however its intersection $\{a\}$ is not $[\gamma, \gamma']$-open.

Definition 2.12. A space (X, τ) is said to be $[\gamma, \gamma']$-regular if for each point x of X and every open neighbourhood U of x there exists open neighbourhoods W and S of x such that $W^\gamma \cap S^{\gamma'} \subset U$.

Proposition 2.13. *For γ-regularlity, γ'-regularlity and $[\gamma, \gamma']$-regularlity of a space (X, τ), the following properties hold.*

(i) *(X, τ) is $[\gamma, \gamma']$-regular space if and only if $\tau_{[\gamma,\gamma']} = \tau$ holds.*
(ii) *(X, τ) is $[\gamma, X]$-regular if and only if it is γ-regular.*
(iii) *If (X, τ) is γ-regular and γ'-regular, then it is $[\gamma, \gamma']$-regular.* □

Remark 2.14. Let (X, τ), γ and γ' be the same space and the same operations as in Example 2.8. This example shows that the converse of Proposition 2.13(iii) is not true in general. Since $\tau_{[\gamma,\gamma']} = \tau$ and $\tau_\gamma \neq \tau$, (X, τ) is $[\gamma, \gamma']$-regular but it is not γ-regular by using Proposition 2.13(i),(ii) and (2.6).

3. $[\gamma, \gamma']$**-closures.** We introduce the $[\gamma, \gamma']$-closure of a set and investigate some properties of $[\gamma, \gamma']$-closed sets.

Definition 3.1. A subset F of (X, τ) is said to be $[\gamma, \gamma']$-*closed* if its complement $X \setminus F$ is $[\gamma, \gamma']$-open. Let $\mathcal{F}_{[\gamma,\gamma']}$ be the set of all $[\gamma, \gamma']$-closed sets of (X, τ).

Definition 3.2. For a subset A of (X, τ) and $\tau_{[\gamma,\gamma']}$, $\tau_{[\gamma,\gamma']}$-Cl(A) denotes the intersection of all $[\gamma, \gamma']$-closed sets of (X, τ) containing A, i.e.,

$\tau_{[\gamma,\gamma']}$-Cl(A)$=\cap\{F \mid A \subset F, F \in \mathcal{F}_{[\gamma,\gamma']}\}$.

Proposition 3.3. *For a point $x \in X, x \in \tau_{[\gamma,\gamma']}$-Cl$(A)$ if and only if $V \cap A \neq \emptyset$ for every $[\gamma, \gamma']$-open set V containing x.* □

Proposition 3.4. *Let A and B be subsets of (X, τ). Then the following hold:*

(i) *$A \subset \tau_{[\gamma,\gamma']}$-Cl$(A)$.*
(ii) *If $A \subset B$, then $\tau_{[\gamma,\gamma']}$-Cl$(A) \subset \tau_{[\gamma,\gamma']}$-Cl$(B)$.*
(iii) *$A \in \mathcal{F}_{[\gamma,\gamma']}$ if and only if $\tau_{[\gamma,\gamma']}$-Cl$(A) = A$.*
(iv) *$\tau_{[\gamma,\gamma']}$-Cl$(A) \in \mathcal{F}_{[\gamma,\gamma']}$.* □

We introduce the following definition of $\mathrm{Cl}_{[\gamma,\gamma']}(A)$.

Definition 3.5. For a subset A of (X, τ), we define $\mathrm{Cl}_{[\gamma,\gamma']}(A)$ as follows:

$\mathrm{Cl}_{[\gamma,\gamma']}(A)=\{x \in X \mid (U^\gamma \cap W^{\gamma'}) \cap A \neq \emptyset$ holds for every open neighbourhoods U and W of $x\}$.

Remark 3.6. In Definitions 3.1,3.2 and 3.5, put $\gamma' = X$. Then, for any subset A of X, the following hold:

(i) $\tau_{[\gamma,X]}$-Cl$(A) = \tau_\gamma$-Cl(A),
(ii) $\mathcal{F}_{[\gamma,X]} = \{F \mid F$ is γ-closed $\}$ and
(iii) $\mathrm{Cl}_{[\gamma,X]}(A)=\mathrm{Cl}_\gamma(A)$,

where γ-closedness,$\mathrm{Cl}_\gamma(A)$ and τ_γ-Cl(A) are defined in [6] and [16] respectively.

Proposition 3.7. *For a subset A of (X, τ), the following hold:*

(i) *$A \subset$Cl$(A) \subset$Cl$_{[\gamma,\gamma']}(A) \subset \tau_{[\gamma,\gamma']}$-Cl$(A)$.*
(ii) *$\mathrm{Cl}_{[\gamma,\gamma']}(A) \subset \mathrm{Cl}_{(\gamma,\gamma')}(A)$, where $\mathrm{Cl}_{(\gamma,\gamma')}(A)$ is defined in* [25]. □

Theorem 3.8. *For a subset A of (X,τ), the following statements are equivalent:*
(a) $\tau_{[\gamma,\gamma']}$-Cl$(A) = A$.
(b) $\mathrm{Cl}_{[\gamma,\gamma']}(A) = A$.
(c) *A is $[\gamma,\gamma']$-closed,i.e.* $A \in \mathcal{F}_{[\gamma,\gamma']}$. □

Theorem 3.9. *For a subset A of (X,τ), the following properties hold:*
(i) *If (X,τ) is $[\gamma,\gamma']$-regular, then* $\mathrm{Cl}(A)=\mathrm{Cl}_{[\gamma,\gamma']}(A) = \tau_{[\gamma,\gamma']}$-Cl$(A)$.
(ii) $\mathrm{Cl}_{[\gamma,\gamma']}(A)$ *is a closed subset of* (X,τ).
(iii) $\mathrm{Cl}_{[\gamma,\gamma']}(\tau_{[\gamma,\gamma']}\text{-Cl}(A)) = \tau_{[\gamma,\gamma']}\text{-Cl}(\mathrm{Cl}_{[\gamma,\gamma']}(A)) = \tau_{[\gamma,\gamma']}\text{-Cl}(A)$. □

Theorem 3.10. *Let γ and γ' be open operations and A a subset of (X,τ). Then, the following statements hold:*
(i) $\mathrm{Cl}_{[\gamma,\gamma']}(A) = \tau_{[\gamma,\gamma']}$-Cl$(A)$.
(ii) $\mathrm{Cl}_{[\gamma,\gamma']}(\mathrm{Cl}_{[\gamma,\gamma']}(A))=\mathrm{Cl}_{[\gamma,\gamma']}(A)$.

Proof. (i) By Proposition 3.7, it suffices to prove that $\tau_{[\gamma,\gamma']}$-Cl$(A) \subset \mathrm{Cl}_{[\gamma,\gamma']}(A)$. Let $x \in \tau_{[\gamma,\gamma']}$-Cl$(A)$ and let W and S be open neighbourhoods of x. By the openness of γ and γ' [16], there exist a γ-open set W' and a γ'-open set S' such that $x \in W' \subset W^\gamma$ and $x \in S' \subset S^{\gamma'}$. By Propositions 2.2(i) and 3.3, $(S'\cap W')\cap A \neq \emptyset$ and hence $(S^\gamma \cap W^{\gamma'})\cap A \neq \emptyset$. This implies that $x \in \mathrm{Cl}_{[\gamma,\gamma']}(A)$.
(ii) This follows immediately from (i) and Theorem 3.9(iii). □

Remark 3.11. Example 2.8 shows that the equalities of Theorem 3.10 are not true without the assumption that both operations are open. The operation γ is not open. However, $\mathrm{Cl}_{[\gamma,\gamma']}(\{a\}) = \{a,c\} \subset \tau_{[\gamma,\gamma']}\text{-Cl}(\{a\}) = X$ and $\mathrm{Cl}_{[\gamma,\gamma']}(\mathrm{Cl}_{[\gamma,\gamma']}(\{a\})) \neq \mathrm{Cl}_{[\gamma,\gamma']}(\{a\})$.

Theorem 3.12. *Let A and B be subsets of X.*
(i) *If $A \subset B$, then* $\mathrm{Cl}_{[\gamma,\gamma']}(A) \subset \mathrm{Cl}_{[\gamma,\gamma']}(B)$.
(ii) $\mathrm{Cl}_{[\gamma,\gamma']}(A \cup B) \subset \mathrm{Cl}_\gamma(A)\cup \mathrm{Cl}_{\gamma'}(B)$.
(iii) *If γ and γ' are regular, then* $\mathrm{Cl}_{[\gamma,\gamma']}(A \cup B)=\mathrm{Cl}_{[\gamma,\gamma']}(A)\cup\mathrm{Cl}_{[\gamma,\gamma']}(B)$. □

Remark 3.13. Example 2.8 shows that the inclusion of Theorem 3.12(ii) is a proper one in general. For a subset $\{c\}$, $\mathrm{Cl}_{[\gamma,\gamma']}(\{c\}) = \{c\} \subset \mathrm{Cl}_\gamma(\{c\})\cup\mathrm{Cl}_{\gamma'}(\{c\}) = X$.

We define the $[\gamma,\gamma']$-interior of a subset A of (X,τ) as follows:

Definition 3.14. (cf.[17;Definition 2.3]) For a subset A of (X,τ) and operations γ and γ' on τ, $\mathrm{Int}_{[\gamma,\gamma']}(A) = \{x \mid U^\gamma \cap V^{\gamma'} \subset A$ for some open neighbourhoods U and V of x $\}$.

Proposition 3.15. *For every subset A of (X,τ), the following holds:*
$\mathrm{Cl}_{[\gamma,\gamma']}(X \setminus A) = X\setminus\mathrm{Int}_{[\gamma,\gamma']}(A)$. □

4. $[\gamma,\gamma']$-generalized closed sets.

Definition 4.1. A subset A of (X,τ) is said to be $[\gamma,\gamma']$-*generalized closed* (briefly $[\gamma,\gamma']$-g.*closed*) if $\mathrm{Cl}_{[\gamma,\gamma']}(A) \subset U$ whenever $A \subset U$ and U is $[\gamma,\gamma']$-open.

Remark 4.2. (i) Every $[\gamma,\gamma']$-closed set is $[\gamma,\gamma']$-g.closed by Theorem 3.8, but its converse is not true as shown in Example 4.3 (below).

(ii) The $[\gamma,X]$-g.closedness coincides with the γ-g.closedness due to [16;Definition 4.4] (cf.Remark 3.6).

(iii) A subset A is $[id,X]$-g.closed if and only if A is g.closed [11], where *id* is the identity operation.

Example 4.3. (cf.[25;Example 4.8]) Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$. Let γ and γ' be operations on a topology τ defined as follows: for every non-empty open set A,

$$A^\gamma = \begin{cases} \mathrm{Int}(\mathrm{Cl}(A)) & \text{if } A = \{a\}, \\ \mathrm{Cl}(A) & \text{if } A \neq \{a\}, \end{cases} \quad \text{and} \quad A^{\gamma'} = X.$$

It follows from Proposition 2.2 that $\tau_{[\gamma,\gamma']} = \tau_\gamma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, d\}, X\}$ and a subset $\{d\}$ is not $[\gamma, \gamma']$-closed. However, $\{d\}$ is a $[\gamma, \gamma']$-g.closed set.

The $[\gamma, \gamma']$-g.closed sets are characterized as the following proposition.

Proposition 4.4. *Let γ and γ' be any operations. A subset A of (X, τ) is $[\gamma, \gamma']$-g.closed if and only if $A \cap \tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(\{x\}) \neq \emptyset$ for every $x \in Cl_{[\gamma,\gamma']}(A)$.* □

The following proposition shows that the regularity on γ in [16;Proposition 4.6] can be omitted.

Proposition 4.5. (cf.[16;Proposition 4.6]) *A subset A of (X, τ) is γ-g.closed if and only if $A \cap \tau_\gamma$-$\mathrm{Cl}(\{x\}) \neq \emptyset$ for every $x \in \mathrm{Cl}_\gamma(A)$.* □

Proposition 4.6. (i) *If a subset A of (X, τ) is $[\gamma, \gamma']$-g.closed, then $\mathrm{Cl}_{[\gamma,\gamma']}(A) \setminus A$ does not contain any non-empty $[\gamma, \gamma']$-closed set.*

(ii) *If both γ and γ' are open operations, then the converse of* (i) *is true.*

Proof. (i) Let F be a $[\gamma, \gamma']$-closed set contained in $\mathrm{Cl}_{[\gamma,\gamma']}(A) \setminus A$. Since $A \subset X \setminus F$ and A is $[\gamma, \gamma']$-g.closed, $\mathrm{Cl}_{[\gamma,\gamma']}(A) \subset X \setminus F$ and hence $F \subset (\mathrm{Cl}_{[\gamma,\gamma']}(A) \setminus A) \cap (X \setminus \mathrm{Cl}_{[\gamma,\gamma']}(A)) = \emptyset$.

(ii) Since γ and γ' are open, it follows from Theorems 3.8 and 3.10 that $\mathrm{Cl}_{[\gamma,\gamma']}(A)$ is $[\gamma, \gamma']$-closed. Let U be a $[\gamma, \gamma']$-open set such that $A \subset U$. Then, $(X \setminus U) \cap \mathrm{Cl}_{[\gamma,\gamma']}(A)$ is a $[\gamma, \gamma']$-closed set by Proposition 2.2(iii) and it is contained in $\mathrm{Cl}_{[\gamma,\gamma']}(A) \setminus A$. It follows from the assumption of the converse of (i) that $(X \setminus U) \cap \mathrm{Cl}_{[\gamma,\gamma']}(A) = \emptyset$. Therefore, we obtain $\mathrm{Cl}_{[\gamma,\gamma']}(A) \subset U$. □

The following example shows that the openness of γ in Proposition 4.6(ii) can not be removed.

Example 4.7. Let (X, τ), γ and γ' be the same space and the same operations as in Example 4.3, respectively. It is shown that γ is not open and γ' is open. Then, $\mathrm{Cl}_{[\gamma,\gamma']}(\{a\}) \setminus \{a\} = \mathrm{Cl}_\gamma(\{a\}) \setminus \{a\} = \{a, d\} \setminus \{a\} = \{d\}$ and $\{d\}$ is not $[\gamma, \gamma']$-closed. However, $\{a\}$ is not $[\gamma, \gamma']$-g.closed in (X, τ).

Definition 4.8. A subset A of (X, τ) is said to be $[\gamma, \gamma']$-*generalized open* (briefly $[\gamma, \gamma']$-g.*open*) if its complement $X \setminus A$ is $[\gamma, \gamma']$-g.closed.

Proposition 4.9. *A subset A of (X, τ) is $[\gamma, \gamma']$-g.open if and only if $F \subset \mathrm{Int}_{[\gamma,\gamma']}(A)$ whenever $F \subset A$ and F is $[\gamma, \gamma']$-closed.* □

5. $[\gamma, \gamma']$-T_i spaces (i=0,1/2,1,2).

In this section we introduce $[\gamma, \gamma']$-T_i spaces (i=0,1/2 ,1,2) and investigate relations among these spaces.

Definition 5.1. A space (X, τ) is said to be $[\gamma, \gamma']$-$T_{1/2}$ if every $[\gamma, \gamma']$-g.closed set of (X, τ) is $[\gamma, \gamma']$-closed. It follows from Remark 4.2(i) that (X, τ) is $[\gamma, \gamma']$-$\mathrm{T}_{1/2}$ if and only if the $[\gamma, \gamma']$-g.closedness coincides with the $[\gamma, \gamma']$-closedness.

Let $X \times X$ be the direct product of X and $\Delta(X) = \{(x, x) \mid x \in X\}$ the diagonal set of X.

Definition 5.2. A space (X,τ) is said to be $[\gamma,\gamma']$-T_2, if for each $(x,y)\in X\times X\setminus\Delta(X)$ there exist open sets U and V containing x and open sets W and S containing y such that $(U^\gamma\cap V^{\gamma'})\cap(W^\gamma\cap S^{\gamma'})=\emptyset$.

Definition 5.3. A space (X,τ) is said to be $[\gamma,\gamma']$-T_1, if for each $(x,y)\in X\times X\setminus\Delta(X)$ there exist open sets U and V containing x and open sets W and S containing y such that $y\notin U^\gamma\cap V^{\gamma'}$ and $x\notin W^\gamma\cap S^{\gamma'}$.

Definition 5.4. A space (X,τ) is said to be $[\gamma,\gamma']$-T_0, if for each $(x,y)\in X\times X\setminus\Delta(X)$ there exist open sets U and V such that $x\in U\cap V$ and $y\notin U^\gamma\cap V^{\gamma'}$, or $y\in U\cap V$ and $x\notin U^\gamma\cap V^{\gamma'}$.

Remark 5.5. (i)(cf.[25;Definition 5.5]) For given two distinct points x and y, the $[\gamma,\gamma']$-T_0-axiom requires that there exist open sets U,V,W and S satisfying one of conditions (a),(b),(c) and (d):

(a) $x\in U\cap V, y\in W\cap S, y\notin U^\gamma\cap V^{\gamma'}$ and $x\notin W^\gamma\cap S^{\gamma'}$,

(b) $x\in U\cap V, x\in W\cap S, y\notin U^\gamma\cap V^{\gamma'}$ and $y\notin W^\gamma\cap S^{\gamma'}$,

(c) $y\in U\cap V, y\in W\cap S, x\notin U^\gamma\cap V^{\gamma'}$ and $x\notin W^\gamma\cap S^{\gamma'}$,

(d) $y\in U\cap V, x\in W\cap S, x\notin U^\gamma\cap V^{\gamma'}$ and $y\notin W^\gamma\cap S^{\gamma'}$.

(ii) A space (X,τ) is $[\gamma,\gamma']$-T_0 if and only if for each $(x,y)\in X\times X\setminus\Delta(X)$, there exists an open set W such that $x\in W$ and $y\notin W^\gamma\cap W^{\gamma'}$, or $y\in W$ and $x\notin W^\gamma\cap W^{\gamma'}$.

To characterize a $T_{1/2}$ space we prepare the following lemma.

Lemma 5.6. *For each $x\in X$,$\{x\}$ is $[\gamma,\gamma']$-closed or its complement $X\setminus\{x\}$ is $[\gamma,\gamma']$-g.closed in (X,τ).* □

Proposition 5.7. *A space (X,τ) is $[\gamma,\gamma']$-$T_{1/2}$ if and only if for each $x\in X,\{x\}$ is $[\gamma,\gamma']$-open or $[\gamma,\gamma']$-closed in (X,τ).*

Proof. (Necessity) It is obtained by Lemma 5.6 and Definition 5.1.

(Sufficiency) Let F be a $[\gamma,\gamma']$-g.closed set. We claim that $\mathrm{Cl}_{[\gamma,\gamma']}(F)\subset F$ holds. Let $x\in\mathrm{Cl}_{[\gamma,\gamma']}(F)$. It suffices to prove it for the following two cases:

Case 1. Suppose that $\{x\}$ is $[\gamma,\gamma']$-open. Since $x\in\tau_{[\gamma,\gamma']}$-$\mathrm{Cl}(F)$ and $\{x\}\in\tau_{[\gamma,\gamma']}$, $\{x\}\cap F\neq\emptyset$ by Proposition 3.3.

Case 2. Suppose that $\{x\}$ is $[\gamma,\gamma']$-closed. By Proposition 4.6(i), $\mathrm{Cl}_{[\gamma,\gamma']}(F)\setminus F$ does not contain the $[\gamma,\gamma']$-closed set $\{x\}$. Since $x\in\mathrm{Cl}_{[\gamma,\gamma']}(F)$, we have $x\in F$.

Therefore, we prove that $\mathrm{Cl}_{[\gamma,\gamma']}(F)\subset F$, and so F is $[\gamma,\gamma']$-closed by Theorem 3.8. □

Proposition 5.8. *A space (X,τ) is $[\gamma,\gamma']$-T_1 if and only if for each $x\in X,\{x\}$ is $[\gamma,\gamma']$-closed in (X,τ).* □

The following proposition is proved by using Definitions 5.2, 5.3, Propositions 5.7 and 5.8.

Proposition 5.9. (i) *If (X,τ) is $[\gamma,\gamma']$-T_2, then it is $[\gamma,\gamma']$-T_1.*

(ii) *If (X,τ) is $[\gamma,\gamma']$-T_1, then it is $[\gamma,\gamma']$-$T_{1/2}$.*

(iii) *If (X,τ) is $[\gamma,\gamma']$-$T_{1/2}$, then it is $[\gamma,\gamma']$-T_0.* □

Remark 5.10. From Proposition 5.9 and Examples 5.11,5.12 and 5.13,the following implications hold and none of the implications is reversible:

$$[\gamma,\gamma']\text{-}T_2\to[\gamma,\gamma']\text{-}T_1\to[\gamma,\gamma']\text{-}T_{1/2}\to[\gamma,\gamma']\text{-}T_0,$$

where $A\to B$ represents that A implies B.

Example 5.11. Let (X,τ) be the double origin topological space, where $X = R^2 \cup \{O^*\}$ and O^* denotes an additional point(eg.[21;p.92]). Let γ be the closure operation, i.e. U^γ=Cl(U) for every $U \in \tau$. Let γ' be operation defined in [25;Example 5.9],i.e. for every non-empty open set A,

$$A^{\gamma'} = \begin{cases} A & \text{if } O \notin A \text{ and } O^* \notin A, \\ \mathrm{Cl}(A) & \text{if } O \in A \text{ or } O^* \in A, \end{cases}$$

where O is the origin of R^2. Then, it is shown directly that each singleton is $[\gamma,\gamma']$-closed in (X,τ). By Proposition 5.8, (X,τ) is $[\gamma,\gamma']$-T_1. Using a fact that the operation γ' is monotone, we can show that $(U^\gamma \cap V^{\gamma'}) \cap (W^\gamma \cap S^{\gamma'}) \neq \emptyset$ holds for any open neighbourhoods U,V of O and any open neighbourhoods W,S of O^*. This implies that (X,τ) is not $[\gamma,\gamma']$-T_2.

Example 5.12. Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,d\},$ $\{a,b,d\},\{a,b,c\},\{a,c,d\},X\}$. Let γ and γ' be operations on a topology τ defined as follows: for every open set A,

$$A^\gamma = \mathrm{Cl}(A) \quad \text{and} \quad A^{\gamma'} = \begin{cases} A & \text{if } a \in A, \\ \mathrm{Cl}(A) & \text{if } a \notin A. \end{cases}$$

Then it is shown that $\tau_{[\gamma,\gamma']} = \tau$ and (X,τ) is $T_{1/2}$. By using Proposition 5.7, (X,τ) is $[\gamma,\gamma']$-$T_{1/2}$. However, by Proposition 5.8, (X,τ) is not $[\gamma,\gamma']$-T_1, in fact, a singleton $\{a\}$ is not $[\gamma,\gamma']$-closed.

Example 5.13. (cf.[25;Example 5.15]) Let $X = R$(the set of the real numbers) and τ be the cofinite topology for X. Let γ and γ' be operations on τ defined as follows: for every open set A,

$$A^\gamma = \mathrm{Cl}(A) \quad \text{and} \quad A^{\gamma'} = \begin{cases} A & \text{if } p \in A, \\ \mathrm{Cl}(A) & \text{if } p \notin A, \end{cases}$$

where p is a specified point of X. Then the topological space (X,τ) is not $[\gamma,\gamma']$-$T_{1/2}$ because a singleton $\{p\}$ is neither $[\gamma,\gamma']$-open nor $[\gamma,\gamma']$-closed. It is shown directly that (X,τ) is $[\gamma,\gamma']$-T_0.

Remark 5.14. (i) In Definitions 5.1,5.2,5.3 and 5.4,put $\gamma' = X$. Then a space (X,τ) is $[\gamma,X]$-T_i if and only if it is γ-T_i where i=0,1/2,1,2.

(ii) Let $\gamma' = X$ in Proposition 5.7. Then by using Proposition 5.7, [16;Definition 4.5], Remark 3.6, Theorem 3.8 and (i) above we have the following:

(5.15) (cf.[25;Corollary 5.14]) A space (X,τ) is γ-$T_{1/2}$ if and only if for each $x \in X, \{x\}$ is γ-open or γ-closed.

The statement (5.15) shows that the regularity on γ in [16;Proposition 4.10(ii)] can be omitted.

6. Comparisons of related separation axioms.

Proposition 6.1. *If (X,τ) is γ-T_i, then it is $[\gamma,\gamma']$-T_i,where i=0,1/2,1,2.*

Proof. The proofs for i=0,1,2 follow from Definitions 5.4,5.3,5.2 ,Remark 5.14(i) and [16;Definitions 4.1-4.3].

The proof for i=1/2 is obtained as follows: Let $x \in X$. Then,$\{x\}$ is γ-open or γ-closed by (5.15). $\{x\}$ is $[\gamma,\gamma']$-open or $[\gamma,\gamma']$-closed because every γ-open is $[\gamma,\gamma']$-open by (2.5). The proof is completed from Proposition 5.7. □

Remark 6.2. The following series of examples show that all converses of Proposition 6.1 cannot be reserved.

Example 6.3. Let (X,τ) be the double origin topological space of Example 5.11. Let γ and γ' be operations on τ defined as follows: for every open set A,

$$A^{\gamma} = \mathrm{Cl}(A) \quad \text{and} \quad A^{\gamma'} = \begin{cases} A & \text{if } O^* \notin A, \\ \mathrm{Cl}(A) & \text{if } O^* \in A. \end{cases}$$

Then (X,τ) is not γ-T_2 because $U^{\gamma} \cap V^{\gamma} \neq \emptyset$ for any open neighbourhoods U and V of O and O^*, respectively. However, it is $[\gamma,\gamma']$-T_2.

Example 6.4. Let (X,τ) be the double origin topological space and γ and γ' operations defined as follows: for every non-empty open set A, $A^{\gamma} = A \cup \{O^*\}$ and $A^{\gamma'}$=Cl(A). Then (X,τ) is not γ-T_1 because a singleton $\{O^*\}$ is not γ-closed;it is $[\gamma,\gamma']$-T_1.

Example 6.5. Let (X,τ) ,γ and γ' be the same space and the same operations as in Example 5.12, respectively. Then (X,τ) is $[\gamma,\gamma']$-$T_{1/2}$. However, it is not γ-$T_{1/2}$ because a singleton $\{a\}$ is neither γ-open nor γ-closed.

Example 6.6. Let (X,τ) ,γ and γ' be the same space and the same operations defined in Example 5.13, respectively. Then, (X,τ) is not γ-T_0;it is $[\gamma,\gamma']$-T_0.

Proposition 6.7. *If (X,τ) is $[\gamma,\gamma']$-T_i, then it is T_i, where i=0,1/2,1,2.*

Proof. The proofs for i=0,2 follow from definitions.

The proof for i=1 (resp. i=1/2) follows from Proposition 5.8 (resp. Proposition 5.7), Remark 2.4 and Definition 3.1. □

Remark 6.8. The following series of examples show that all converses of Proposition 6.7 cannot be reserved.

Example 6.9. Let (X,τ) , γ and γ' be the same space and the same operations as in Example 5.11. Then, (X,τ) is not $[\gamma,\gamma']$-T_2(Example 5.11); it is T_2([21]).

Example 6.10. Let (X,τ) , γ and γ' be the same space and the same operations as in Example 5.13. Then, (X,τ) is T_1 and hence $T_{1/2}$. However, it is not $[\gamma,\gamma']$-T_1 because it is not $[\gamma,\gamma']$-$T_{1/2}$.

Example 6.11. Let (X,τ) , γ and γ' be the same space and the same operations as in [17;Example 5], that is, $X = \{0,1\}$, τ is the Sierpinski topology on X, γ is the closure operation and γ' is the interior-closure operation(i.e.$A^{\gamma'}$=Int(Cl(A)) for any $A \in \tau$). Then, (X,τ) is not $[\gamma,\gamma']$-T_0; it is T_0.

Proposition 6.12. *If (X,τ) is (γ,γ')-T_i, then it is $[\gamma,\gamma']$-T_i , where i=0,1/2,1,2.*

Proof. The proofs for i=0,2 follow from Definitions 5.2 and 5.4 and [25;Definitions 5.1,5.5]. The proof for i=1 (resp. i=1/2) follows from [25;Proposition 5.12(i)] (resp.[25;Proposition 5.12(ii)]) and Proposition 6.1. □

Remark 6.13. The converses of Proposition 6.12 for i=0,2 are not true as showing by the following examples. The converse of Proposition 6.12 for i=1,1/2 can not reversible by Proposition 6.1, [25;Proposition 5.12] and Examples 6.4 and 6.5.

Example 6.14. Let (X,τ) ,γ and γ' be the same space and the same operations as in Example 5.13. Then, (X,τ) is not (γ,γ')-T_0;it is $[\gamma,\gamma']$-T_0.

Example 6.15. Let (X,τ) ,γ and γ' be the same space and the same operations as in Example 6.3. Then, (X,τ) is not (γ,γ')-T_2;it is $[\gamma,\gamma']$-T_2.

Remark 6.16. From Propositions 6.1 and 6.7,Remark 5.10,[10;Corollary 5.6],Proposition 6.12 and [16; p.180], for distinct operations γ and γ' we have the following diagram (cf.[25;

p.57]). We note that none of the implications in the following diagram is reversible by Remarks 5.10,6.2,6.8 and 6.13 and [17;Theorem 1]:

$$\begin{array}{ccccccccccccc} & & \gamma\text{-}T_2 & \longrightarrow & \gamma\text{-}T_1 & \longrightarrow & \gamma\text{-}T_{1/2} & \longrightarrow & \gamma\text{-}T_0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ (\gamma,\gamma')\text{-}T_2 & \longrightarrow & [\gamma,\gamma']\text{-}T_2 & \longrightarrow & [\gamma,\gamma']\text{-}T_1 & \longrightarrow & [\gamma,\gamma']\text{-}T_{1/2} & \longrightarrow & [\gamma,\gamma']\text{-}T_0 & \longleftarrow & (\gamma,\gamma')\text{-}T_0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & T_2 & \longrightarrow & T_1 & \longrightarrow & T_{1/2} & \longrightarrow & T_0, & & \end{array}$$

where $A \to B$ represents that A implies B.

7. $([\gamma,\gamma'],[\beta,\beta'])$ -continuous functions. Throughout this section, let $f{:}(X,\tau)\to (Y,\sigma)$ be a function and let $\gamma,\gamma' : \tau \to P(X)$ be operations on τ and $\beta,\beta' : \sigma \to P(Y)$ be operations on σ.

Definition 7.1. A function $f{:}(X,\tau)\to (Y,\sigma)$ is said to be $([\gamma,\gamma'],[\beta,\beta'])$-*continuous* if for each point $x \in X$ and each open neighbourhoods W and S of $f(x)$ there exist open neighbourhoods U and V of x such that $f(U^\gamma \cap V^{\gamma'}) \subset W^\beta \cap S^{\beta'}$. A function $f{:}(X,\tau)\to (Y,\sigma)$ is called a $([\gamma,\gamma'],[\beta,\beta'])$-*homeomorphism* if f is a $([\gamma,\gamma'],[\beta,\beta'])$-continuous bijection and $f^{-1} : (Y,\sigma) \to (X,\tau)$ is $([\beta,\beta'],[\gamma,\gamma'])$-continuous.

Theorem 7.2. *Let* (a),(b_1),(b_2),(c_1),(c_2),(c_3) and (c_4) *be the following properties for a function* $f{:}(X,\tau)\to (Y,\sigma)$.

(a) $f{:}(X,\tau)\to (Y,\sigma)$ *is* $([\gamma,\gamma'],[\beta,\beta'])$-*continuous,*

(b_1) $f(\mathrm{Cl}_{[\gamma,\gamma']}(A)) \subset \mathrm{Cl}_{[\beta,\beta']}(f(A))$ *for every subset* A *of* (X,τ),

(b_2) $\mathrm{Cl}_{[\gamma,\gamma']}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_{[\beta,\beta']}(B))$ *for every subset* B *of* (Y,σ),

(c_1) $f^{-1}(B)$ *is* $[\gamma,\gamma']$-*closed for every* $[\beta,\beta']$-*closed set* B *of* (Y,σ),

(c_2) $f(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(A)) \subset \tau_{[\beta,\beta']}\text{-}\mathrm{Cl}(f(A))$ *for every subset* A *of* (X,τ),

(c_3) $f^{-1}(V) \in \tau_{[\gamma,\gamma']}$ *for every set* $V \in \sigma_{[\beta,\beta']}$,

(c_4) *for each point* $x \in X$ *and each set* $W \in \sigma_{[\beta,\beta']}$ *containing* $f(x)$ *there exist a set* $U \in \tau_{[\gamma,\gamma']}$ *containing* x *such that* $f(U) \subset W$.

Then (a) $\Rightarrow$(b_1) $\Leftrightarrow$(b_2) $\Rightarrow$(c_1) $\Leftrightarrow$(c_2) $\Leftrightarrow$(c_3) $\Leftrightarrow$(c_4) *hold.* $\Box$

Proof. (a) $\Rightarrow$(b_1). Let $f(x) \in f(\mathrm{Cl}_{[\gamma,\gamma']}(A))$ and W, S be open neighbourhoods of $f(x)$. There exist open neighbourhoods U and V of x such that $f(U^\gamma \cap V^{\gamma'}) \subset W^\beta \cap S^{\beta'}$. Since $x \in \mathrm{Cl}_{[\gamma,\gamma']}(A)$, $(U^\gamma \cap V^{\gamma'}) \cap A \neq \emptyset$ by Definitioin 3.5. Therefore, we have $f(A) \cap (W^\beta \cap S^{\beta'}) \neq \emptyset$. This implies that $f(x) \in \mathrm{Cl}_{[\beta,\beta']}(f(A))$.

(b_1) $\Leftrightarrow$(b_2). This follows from Definition 3.4 and usual arguments.

(b_2) $\Rightarrow$(c_1). Let B be a $[\beta,\beta']$-closed set of (Y,σ). By (b_2) and Theorem 3.8, $\mathrm{Cl}_{[\gamma,\gamma']}(f^{-1}(B)) \subset f^{-1}(B)$ and hence $f^{-1}(B)$ is $[\gamma,\gamma']$-closed.

(c_1) $\Rightarrow$(c_2). For every subset A of (X,τ), by using (c_1) and Proposition 3.4(iv), $f^{-1}(\tau_{[\beta,\beta']}\text{-}\mathrm{Cl}(f(A)))$ is $[\gamma,\gamma']$-closed in (X,τ). Using Definition 3.2 and Proposition 3.4(iii) we obtain (c_2).

(c_2) $\Rightarrow$(c_1). Let B be a $[\beta,\beta']$-closed set of (Y,σ). By (c_2) and Proposition 3.4, $\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(f(\tau_{[\gamma,\gamma']}\text{-}\mathrm{Cl}(f^{-1}(B)))) \subset f^{-1}(\tau_{[\beta,\beta']}\text{-}\mathrm{Cl}(f(f^{-1}(B)))) \subset f^{-1}(B)$. Therefore, by Proposition 3.4(iii), $f^{-1}(B)$ is $[\gamma,\gamma']$-closed.

(c_2) $\Rightarrow$(c_3). This follows from Definition 3.1 and the equivalence of (c_1) $\Leftrightarrow$(c_2) above.

(c_3) ⇒(c_4). It is obvious from Definition 2.1.

(c_4) ⇒(c_3). Let $V \in \sigma_{[\beta,\beta']}$. For each $x \in f^{-1}(V)$, by (c_4), there exists a $[\gamma,\gamma']$-open set U_x containing x such that $f(U_x) \subset V$. Then we have $f^{-1}(V) = \cup\{U_x \in \tau_{[\gamma,\gamma']} \mid x \in f^{-1}(V)$ $\}$ and hence $f^{-1}(V) \in \tau_{[\gamma,\gamma']}$ using Proposition 2.2(ii). □

Corollary 7.3. *If (Y,σ) is a $[\beta,\beta']$-regular space, or operations β and β' are open on σ, then all properties of Theorem 7.2 are equivalent.*

Proof. By Theorem 7.2 it is sufficient to prove the implication (c_1) ⇒ (a), where (a) and (c_1) are the properties of Theorem 7.2.

First, we show the implication under the assumption that (Y,σ) is a $[\beta,\beta']$-regular space. Let $x \in X$ and W, S be open neighbourhoods of $f(x)$. By Proposition 2.13(i), $Y \setminus (W \cap S)$ is $[\beta,\beta']$-closed. Then, $f^{-1}(Y \setminus (W \cap S))$ is $[\gamma,\gamma']$-closed by (c_1) and hence $f^{-1}(W \cap S)$ is a $[\gamma,\gamma']$-open set containing x. Therefore, there exist open neighbourhoods U and V of x such that $U^\gamma \cap V^{\gamma'} \subset f^{-1}(W \cap S)$ and so $f(U^\gamma \cap V^{\gamma'}) \subset W^\beta \cap S^{\beta'}$. This implies that f is ($[\gamma,\gamma']$,$[\beta,\beta']$)-continuous.

Second, we suppose that the operations β and β' are open. Let $x \in X$ and W, S be open neighbourhoods of $f(x)$. By using openness of β and β' (cf.[16;Definition 2.6]), there exist a β-open set A and a β'-open set B such that $f(x) \in A \cap B$ and $A \cap B \subset W^\beta \cap S^{\beta'}$. By Proposition 2.2(i), $Y \setminus (A \cap B)$ is $[\gamma,\gamma']$-closed and hence $f^{-1}(Y \setminus (A \cap B))$ is $[\beta,\beta']$-closed. Therefore, there exist open neighbourhoods U and V of x such that $U^\gamma \cap V^{\gamma'} \subset f^{-1}(A \cap B)$ and so $f(U^\gamma \cap V^{\gamma'}) \subset W^\beta \cap S^{\beta'}$. This implies that f is ($[\gamma,\gamma']$,$[\beta,\beta']$)-continuous. □

Remark 7.4. (i) As known by [24;p.67] the interior-closure operation (ie. U^β=-Int(Cl(U))) is a typical example of the open operation. Moreover the identity operation and the operation $X : \tau \rightarrow (X,\tau)$ are open on τ. Therefore,in Corollary 7.3, if β and β' are choosen from these operations above, then all properties of Theorem 7.2 are equivalent.

(ii) The converses of implications (a) ⇒(b_1) and (b_2) ⇒(c_1) in Theorem 7.2 are not true in general as shown by the following examples.

Example 7.5. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$ be a topology on X. Let f:(X,τ)→(X,τ) be a function defined by $f(a) = b, f(b) = c$ and $f(c) = a$. Let $\gamma = \beta$ be the closure operation on τ and $\gamma' = \beta' = X : \tau \rightarrow P(X)$. Then, (X,τ) is not $[\gamma,X]$-regular, because of $\tau_{[\gamma,X]} = \{\emptyset,X\} \neq \tau$ (cf. Proposition 2.13 and (2.6)). It is shown that f is not $([\gamma,X],[\gamma,X])$-continuous. However, f satisfies the condition (b_1) in Theorem 7.2.

Example 7.6. Let (X,τ) and τ be the same space and the same topology as in Example 7.5 above. Let f:(X,τ)→(X,τ) be the identity. Let $\gamma = \gamma' = \beta' = X : \tau \rightarrow P(X)$ be the operations on τ and β the closure operation on τ. Then, the condition (c_1) in Theorem 7.2 is true. The condition (b_2) is not true in general. In fact, $\tau_{[\gamma,X]} = \{\emptyset,X\}$ and $f(\mathrm{Cl}_{[\gamma,X]}(\{a\})) = f(X) = X \not\subseteq \mathrm{Cl}_{[\beta,X]}(\{a\}) = \{a,c\}$ holds.

Remark 7.7. By Theorem 7.2 and Proposition 2.9 we have the following:

(7.8) if the operations γ,γ' and β,β' are regular on τ and σ, respectively, and if f:(X,τ)→ (Y,σ) is $([\gamma,\gamma'],[\beta,\beta'])$-continuous, then the induced function $f : (X,\tau_{[\gamma,\gamma']}) \rightarrow (Y,\sigma_{[\beta,\beta']})$ is continuous.

However, the converse of (7.8) above is not true in general as shown by Example 7.5.

Let (X,τ),(Y,σ) and (Z,η) be spaces and $\gamma,\gamma' : \tau \rightarrow P(X), \beta,\beta' : \sigma \rightarrow P(Y)$ and $\delta,\delta' : \eta \rightarrow P(Z)$ be operations on τ,σ and η, respectively.

Let $\mathrm{h}_{[\gamma,\gamma']}(X,\tau)$ be the family of all $([\gamma,\gamma'],[\gamma,\gamma'])$-homeomorphisms from (X,τ) onto itself.

Theorem 7.9. (i) *If $f : (X,\tau) \to (Y,\sigma)$ is $([\gamma,\gamma'],[\beta,\beta'])$-continuous and $g : (Y,\sigma) \to (Z,\eta)$ is $([\beta,\beta'],[\delta,\delta'])$ -continuous, then its composition $g \circ f : (X,\tau) \to (Z,\eta)$ is $([\gamma,\gamma'],[\delta,\delta'])$-continuous.*

(ii) *The set $h_{[\gamma,\gamma']}(X,\tau)$ is a group.*

(iii) *A $(([\gamma,\gamma'],[\beta,\beta'])$-homeomorphism $f : (X,\tau) \to (Y,\sigma)$ induces an isomorphism $f_* : h_{[\gamma,\gamma']}(X,\tau) \to h_{[\beta,\beta']}(Y,\sigma)$ and the identity $1_X : (X,\tau) \to (X,\tau)$ induces the identity from $h_{[\gamma,\gamma']}(X,\tau)$ onto itself.*

Proof. (i) This follows from Definition 7.1.

(ii) A binary operation $\mu : h_{[\gamma,\gamma']}(X,\tau) \times h_{[\gamma,\gamma']}(X,\tau) \to h_{[\gamma,\gamma']}(X,\tau)$ is defined by $\mu(f,g) = g \circ f$ (the composition) for every $f,g \in h_{[\gamma,\gamma']}(X,\tau)$. Then $(h_{[\gamma,\gamma']}(X,\tau),\mu)$ is a group using Definition 7.1 and (i) above.

(iii) It is evidently shown from (i) that an isomorphism f_* is defined by $f_*(h) = f \circ h \circ f^{-1}$ for every $h \in h_{[\gamma,\gamma']}(X,\tau)$. The induced isomorphism $(1_X)_*$ is the identity by definitions. □

In the end of this section, we define the notion of bioperation-closed functions (Definition 7.14 below). The characterization will be obtained as a corollary of Proposition A.3 below which study some functions by a general point of view (i.e.,Definition A.4 and Proposition A.5 in section 8).

(A.1) Let $\mathcal{E}_X$ and $\mathcal{E}_Y$ be given two collections of subsets of (X,τ) and (Y,σ), respectively, satisfying the following conditions: $\emptyset, X \in \mathcal{E}_X$ and $\emptyset, Y \in \mathcal{E}_Y$. For $\mathcal{E}_X$ and $\mathcal{E}_Y$, we define two collections of subsets as follows: $\mathcal{E}_X^C = \{U \mid X \setminus U \in \mathcal{E}_X\}$ and $\mathcal{E}_Y^C = \{V \mid Y \setminus V \in \mathcal{E}_Y\}$.

Definition A.2. If a function $f : (X,\tau) \to (Y,\sigma)$ is said to be

(a) $(\mathcal{E}_X,\mathcal{E}_Y)$-*closed* , if for every $F \in \mathcal{E}_X^C$, $f(F) \in \mathcal{E}_Y^C$,

(b) $(\mathcal{E}_X,\mathcal{E}_Y)$-*continuous* , if for every $V \in \mathcal{E}_Y$, $f^{-1}(V) \in \mathcal{E}_X$.

Proposition A.3. (i) *If $f : (X,\tau) \to (Y,\sigma)$ is $(\mathcal{E}_X,\mathcal{E}_Y)$-closed, then the following condition holds:*

(*) *for each subset B of (Y,σ) and each $U \in \mathcal{E}_X$ satisfying $f^{-1}(B) \subset U$, there exists a set $V \in \mathcal{E}_Y$ such that $B \subset V$ and $f^{-1}(V) \subset U$.*

(ii) *Conversely, if $f : (X,\tau) \to (Y,\sigma)$ satisfies the condition (*) in* (i), *then f is $(\mathcal{E}_X,\mathcal{E}_Y)$-closed.*

Proof. (i) Let $B \in P(Y)$ and $U \in \mathcal{E}_X$ such that $f^{-1}(B) \subset U$. Put $V = Y \setminus f(X \setminus U)$. Then it is shown that $V \in \mathcal{E}_X^C, B \subset V$ and $f^{-1}(V) \subset U$.

(ii) Let $F \in \mathcal{E}_X^C$. Put $B = Y \setminus f(F)$. Then, it is shown that $f^{-1}(B) \subset X \setminus F$ and $X \setminus F \in \mathcal{E}_X$. It follows from (*) that there exists a set $V \in \mathcal{E}_Y$ such that $Y \setminus f(F) \subset V$ and $f^{-1}(V) \subset X \setminus F$. Then we have $Y \setminus V \subset f(F) \subset f(X \setminus f^{-1}(V)) \subset f(f^{-1}(Y \setminus V)) \subset Y \setminus V$. Therefore, we can obtain $f(F) = Y \setminus V$ and hence $f(F) \in \mathcal{E}_Y^C$. This implies that f is $(\mathcal{E}_X,\mathcal{E}_Y)$-closed. □

In Proposition A.3, set $\mathcal{E}_X = \tau$ and $\mathcal{E}_Y = GO(Y,\sigma)$(ie.the set of all g-open sets of (Y,σ) [11;Definition 4.1]), $RO(Y,\sigma)$(ie.the set of all regular-open sets of (Y,σ)), $SGO(Y,\sigma)$ (ie. the set of all sg-open sets of (Y,σ) [2;Definitions 1,2]), $GSO(Y,\sigma)$(ie. the set of all gs-open sets of (Y,σ) [1;Definition 1]) and $PO(Y,\sigma)$(ie. the set of all preopen sets of (Y,σ) [13]), respectively. Then we obtain, respectively, the following characterization of g-closed functions, $(\tau, RO(Y,\sigma))$-closed functions, sg-closed functions and gs-closed functions and $(\tau, PO(Y,\sigma))$-closed functions:

(7.10) [12;Theorem 1.3] A function $f:(X,\tau)\to(Y,\sigma)$ is g-closed if and only if for each subset B of (X,τ) and each open set U containing $f^{-1}(B)$ there is a g-open set V of (Y,σ) such that $B\subset V$ and $f^{-1}(V)\subset U$.

(7.11) A function $f:(X,\tau)\to(Y,\sigma)$ is $(\tau, GO(Y,\sigma))$-closed if and only if for each subset B of (X,τ) and each open set U containing $f^{-1}(B)$ there is a regular-open set V of (Y,σ) such that $B\subset V$ and $f^{-1}(V)\subset U$.

(7.12) [4;Theorem 3.3] (resp. [4;Theorem 4.5]) A function $f:(X,\tau)\to(Y,\sigma)$ is sg-closed (resp. gs-closed) if and only if for each subset B of (X,τ) and each open set U containing $f^{-1}(B)$ there is a sg-open (resp. gs-open) set V of (Y,σ) such that $B\subset V$ and $f^{-1}(V)\subset U$.

(7.13) A function $f:(X,\tau)\to(Y,\sigma)$ is $(\tau, PO(Y,\sigma))$-closed if and only if for each subset B of (X,τ) and each open set U containing $f^{-1}(B)$ there is a preopen set V of (Y,σ) such that $B\subset V$ and $f^{-1}(V)\subset U$.

Now we define some bioperation-closed functions as follows.

Definition 7.14. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $([\gamma,\gamma'],[\beta,\beta'])$-*closed* (resp. $([\gamma,\gamma'],[\beta,\beta'])$-*generalized closed*) if f is a $(\tau_{[\gamma,\gamma']},\sigma_{[\beta,\beta']})$-closed (resp. $(\tau_{[\gamma,\gamma']}, GO_{[\beta,\beta']}(Y,\sigma))$-closed) function, where $GO_{[\beta,\beta']}(Y,\sigma)$ is the set of all $[\beta,\beta']$-g.open sets of (Y,σ).

The following proposition is a characterization of $([\gamma,\gamma'],[\beta,\beta'])$-closed functions by setting $\mathcal{E}_X=\tau_{[\gamma,\gamma']}$ and $\mathcal{E}_Y=\sigma_{[\beta,\beta']}$ in Proposition A.3.

Proposition 7.15. *A function $f:(X,\tau)\to(Y,\sigma)$ is $([\gamma,\gamma'],[\beta,\beta'])$-closed if and only if for each subset B of (Y,σ) and each $[\gamma,\gamma']$-open set U containing $f^{-1}(B)$, there exists a $[\beta,\beta']$-open set V such that $B\subset V$ and $f^{-1}(V)\subset U$.* □

Proposition 7.16. (i) *If $f:(X,\tau)\to(Y,\sigma)$ is bijective and $f^{-1}:(Y,\sigma)\to(X,\tau)$ is $([\beta,\beta'],[\gamma,\gamma'])$-continuous, then f is $([\gamma,\gamma'],[\beta,\beta'])$-closed.*

(ii) *If $f:(X,\tau)\to(Y,\sigma)$ is $([\gamma,\gamma'],[\beta,\beta'])$-continuous and $([id,id],[\beta,\beta'])$-closed, then*

(a) *$f(A)$ is $[\beta,\beta']$-g.closed for every $[\gamma,\gamma']$-g.closed set A of (X,τ), and*

(b) *$f^{-1}(B)$ is $[\gamma,\gamma']$-g.closed for every $[\beta,\beta']$-g.closed set B of (Y,σ).*

Proof. (i) This follows from definitions and Theorem 7.2.

(ii) (a) Let V be a $[\beta,\beta']$-open set containing $f(A)$. By using Theorem 7.2,$f^{-1}(V)$ is a $[\gamma,\gamma']$-open set containing A and so $\mathrm{Cl}_{[\gamma,\gamma']}(A)\subset f^{-1}(V)$. It follows from Definition 7.14 and Theorem 7.2 that $f(\mathrm{Cl}_{[\gamma,\gamma']}(A))$ is a $[\beta,\beta']$-closed set and hence $\mathrm{Cl}_{[\beta,\beta']}(f(A))\subset\mathrm{Cl}_{[\beta,\beta']}(f(\mathrm{Cl}_{[\gamma,\gamma']}(A)))=f(\mathrm{Cl}_{[\gamma,\gamma']}(A))\subset V$. This implies that $f(A)$ is $[\gamma,\gamma']$-g.closed.

(b) Let U be a $[\gamma,\gamma']$-open set containing $f^{-1}(B)$. Since $\mathrm{Cl}_{[\gamma,\gamma']}(f^{-1}(B))\cap(X\setminus U)$,say F, is closed (cf.Theorems 3.8,3.9), we have $f(F)\subset f((\mathrm{Cl}_{[\gamma,\gamma']}(f^{-1}(B)))\cap f^{-1}(Y\setminus B))\subset \mathrm{Cl}_{[\beta,\beta']}(f(f^{-1}(B)))\cap f(f^{-1}(Y\setminus B))$ by using assumptions and Theorem 7.2. Therefore, $\mathrm{Cl}_{[\gamma,\gamma']}(B)\setminus B$ contains a $[\beta,\beta']$-closed set $f(F)$. It follows from Proposition 4.6(i) that $f(F)=\emptyset$ and hence $\mathrm{Cl}_{[\gamma,\gamma']}(f^{-1}(B))\subset U$. This shows that $f^{-1}(B)$ is $[\gamma,\gamma']$-g.closed. □

Regarding Definition 7.1, Theorem 7.2 , Corollary 7.3 and Proposition 7.16(ii) we note the following in the end of this section.

Remark 7.17. (i) In Definition 7.1, put $\gamma'=X$ and $\beta'=Y$. Then, it is shown that

(7.18) a function $f:(X,\tau)\to(Y,\sigma)$ is $([\gamma,X],[\beta,Y])$-continuous if and only if f is (γ,β)-continuous (cf.[16;Definition 4.12]).

Under this setting, we obtain immediately [16;Proposition 4.13] and [16;Remarks 4.14,4.15] from Theorem 7.2 and Corollary 7.3, respectively.

(ii) It is shown that if $f:(X,\tau)\to(Y,\sigma)$ is ($[\gamma,\gamma']$,$[\beta,\beta']$)-continuous, then f is (id,β)-continuous and (id,β')-continuous. Conversely, if f is (γ,β)-continuous and (γ',β')-continuous, then f is ($[\gamma,\gamma']$,$[\beta,\beta']$)-continuous. Therefore, we have the following:

(7.19) A function $f:(X,\tau)\to(Y,\sigma)$ is ($[id,id],[\beta,\beta]$)-continuous if and only if f is (id,β)-continuous.

(iii) Let $\gamma'=X$ and $\beta'=Y$ in Proposition 7.16(ii). Then, using (i) above we have the following:

(7.20) (cf.[16;Proposition 4.18]) Suppose that $f:(X,\tau)\to(Y,\sigma)$ is (γ,β)-continuous and f is (id,β)-closed. Then,

(a) for every γ-g.closed set A of (X,τ), the image $f(A)$ is β-g.closed, and

(b) for every β-g.closed set B of (Y,β), $f^{-1}(B)$ is γ-g.closed.

The statement (7.20)(b) above shows that the regularity on β in [16;Proposition 4.18(ii)] can be omitted.

(iv) In Proposition 7.16(ii), put $\gamma=\gamma'=id$ and $\beta=\beta'=id$. Then we can obtain [11;Theorems 6.1,6.3] as a corollary of Proposition 7.16(ii) because of $\tau_{[id,id]}=\tau$ and $\mathrm{Cl}_{[id,id]}(A)=\mathrm{Cl}\,(A)$ for every set A. .

8. Some relations amongs ($[\gamma,\gamma']$, $[\beta,\beta']$)-continuous functions,$[\gamma,\gamma']$- T_i spaces and $[\gamma,\gamma']$-normal spaces.

Theorem 8.1. (i) *If $f:(X,\tau)\to(Y,\sigma)$ is a ($[\gamma,\gamma']$,$[\beta,\beta']$)-continuous injection and if (Y,σ) is $[\beta,\beta']$-T_i, then (X,τ) is $[\gamma,\gamma']$-T_i, where i=0,1/2,1,2.*

(ii) *If $f:(X,\tau)\to(Y,\sigma)$ is a ($[\gamma,\gamma']$,$[\beta,\beta']$)-continuous and ($[id,id],[\beta,\beta']$)-closed surjection and if (X,τ) is $[\gamma,\gamma']$-$T_{1/2}$,then (Y,σ) is $[\beta,\beta']$-$T_{1/2}$.*

(iii) *Suppose that $f:(X,\tau)\to(Y,\sigma)$ is a ($[\gamma,\gamma']$,$[\beta,\beta']$)-homeomorphism. Then, (X,τ) is $[\gamma,\gamma']$-T_i if and only if (Y,σ) is $[\beta,\beta']$-T_i,where i=0,1/2,1,2.*

Proof. (i) The proof for i=1/2 is as follows: Let $x\in X$. Then, by Proposition 5.7, $\{f(x)\}$ is $[\beta,\beta']$-closed or $[\beta,\beta']$-open in (Y,σ). By Theorem 7.2 and Proposition 5.7, $\{x\}$ is $[\gamma,\gamma']$-closed or $[\gamma,\gamma']$-open and hence (X,τ) is $[\gamma,\gamma']$-$T_{1/2}$. The proof for i=1 is similar as the proof for i=1/2 by Proposition 5.8 in place of Proposition 5.7. The proofs for i=0,2 follow from Definitions 5.4,5.2 and Theorem 7.2.

(ii) Let B be a $[\beta,\beta']$-g.closed set in (Y,σ). By Proposition 7.16(ii)(b), $f^{-1}(B)$ is $[\gamma,\gamma']$-g.closed in (X,τ). Then, by using Definitions 5.1,3.1 and Proposition 2.2(ii), $f^{-1}(B)$ is closed and hence $f(f^{-1}(B))=B$ is $[\beta,\beta']$-closed. This implies that (Y,σ) is $[\beta,\beta']$-$T_{1/2}$.

(iii) This follows from (i). □

Proposition 8.2. (i) *Suppose that γ and γ' are regular operations on τ. A space (X,τ) is $[\gamma,\gamma']$-T_i if and only if an associated space $(X,\tau_{[\gamma,\gamma']})$ is T_i,where i=1,1/2.*

(ii) *If γ and γ' are regular operations on τ and $(X,\tau_{[\gamma,\gamma']})$ is T_2, then (X,τ) is $[\gamma,\gamma']$-T_2.*

(iii) *If γ and γ' are regular and open and if (X,τ) is $[\gamma,\gamma']$-T_2, then $(X,\tau_{[\gamma,\gamma']})$ is T_2.*

Proof. (i) It follows from Proposition 2.9 that a subset A is $[\gamma,\gamma']$-open in (X,τ) if and only if A is open in $(X,\tau_{[\gamma,\gamma']})$. Therefore, the proof for i=1/2 (resp. i=1) follows from Propositions 5.7 (resp. Proposition 5.8).

(ii) This follows from the Hausdorffness of $(X,\tau_{[\gamma,\gamma']})$ and Definitions 5.2,2.1.

(iii) Let x and y be distinct points of X. By assumptions there exist γ-open sets U,W and γ'-sets V,S such that $x\in U\cap V, y\in W\cap S$ and $(U\cap V)\cap(W\cap S)=\emptyset$. It follows from Proposition 2.2(i) that $U\cap V\in\tau_{[\gamma,\gamma']}$ and $W\cap S\in\tau_{[\gamma,\gamma']}$. This implies that $(X,\tau_{[\gamma,\gamma']})$ is T_2. □

Theorem 8.3. *(i) Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(X,\tau)\to(Y,\sigma)$ be $([\gamma,\gamma'],[\beta,\beta'])$-continuous functions. If (Y,σ) is $[\beta,\beta']$-T_2 and γ and γ' are regular on τ, then the following set $A=\{x \mid x\in X, f(x)=g(x)\}$ is $[\gamma,\gamma']$-closed.*

(ii) If $f:(X,\tau)\to(Y,\sigma)$ is a $([\gamma,\gamma'],[\beta,\beta'])$-continuous function and (Y,σ) is $[\beta,\beta']$ -T_2, then the following set $B=\{(x,y)\mid(x,y)\in X\times X, f(y)=f(x)\}$ is closed.

Proof. (i) We show that $\mathrm{Cl}_{[\gamma,\gamma']}(A)\subset A$. Suppose that $x\notin A$ for some $x\in X$. Since (Y,σ) is $[\beta,\beta']$- T_2, for $f(x)\neq f(y)$ there exist open sets U and V containing $f(x)$ and open sets W and S containing $g(x)$ such that $(U^\beta\cap V^{\beta'})\cap(W^\beta\cap S^{\beta'})=\emptyset$. Then, by assumptions there exist open sets U_1,V_1 containing x and open sets W_1,S_1 containing y such that $f(U_1^\gamma\cap V_1^{\gamma'})\subset U^\beta\cap V^{\beta'}$ and $f(W_1^\gamma\cap S_1^{\gamma'})\subset W^\beta\cap S^{\beta'}$. By using regulalities of γ and γ' there exist open sets U_2,V_2 containing x such that $f(U_2^\gamma\cap V_2^{\gamma'})\cap g(U_2^\gamma\cap V_2^{\gamma'})=\emptyset$ and hence $A\cap(U_2^\gamma\cap V_2^{\gamma'})=\emptyset$. This implies that $x\notin\mathrm{Cl}_{[\gamma,\gamma']}(A)$ and hence A is $[\gamma,\gamma']$-closed.

(ii) We claim that $X\times X\setminus B$ is an open set. Let $(x,y)\in X\times X\setminus B$. Then, there exist open sets U,V containing $f(x)$ and open sets W,S containing $f(y)$ such that $(U^\beta\cap V^{\beta'})\cap(W^\beta\cap S^{\beta'})=\emptyset$. It follows from the assumption that there exist open sets U_1,V_1 containing x and W_1,S_1 containing y such that $f(U_1\cap V_1)\subset U^\beta\cap V^{\beta'}$ and $f(W_1\cap S_1)\subset W^\beta\cap S^{\beta'}$. Therefore, we have $(U_1\cap V_1)\times(W_1\cap S_1)\subset X\setminus B$ and hence $X\setminus B$ is open. □

Remark 8.4. (i) Put $\gamma'=X$ and $\gamma'=Y$ in Theorem 8.1. Then we obtain a slight improvement of [16;Theorem 4.20] and [16;Theorem 4.22] as follows:

(8.5) (cf.[16;Theorem 4.20(i),Proposition 4.25]) If $f:(X,\tau)\to(Y,\tau)$ is a (γ,β)-continuous injection and if (Y,σ) is β-T_i,then (X,τ) is γ-T_i where i=0,1/2,1,2.

(8.6) (cf.[16;Theorem 4.20(ii)) If $f:(X,\tau)\to(Y,\tau)$ is a (γ,β)-continuous and (id,β)-closed surjection and if (X,τ) is γ-$\mathrm{T}_{1/2}$, then $(X,\tau)$is β-$\mathrm{T}_{1/2}$.

(8.7) (cf.[16;Theorem 4.22]) Suppose that $f:(X,\tau)\to(Y,\tau)$ is a (γ,β)-homeomorphism. Then, (X,τ) is γ-T_i if and only if (Y,σ) is β-T_i where i=0,1/2,1,2.

The statements (8.6) and (8.7) show that the regularity on β in [16;Theorem 4.20(ii)] and [16;Theorem 4.22] can be omitted. The statement (8.5) for i=1/2 shows that the (id,β)-closedness of f in [16;Theorem 4.20(i)] can be omitted.

(ii) Let $\gamma'=X$ in Proposition 8.2(ii) and (iii). Then we obtain [16;Lemma 4.26(i)] and [16;Lemma 4.26(ii)], respectively. Let $\gamma'=X$ in Proposition 8.2(i). Moreover, then under the assumption that γ is regular, (X,τ) is γ-T_i if and only if (X,τ_γ) is T_i where i=1,1/2. The result for i=1/2 is shown in [16;Proposition 4.24].

(iii) Let $\gamma'=X$ and $\beta'=Y$ in Theorem 8.3(i),(ii). Then we have the following:

(8.8) (cf.[16;Proposition 4.27]) Let $f:(X,\tau)\to(Y,\tau)$ and $g:(X,\tau)\to(Y,\sigma)$ be (γ,β)-continuous. If (Y,σ) is β-T_2 and γ is regular on τ, then the set $A=\{x\in X|f(x)=g(x)\}$ is γ-closed.

(8.9) (cf.[16;Proposition 5.2]) If $f:(X,\tau)\to(Y,\tau)$ is (γ,β)-continuous and (Y,σ) is β-T_2, then the set $B=\{(x,y)\in X\times X|f(x)=f(y)\}$ is closed.

The statement (8.8) shows that the openness on γ and β and the regularity on β in [16;Proposition 4.27] can be omitted.

In Definition 8.11 below we define the notion of $[\gamma,\gamma']$-normal spaces preparing the following general point of view. Let $\mathcal{O}_X$ and $\mathcal{O}_Y$ be collections of subsets of (X,τ) and (Y,σ), respectively, such that $\emptyset,X\in\mathcal{O}_X$ and $\emptyset,Y\in\mathcal{O}_Y$ (cf.(A.1)). As similarly in (A.1) we define collections of subsets $\mathcal{O}_X^C$ and $\mathcal{O}_Y^C$: $\mathcal{O}_X^C=\{A\mid X\setminus A\in\mathcal{O}_X\}$ and $\mathcal{O}_Y^C=\{F\mid Y\setminus F\in\mathcal{O}_Y\}$.

Definition A.4. A space (X,τ) is said to be $(\mathcal{O}_X,\mathcal{E}_X)$-*normal* if for any pair of disjoint sets $A,B\in\mathcal{O}_X^C$, there exist disjoint sets $V,V'\in\mathcal{E}_X$ such that $A\subset V$ and $B\subset V'$.

Remark 8.10. In Definitioin A.4 above, put $\mathcal{O}_X=\tau$ and $\mathcal{E}_X=SO(X,\tau),PO(X,\tau)$, where $SO(X,\tau)$ and $PO(X,\tau)$ denote the set of all semiopen sets [10] of (X,τ) and the set of all preopen sets [13] of (X,τ), respectively. Then, (τ,τ)-normal spaces, $(\tau, SO(X,\tau))$-normal spaces and $(\tau, PO(X,\tau))$-normal spaces are called as normal spaces, s-normal spaces [14] and pre-normal spaces [15],respectively.

Proposition A.5. *If $f:(X,\tau)\to (Y,\sigma)$ is an $(\mathcal{O}_X,\mathcal{O}_Y)$-continuous and $(\mathcal{E}_X,\mathcal{E}_Y)$-closed surjection and (X,τ) is $(\mathcal{O}_X,\mathcal{E}_X)$-normal, then (Y,σ) is $(\mathcal{O}_Y,\mathcal{E}_Y)$-normal.*

Proof. Let $A,B\in\mathcal{O}_Y^C$ and $A\cap B=\emptyset$. Since f is surjective and $(\mathcal{O}_X,\mathcal{O}_Y)$-continuous, we have that $f^{-1}(A)$ and $f^{-1}(B)$ are not empty and $f^{-1}(A), f^{-1}(B)\in\mathcal{O}_X^C$. Since (X,τ) is $(\mathcal{O}_X,\mathcal{E}_X)$-normal, there exist disjoint sets $U\in\mathcal{E}_X$ and $U'\in\mathcal{E}_X$ such that $f^{-1}(A)\subset U$ and $f^{-1}(B)\subset U'$. By Proposition A.3(i), there exist subsets $V\in\mathcal{E}_Y$ and $V'\in\mathcal{E}_Y$ such that $A\subset V$, $B\subset V'$, $f^{-1}(V)\subset U$ and $f^{-1}(V')\subset U'$. Then, $f^{-1}(V)\cap f^{-1}(V')=\emptyset$ and hence $V\cap V'=\emptyset$. Therefore, this implies that (Y,σ) is $(\mathcal{O}_Y,\mathcal{E}_Y)$-normal. □

Definition 8.11. A $(\tau,\tau_{[\gamma,\gamma']})$ -normal space is called shortly as a $[\gamma,\gamma']$-*normal space.*

Proposition 8.12. (i) *If (X,τ) is $[\gamma,\gamma']$-normal, then it is $(\tau, GO_{[\gamma,\gamma']}(X,\tau))$-normal, that is the following condition holds:*

(*) *for any pair of disjoint closed sets A,B of (X,τ), there exist disjoint $[\gamma,\gamma']$-g.open sets U,U' such that $A\subset U$ and $B\subset U'$.*

(ii) *If γ and γ' are open operations on τ and if (X,τ) is $(\tau, GO_{[\gamma,\gamma']}(X,\tau))$-normal, then it is $[\gamma,\gamma']$-normal.*

Proof. (i) By Proposition 4.9, every $[\gamma,\gamma']$-open set is $[\gamma,\gamma']$-g.open. Therefore, the proof of (i) is proved.

(ii) Let A and B be disjoint closed sets of (X,τ). By assumptions and Proposition 4.9, there exist disjoint $[\gamma,\gamma']$-g.open sets U and U' such that $A\subset \mathrm{Int}_{[\gamma,\gamma']}(U)$ and $B\subset \mathrm{Int}_{[\gamma,\gamma']}(U')$. Then, by Proposition 3.15, Definition 3.1, Theorems 3.8 and 3.10(ii), $\mathrm{Int}_{[\gamma,\gamma']}(U)$ and $\mathrm{Int}_{[\gamma,\gamma']}(U')$ are disjoint $[\gamma,\gamma']$-open sets. This implies that (X,τ) is $[\gamma,\gamma']$-normal. □

Theorem 8.13. (i) *If $f:(X,\tau)\to (Y,\sigma)$ is a continuous $(\tau_{[\gamma,\gamma']},\sigma_{[\beta,\beta']})$-closed surjection and (X,τ) is $[\gamma,\gamma']$-normal, then (Y,σ) is $[\beta,\beta']$-normal.*

(ii) *If β and β' are open operations and if $f:(X,\tau)\to (Y,\sigma)$ is a continuous $([\gamma,\gamma'],[\beta,\beta'])$ -generalized closed surjection and (X,τ) is $[\gamma,\gamma']$-normal, then (Y,σ) is $[\beta,\beta']$-normal.* □

Proof. (i) This follows from Proposition A.5 setting $\mathcal{O}_X=\tau, \mathcal{O}_Y=\sigma$, $\mathcal{E}_X=\tau_{[\gamma,\gamma']}$ and $\mathcal{E}_Y=\sigma_{[\beta,\beta']}$.

(ii) In Proposition A.5, put $\mathcal{O}_X=\tau,\mathcal{O}_Y=\sigma$, $\mathcal{E}_X=\tau_{[\gamma,\gamma']}$ and $\mathcal{E}_Y=GO_{[\beta,\beta']}(Y,\sigma)$. By Proposition A.5 and Definition 7.14, (Y,σ) is $(\sigma, GO_{[\beta,\beta']}(Y,\sigma))$-normal. Therefore, by using Proposition 8.12(ii), (Y,σ) is $[\beta,\beta']$-normal. □

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