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ON ANALYTIC T–ALGEBRAS

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Abstract. In this paper we give an analytic method for constructing proper examples of a great variety of non-associative algebras of the BCK-type and generalizations of these. It should be noted that good examples of some of theses types have not been well-known until now.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([1, 2]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3, 4] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. The present authors ([7]) introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([5]) introduced a new notion, called an BH-algebra, which is also a generalization of $BCH/BCI/BCK$-algebras, and defined the notions of ideals and boundedness in BH-algebras, showing that there is always a maximal ideal in bounded BH-algebras. Furthermore, they constructed quotient BH-algebras via translation ideals and they obtained the fundamental theorem of homomorphisms for BH-algebras as a consequence. In this paper we give an analytic method for constructing proper examples of a great variety of non-associative algebras of the BCK-type and generalizations of these. It should be noted that good examples of some of theses types have not been well-known until now.

2. Preliminaries.

A *d-algebra* ([7]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all x, y in X.

A d-algebra $(X; *, 0)$ is a BCK-*algebra* if it satisfies the following axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

(V) $(x * (x * y)) * y = 0$

for all x, y, z in X.

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Example 2.1 ([7])**.** Let R be the set of all real numbers and define $x * y := x \cdot (x - y)$, $x, y \in R$, where "$\cdot$" and "$-$" are ordinary product and substraction of real numbers. Then $x * x = 0, 0 * x = 0, x * 0 = x^2$. If $x * y = y * x = 0$, then $x(x - y) = 0$ and $x^2 = xy$, $y(y - x) = 0$, $y^2 = xy$. Thus if $x = 0$, $y^2 = 0$, $y = 0$; if $y = 0$, $x^2 = 0$, $x = 0$ and if $xy \neq 0$, then $x = y$. Hence $(R; *, 0)$ is a d-algebra, but not BCK-algebra, since $(2 * 0) * 2 \neq 0$.

Let $(X; *, 0)$ be a BCK-algebra. Then the following hold for any x, y and z in X,
(VI) $x * 0 = x$,
(VII) $(x * y) * x = 0$,
(VIII) $(x * y) * z = (x * z) * y$.

An algebra $(X; *, 0)$ is a BCH-*algebra* if it satisfies the axioms (I), (III) and (VIII). It is well known that for any BCH-algebra X the axiom (VI) holds. An algebra $(X; *, 0)$ is a BH-*algebra* ([5]) if it satisfies the axioms (I), (III) and (VI).

Example 2.2 ([5])**.** (a) Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

It is easy to verify that $(X; *, 0)$ is a BH-algebra, but not a BCH-algebra, since $(2 * 3) * 2 = 1 \neq 2 = (2 * 2) * 3$.

(b) Let $\mathbb{R}$ be the set of all real numbers and define

$$x * y := \begin{cases} 0 & \text{if } x = 0, \\ \frac{(x-y)^2}{x} & \text{otherwise,} \end{cases}$$

for all $x, y \in \mathbb{R}$, where "$-$" is the usual subtraction of real numbers. Then it is easy to check that $(\mathbb{R}; *, 0)$ is a BH-algebra, but not a BCH-algebra.

We have seen that depending on our choice of proper axioms described above or otherwise there are a great variety of algebras which have proven interesting. In order to discuss them simultaneoulsy and via systematic manner we call such algebras T-*algebras*. Thus, for example, T means BH if we select the axioms (I), (III) and (VI). We list some other axioms which prove to be of interest as well:
(IX) $(x * y) * x = 0 * y$,
(X) $0 * (x * y) = (0 * x) * (0 * y)$.
In particular, we call a T-algebra $(X; *, 0)$ a BHN-*algebra* if it satisfies the axioms (I), (III), (V), (VI), (IX) and (X). A BHN-algebra X is called a BHK-*algebra* if it does not satisfy (VIII).

3. Analytic construction.

Let $X := [0, \infty)$ be the set of all non-negative real numbers unless otherwise specified. We define a general binary operation "$*$" on X as follows:

$$x * y = \max\{0, f(x, y)(x - y)\} = \max\{0, \lambda(x, y)x\},$$

where $f(x,y)$ and $\lambda(x,y)$ are non-negative real valued functions on $X \times X$ with

$$\lambda(0,y) = 0 \tag{a}$$

By manipulation of the functions $f(x,y)$ and $\lambda(x,y)$ we will seek to control the axiom set satisfied by the corresponding "analytic T-algebras" in order to produce interesting and unusual examples of classes which have been studied but whose membership sets have not always seen to be non-empty or if so, particularly large. Beginning with some basic information the following proposition is useful though elementary.

Proposition 3.1. *If $x, y \in X$ with $x > 0$, then*

$$x * y = 0 \Longleftrightarrow x \leq y \Longleftrightarrow \lambda(x,y) = 0.$$

Proof. If $x \leq y$ then $x - y \leq 0$ and $x * y = \max\{0, (x-y)f(x,y)\} = 0$, since $f(x,y) \geq 0$. Conversely, if $x * y = 0$, then $(x-y)f(x,y) \leq 0$ and hence $x - y \leq 0$.

If $x * y = 0$ then $\lambda(x,y)x \leq 0$ and hence $\lambda(x,y) \leq 0$, since $x > 0$. Thus $\lambda(x,y) = 0$. Conversely, if $\lambda(x,y) = 0$, then $x * y = \max\{0, \lambda(x,y)x\} = 0$. $\square$

For the case: $x = 0$, we know that $0 * y = 0, 0 \leq y$ holds always, but we have no information on the case of $\lambda(0,y)$, and so we assume that $\lambda(0,y) = 0$. Along the same lines as Proposition 3.1 we find the next proposition of use also.

Proposition 3.2. *The function $\lambda(x,y)$ can be described as follows:*

$$\lambda(x,y) := \begin{cases} 0 & \text{if } x \leq y, \\ \frac{x-y}{x} f(x,y) > 0 & \text{otherwise.} \end{cases}$$

Proof. If $x > y$, then by Proposition 3.1 $\lambda(x,y) > 0$. Since $x > 0$, we have $f(x,y)(x-y) \geq 0$ and $\lambda(x,y)x \geq 0$. Hence $x * y = f(x,y)(x-y) = \lambda(x,y)x$, and so we obtain $\lambda(x,y) = \frac{x-y}{x} f(x,y)$.

If $x \leq y$ and $x > 0$, then by Proposition 3.1 $\lambda(x,y) = 0$. If $x \leq y$ and $x = 0$, then $\lambda(x,y) = 0$ by assumption. $\square$

Similarly the following propositions are easily checked but necessary for the development of the theory.

Proposition 3.3. *If the function $\lambda(x,y)$ satisfies the conditon*

$$\lambda(x,x) = 0 \tag{b}$$

then axiom (I) holds.

Proposition 3.4. *Axiom (II) holds for any function $\lambda(x,y)$.*

Proposition 3.5. *Axiom (III) holds for any function $\lambda(x,y)$.*

Proof. Assume that there are $x, y \in X$ such that $x \neq y$ and $x * y = 0 = y * x$. We may assume that $y < x$. Then by Proposition 3.2 we have $x * y = \max\{0, \lambda(x,y)x\} = \lambda(x,y)x > 0$, a contradiction. $\square$

Proposition 3.6. *If the function $\lambda(x, y)$ satisfies the condition*

$$\text{(c)} \qquad \lambda(x, 0) = 1$$

then axiom (VI) holds.

Proof. $x * 0 = \max\{0, \lambda(x, 0)x\} = \lambda(x, 0)x = x$. □

Proposition 3.7. *If the function $\lambda(x, y)$ satisfies the condition*

$$\text{(d)} \qquad 0 < \lambda(x, y) \leq 1 \ \textit{ when } \ x > y$$

then axiom (VII) holds.

Proof. If $x \leq y$, then $x * y = \max\{0, (x - y)f(x, y)\} = 0$, and hence $(x * y) * x = 0 * x = 0$.
Let $x > y$. If $q := x * y$ then $q = \lambda(x, y)x$ and $(x * y) * x = q * x = \max\{0, \lambda(q, x)q\}$. In order to satisfy the axiom (VII), $\lambda(q, x)q$ should be zero. Hence either $q = 0$ or $\lambda(q, x) = 0$. If $q = 0$, then $\lambda(x, y) = 0$ and hence $x \leq y$, a contradiction. Thus $0 = \lambda(q, x) = \lambda(x * y, x)$ and $x * y \leq x$. This means that $\lambda(x, y)x \leq x$. Since $0 < x$, we obtain the condition $0 < \lambda(x, y) \leq 1$. □

Proposition 3.8. *If the function $\lambda(x, y)$ satisfies the conditions (a) $\sim$ (d) and*

$$\text{(e)} \qquad \lambda(x, \lambda(x, y)x) \leq \frac{y}{x} < 1 \ \textit{ when } \ x > y$$

then axiom (V) holds.

Proof. If $x \leq y$ then, by Proposition 3.1 and the conditon (c), we obtain $(x * (x * y)) * y = (x * 0) * y = x * y = 0$.
If $x > y$ then $0 < \lambda(x, y) \leq 1$ by the condition (d). If $\lambda(x, y) = 1$, then $x * y = x$ and hence $(x * (x * y)) * y = 0 * y = 0$. If $0 < \lambda(x, y) < 1$, then $x * y = \max\{0, \lambda(x, y)x\} = \lambda(x, y)x < x$, since $x > 0$. If we take $q := x * (x * y)$, then $q > 0$ by Proposition 3.1 and the axiom (III). Hence we obtain

$$\begin{aligned}(x * (x * y)) * y = 0 &\Longleftrightarrow x * (x * y) \leq y \\ &\Longleftrightarrow \lambda(x, \lambda(x, y)x) \leq y \\ &\Longleftrightarrow \lambda(x, \lambda(x, y)x) \leq \frac{y}{x} < 1\end{aligned}$$

when $y < x$. □

We summarize the results described above as follows:

Theorem 3.9. *If we define non-negative real valued functions $f(x, y)$ and $\lambda(x, y)$ satisfying the conditions (a) $\sim$ (e), then $(X; *, 0)$ is a BHN-algebra.*

Example 3.10. Define a binary operation "$*$" on X as follows:

$$x * y := \begin{cases} x & (y = 0) \\ 0 & (x \leq y) \\ x - 1 & (0 < y < x,\ y > 1,\ x - y > 1) \\ x - y & (0 < y < x,\ \ \text{not}\{y > 1,\ x - y > 1\}) \end{cases}$$

Then $(X; *, 0)$ is a BHK-algebra.

Proof. If we define a non-negative real valued function $\lambda(x, y)$ on X as follows:

$$\lambda(x, y) := \begin{cases} 1 & (y = 0) \\ 0 & (x \leq y) \\ \frac{x-1}{x} & (0 < y < x,\ y > 1,\ x - y > 1) \\ \frac{x-y}{x} & (0 < y < x,\ \text{not}\{y > 1,\ x - y > 1\}) \end{cases}$$

then the associated binary operation $*$ as defined above is precisely the operation in the example. To complete the argument we need to check the condition (e), i.e., $\lambda(x, \lambda(x, y)x) \leq \frac{y}{x} < 1$ when $0 < y < x$.

If $1 < y, 1 < x - y$, then $\lambda(x, y)x = x - 1$ and hence $\lambda(x, \lambda(x, y)x) = \lambda(x, x - 1) = \frac{x-(x-1)}{x} = \frac{1}{x} < \frac{y}{x} < 1$.

If $1 \leq y$ or $x - y \leq 1$, then $\lambda(x, y)x = \frac{x-y}{x} = x - y$. Hence $\lambda(x, \lambda(x, y)x) = \lambda(x, x - y) = \frac{x-(x-y)}{x} = \frac{y}{x} < 1$. Hence $(X; *, 0)$ is a BHN-algebra.

We can easily see that $(x * y) * z \neq (x * z) * y$. If we take $x := 2.2, y := 1.1$ and $z := 1$, then $(x * y) * z = 0.2$, while $(x * z) * y = 0.1$. Moreover, if we take $x := 9.8, y := 3.5$ and $z := 8.5$, then $(x * y) * z = 7.8$, while $(x * z) * y = 0.3$. This means that $(X; *, 0)$ is a BHK-algebra. $\square$

We know that a T-algebra can be defined according to our selection of axioms. If we select fewer axioms for constructing a T-algebra, then we give fewer restrictions on the conditions for $\lambda(x, y)$, and hence we may considerably enlarge the class of functions which satisfy the given axioms.

To illustrate, define a binary operation "$*$" on X by

$$x * y; = \max\{0, \lambda(x, y)x\}$$

where $\lambda(x, y)$ is a non-negative real valued function on X such that $\lambda(x, y) > 0$ if $x > y$ and $\lambda(x, y) = 0$ otherwise. Then it is easy to demonstrate that $(X; *, 0)$ is a d-algebra.

Using the approach developed above which led to some important examples of certain algebras, we show here how certain axioms sets are independent and how the corresponding classes are distinct in a manner which obviates the need for Cayley tables whose properties may be difficult to verify in any case.

Example 3.11. Define a binary operation "$*$" on X by

$$x * y := \max\{0, \lambda(x, y)x\}$$

where $\lambda(x, y)$ is a non-negative real valued function on X such that

$$\lambda(x, y) := \begin{cases} 0 & \text{if } \ x \leq y \\ \frac{(x-y)^2}{x} & \text{otherwise} \end{cases}$$

Then $(X; *, 0)$ is a d-algebra, but not a BH-algebra, since $2 * 0 = 4 \neq 0$. $\square$

A d-algebra $(X; *, 0)$ is called a *$d - BH$-algebra* if it satisfies axiom (VI).

Example 3.12. If we define $x * y := \max\{0, \frac{x(x-y)}{x+y}\}$ on X, then $(X; *, 0)$ is a $d-BH$-algebra.

Example 3.13. If we define $x * y := |\frac{x-y}{1+y}|$ on X, then $(X; *, 0)$ is a BH-algebra, but not a d-algebra, since $0 * 3 = \frac{3}{4} \neq 0$.

As another illustration of our technique, leading to a very large class of BCK-algebras whose special properties may themselves be quite interesting in view of the way they have been defined, consider the following. We know that a BHN-algebra will be a BCK-algebra if it satisfies the axiom (IV). Thus it becomes necessary that we investigate equivalent conditions on $\lambda(x, y)$ to the axiom (IV).

Assume $(X; *, 0)$ is a d-algebra. Consider $((x * y) * (x * z)) * (z * y)$. If $x \leq y$, then by Proposition 3.1 $x*y = 0$, and hence $((x*y)*(x*z))*(z*y) = (0*(x*z))*(z*y) = 0*(z*y) = 0$ by (II).

Let $z \leq y < x$. Then $((x * y) * (x * z)) * (z * y) = (x * y) * (x * z)$. Now

$$\begin{aligned}(x * y) * (x * z) = 0 &\Longleftrightarrow x * y \leq x * z \\ &\Longleftrightarrow \lambda(x, y) \leq \lambda(x, z)x \\ &\Longleftrightarrow \lambda(x, y) \leq \lambda(x, z)\end{aligned}$$

It follows that if we take the function $\lambda(x, u)$ as a non-increasing function for $0 \leq u \leq x$, then the axiom (IV) holds.

Let $y < z \leq x$. Since we already take the function $\lambda(x, y)$ to be a non-increasing function, $\lambda(x, z) \leq \lambda(x, y)$ and thus $x * z \leq x * y$. Hence

$$\begin{aligned}(x * y) * (x * z) &= \lambda(x * y, x * z)(x * y) \\ &= \lambda(x * y, x * z)\lambda(x, y)x,\end{aligned}$$

since $x * y = \lambda(x, y)x \geq 0$. It follows that:

$$\begin{aligned}(IV) &\Longleftrightarrow (x * y) * (x * z) \leq z * y \\ &\Longleftrightarrow \lambda(x * y, x * z)\lambda(x, y)x \leq \lambda(z, y)z \\ &\Longleftrightarrow \frac{\lambda(x * y, x * z)\lambda(x, y)}{\lambda(z, y)} \leq \frac{z}{x} \cdots (f)\end{aligned}$$

Let $y < x < z$. Then $((x * y) * (x * z)) * (z * y) = (x * y) * (z * y)$. It follows that:

$$\begin{aligned}(IV) &\Longleftrightarrow (x * y) * (z * y) = 0 \\ &\Longleftrightarrow x * y \leq z * y \\ &\Longleftrightarrow \lambda(x, y)x \leq \lambda(z, y)z \\ &\Longleftrightarrow \frac{x}{z} \leq \frac{\lambda(z, y)}{\lambda(x, y)} \cdots (g)\end{aligned}$$

We summarize:

Theorem 3.14. *If we define $x*y := \max\{0, \lambda(x, y)x\}$ on X, where $\lambda(x, y)$ is a non-negative real valued function on $X \times X$, and $\lambda(x, u)$ is a non-increasing function for $0 \leq u \leq x$ satisfying the conditions (a) $\sim$ (g), then $(X; *, 0)$ is a BCK-algebra.*

Example 3.15. If we define a binary operation "$*$" on X by $x * y := \max\{0, \lambda(x, y)x\}$, where $\lambda(x, y) = \frac{x-y}{x}$ if $x > y$, and $\lambda(x, y) = 0$ otherwise, then $(X; *, 0)$ is a BCK-algebra. Obviously this is only a simplest example where others may be constructed.

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