THE LAURENT EXTENSION OF A NOETHERIAN INTEGRAL DOMAIN

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Received February 7, 2000

Abstract. Let $R[\alpha,\ \alpha^{-1}]$ be an extension of a Noetherian integral domain R where α is an element of an algebraic field extension over the quotient field of R. In the case α is an anti-integral element over R we will give a condition for a prime ideal p of R to be $pR[\alpha,\ \alpha^{-1}] = R[\alpha,\ \alpha^{-1}]$. By making use of this we will proceed mainly with the study of flatness and faithful flatness of the extension $R[\alpha,\ \alpha^{-1}]/R$. Let $\eta_1,\ \cdots,\ \eta_d$ be the coefficients of the minimal polynomial of α over the quotient field of R. Then we will also investigate the extension $R[\eta_1,\ \cdots,\ \eta_d]/R$.

§1. Laurent extensions and ideals $J_{[\alpha],\ i}$.

Let R be a Noetherian integral domain with the quotient field K. Let α be an element which is algebraic over K and set $d = [K(\alpha) : K]$. We denote the minimal polynomial of α over K by

$$\phi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d,$$
$$\eta_1,\ \cdots,\ \eta_d \in K.$$

Set $I_{\eta_i} = R :_R \eta_i$ for $1 \le i \le d$ and $I_{[\alpha]} = \cap_{i=1}^d I_{\eta_i}$. We call $I_{[\alpha]}$ the *generalized denominator ideal* of α. Furthermore we will set

$$J_{[\alpha],\ 0} = I_{[\alpha]}(\eta_1,\ \cdots,\ \eta_d)$$

where $(\eta_1,\ \cdots,\ \eta_d)$ is a fractional ideal of R generated by the elements $\eta_1,\ \cdots,\ \eta_{d-1},\ \eta_d$ and

$$J_{[\alpha],\ i} = I_{[\alpha]}(1,\ \eta_1,\ \cdots,\ \eta_{i-1},\ \eta_{i+1},\ \cdots,\ \eta_d)$$

for $1 \le i \le d$. Sometimes we will use the notation $\widetilde{J_{[\alpha]}}$ instead of $J_{[\alpha],\ d}$. Set $J_{[\alpha]} = I_{[\alpha]} + J_{[\alpha],\ 0} = I_{[\alpha]}(1,\ \eta_1,\ \eta_2,\ \cdots,\ \eta_d)$.

We call $R[\alpha,\ \alpha^{-1}]$ the *Laurent extension* of α over R.

Let $R[X]$ be a polynomial ring over R in an indeterminate X and $\pi :\ R[X] \longrightarrow R[\alpha]$ the $R-$algebra homomorphism defined by $\pi(X) = \alpha$. We say that α is an *anti-integral element* over R of degree d if $\mathrm{Ker}(\pi) = I_{[\alpha]}\phi_\alpha(X)R[X]$. Set

$$\Gamma_{J[\alpha]} = \{\ p \in \mathrm{Spec}(R) \mid\ p + J_{[\alpha]} = R\ \}$$

and

$$\mathrm{V}(\widetilde{J_{[\alpha]}}) = \{\ p \in \mathrm{Spec}(R)\ \mid\ p \supset \widetilde{J_{[\alpha]}}\ \}.$$

1991 *Mathematics Subject Classification.* Primary 13B02, Secondary 13B22, 13C11.
Key words and phrases. denominator ideal, anti-integral element, Laurent extension.

Our notation is standard and our general reference for unexplained technical terms is H.Matsumura: [2].

We will list the following results for later use.

Lemma 1.1 (M. Kanemitsu and K. Yoshida [1, Theorem 7 (2)]). *Assume that α is an anti-integral element over R of degree d. Then*

$$\{\ p \in \mathrm{Spec}(R)\ \mid\ pR[\alpha] = R[\alpha]\ \} = \mathrm{V}(\widetilde{J_{[\alpha]}}) \cap \Gamma_{J_{[\alpha]}}.$$

An element γ in $R[\alpha]$ is said to be an *excellent element* if there exist elements $c_0,\ c_1,\ \cdots,\ c_n \in R$ such that

$$\gamma = c_0 + c_1\alpha + \cdots + c_n\alpha^n \text{ and } (c_0,\ c_1,\ \cdots,\ c_n)R = R.$$

Lemma 1.2 (J. Sato, S. Oda and K. Yoshida [5, Corollary 5]). *Assume that α is an anti-integral element over R of degree d. Then the following statements are equivalent.*

(i) $R[\alpha]/R$ *is a flat extension.*

(ii) $R[\alpha,\ \alpha^{-1}]/R$ *is a flat extension.*

(iii) $\alpha \in \mathrm{rad}(J_{[\alpha]}R[\alpha])$.

(iv) $J_{[\alpha]} = R$.

(iv) *Every excellent element belongs to* $\mathrm{rad}(J_{[\alpha]}R[\alpha])$.

Our key result is the following.

Theorem 1.3. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree $d \geq 2$. For a prime ideal p of R the following are equivalent to each other.*

(i) $pR[\alpha,\ \alpha^{-1}] = R[\alpha,\ \alpha^{-1}]$.

(ii) $p + J_{[\alpha]} = R$ *and there exists an integer i $(1 \leq i \leq d)$ such that* $p \supset J_{[\alpha],\ i}$.

Proof. Set $A = R[\alpha,\ \alpha^{-1}]$.

(i) $\Longrightarrow$ (ii). First we will prove that $p + J_{[\alpha]} = R$. The condition $pA = A$ implies that α is in $\mathrm{rad}(pR[\alpha])$. Then there exists a natural number n such that $\alpha^n = a_0 + a_1\alpha + \cdots + a_m\alpha^m$ and $a_0,\ a_1,\ \cdots,\ a_m \in p$ for some m. Let

$$f(X) = X^n - (a_0 + a_1X + \cdots + a_mX^m).$$

Then $f(X)$ is in $\mathrm{Ker}(\pi)$. This shows that there exist elements $b_0,\ \cdots,\ b_d \in J_{[\alpha]}$ and $g(X) \in R[X]$ satisfying

$$f(X) = (b_0 + b_1X + \cdots + b_dX^d)g(X).$$

Hence 1 is in $p + J_{[\alpha]}$, and so $p + J_{[\alpha]} = R$.

Secondly we will show that there exists an integer i $(1 \leq i \leq d)$ with $p \supset J_{[\alpha],\ i}$. Suppose that $p \not\supset J_{[\alpha],\ i}$ for every i with $1 \leq i \leq d$. We will prove that $pR[\alpha] \neq R[\alpha]$. If $pR[\alpha] = R[\alpha]$, then by Lemma 1.1, we get $p \supset \widetilde{J_{[\alpha]}} = J_{[\alpha],\ d}$. This is a contradiction.

We know that

$$\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0.$$

We will prove that there exists an element c of $I_{[\alpha]}$ such that $c\eta_i$ and $c\eta_j$ are not in p for some $i \neq j$ $(1 \leq i,\ j \leq d)$. Note that $J_{[\alpha],\,1} = I_{[\alpha]}(1,\ \eta_2,\ \cdots,\ \eta_d)$. Since $J_{[\alpha],\,i} \not\subset p$, there exists an integer i $(2 \leq i \leq d)$ such that $a \in I_{[\alpha]}$ and $a\eta_i \notin p$. If we can take an integer j $(j \neq i$ and $1 \leq j \leq d)$ satisfying $a\eta_j \notin p$, then the assertion is proved. So we may assume that

$$a(\eta_1,\ \cdots,\ \eta_{i-1},\ \eta_{i+1},\ \cdots,\ \eta_d) \subset p.$$

Furthermore, there exsits an integer j $(j \neq i$ and $1 \leq j \leq d)$ such that $b \in I_{[\alpha]}$ and $b\eta_j \notin p$. Similarly as above we may assume that

$$b(\eta_1,\ \cdots,\ \eta_{i-1},\ \eta_{i+1},\ \cdots,\ \eta_d) \subset p.$$

Set $c = a + b$. Then $c\eta_i \notin p$ and $c\eta_j \notin p$.

From the argument above we see that there are at least two non-zero terms in $c\phi_\alpha(\alpha)$ modulo $pR_p[\alpha]$. Since $J_{[\alpha]}R_p = R_p$, the ideal $I_{[\alpha]}R_p$ is invertible, so $I_{[\alpha]}R_p$ is principal. Hence we may assume that c is a generator of $I_{[\alpha]}$. Let π' be the $R-$algebra homomorphosm of $R_p[X]$ into $R_p[\alpha]$ defined by $\pi'(X) = \alpha$. Then $\mathrm{Ker}(\pi') = I_{[\alpha]}\phi_\alpha(X)R_p[X] = (c\phi_\alpha(X))$. Hence $R_p[\alpha] \cong R_p[X]/(c\phi_\alpha(X))$. On the other hand

$$R_p[\alpha]/pR_p[\alpha] \cong k(p)[\overline{\alpha}] \cong k(p)[X]/(\overline{c\phi_{\alpha(X)}})$$

where $k(p)$ is the residue field of p. Let Q be the prime ideal of $R_p[\alpha]$ which corresponds to the irreducible factor of $\overline{c\phi_\alpha(X)}$ different from $\overline{X}$. Then Q does not contain α and $Q \supset pR_p[\alpha]$. Set $P = Q \cap R[\alpha]$. Then $P \supset pR[\alpha]$ and $P \not\ni \alpha$. This is absurd from the fact $P \supset \mathrm{rad}(pR[\alpha]) \ni \alpha$.

(ii) $\Longrightarrow$ (i). Assume that $pA \neq A$. Then there exists a prime ideal P of A such that $P \supset pA$. By the condition $p + J_{[\alpha]} = R$, there exists elements b of p and c of $J_{[\alpha]}$ such that $b + c = 1$. Since c is in $J_{[\alpha]} = J_{[\alpha]}(1,\ \eta_1,\ \cdots,\ \eta_d)$, we can write

$$c = c_0 + c_1\eta_1 + \cdots + c_d\eta_d \text{ and } c_0,\ c_1,\ \cdots,\ c_d \in I_{[\alpha]}.$$

By the condition (ii), there exists an integer i $(1 \leq i \leq d)$ such that $p \supset J_{[\alpha],\,i}$. Multiplying the equality

$$\alpha^d + \eta_1\alpha^{d-1} + \cdots + \eta_d = 0.$$

by c_i, we have

$$c_i\alpha^d + c_i\eta_1\alpha^{d-1} + \cdots + c_i\eta_d = 0.$$

We know that $c_i,\ c_i\eta_1,\ \cdots,\ c_i\eta_d$ other than $c_i\eta_i$ are in $J_{[\alpha],\,i}$, and so in P. Hence $c_i\eta_i\alpha^{d-i}$ is in P. Then in the equation

$$c\alpha^{d-i} = c_0\alpha^{d-i} + c_1\eta_1\alpha^{d-i} + \cdots + c_d\eta_d\alpha^{d-i},$$

$c_0,\ c_1\eta_1,\ \cdots,\ c_d\eta_d$ other than $c_i\eta_i$ are in $J_{[\alpha],\,i}$, and so in P. Therefore $c\alpha^{d-i}$ is in P. Using $c = 1 - b$, we get $\alpha^{d-i} \in P$. If $p \supset J_{[\alpha],\,d} = \widetilde{J}_{[\alpha]}$, then by Lemma 1.1, we know $pR[\alpha] = R[\alpha]$. This claims that $pA = A$. This is absurd. Hence $i \neq d$. This shows that α is in P. This contardicts to the fact α is a unit of A. $\square$

Remark 1.4. (1) By Theorem 1.3, we obtain:

$$\{ \, p \in \mathrm{Spec}(R) \ | \ pR[\alpha, \ \alpha^{-1}] = R[\alpha, \ \alpha^{-1}] \, \} = (\cup_{i=1}^{d} \mathrm{V}(J_{[\alpha], \ i})) \cap \Gamma_{J_{[\alpha]}}.$$

If $J_{[\alpha]} = R$, then we have

$$\{ \, p \in \mathrm{Spec}(R) \ | \ pR[\alpha, \ \alpha^{-1}] = R[\alpha, \ \alpha^{-1}] \, \} = \cup_{i=1}^{d} \mathrm{V}(J_{[\alpha], \ i}).$$

It is a closed set.

(2) In the case $d = 1$, we have the following Theorem $1.3'$.

Theorem $1.3'$. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree 1. For a prime ideal p of R, the following are equivalent to each other:*

(i) $pR[\alpha, \ \alpha^{-1}] = R[\alpha, \ \alpha^{-1}]$,

(ii) $p + J_{[\alpha]} = R$ *and there exists an integer* i $(i = 0, \ 1)$ *such that* $p \supset J_{[\alpha], \ i}$.

Proof. We will prove that $p \supset J_{[\alpha], \ i}$ for some i $(i = 0, \ 1)$ in the proof (i) $\Longrightarrow$ (ii) because the rest of the proof is the same argument as in Theorem 1.3. Assume that $p \not\supset I_{[\alpha]}\eta_1 \cap I_{[\alpha]}$. Then there exists an element c of $I_{[\alpha]}$ such that $c\eta_1 \notin p$ and $c\eta_1 \in I_{[\alpha]}$. The fact $c\eta_1 \in I_{[\alpha]}$ implies $c\eta_1^2 \in R$. Since $c \cdot c\eta_1^2 = (c\eta_1)^2$ is not in p, we see that c is not in p. Hence $\alpha = -c\eta_1/c$ is in R_p. Furthermore, $\alpha^{-1} = -c/c\eta_1$ is in R_p because $c\eta_1 \notin p$. Therefore $R_p \supset R[\alpha, \ \alpha^{-1}] = A$. Thus $pR_p \supset pA = A$. This is a contradiction. □

(3) For another characterization of $pR[\alpha, \ \alpha^{-1}] = R[\alpha, \ \alpha^{-1}]$ in the case $d = 1$, see [1, p. 55, Remark].

(4) From now on we will assume $d \geq 2$.

Proposition 1.5. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree d. Let $A = R[\alpha, \ \alpha^{-1}]$ and ϕ the contraction mapping of* $\mathrm{Spec}(A)$ *into* $\mathrm{Spec}(R)$. *Then the following are equivalent.*

(i) *The contraction mapping ϕ is surjective.*

(ii) *For every i with $1 \leq i \leq d$, the equality* $\mathrm{rad}(J_{[\alpha]}) = \mathrm{rad}(J_{[\alpha], \ i})$ *holds.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that there exists an integer i $(1 \leq i \leq d)$ such that $\mathrm{rad}(J_{[\alpha]}) \neq \mathrm{rad}(J_{[\alpha], \ i})$. By the definitions of $J_{[\alpha]}$ and $J_{[\alpha], \ i}$, we have $J_{[\alpha], \ i} \subset J_{[\alpha]}$. Hence $\mathrm{rad}(J_{[\alpha], \ i}) \underset{\neq}{\subset} \mathrm{rad}(J_{[\alpha]})$. Then there exists a prime ideal p of R such that $J_{[\alpha], \ i} \subset p$ and $J_{[\alpha]} \not\subset p$. This implies that $pR_p \supset J_{[\alpha], \ i}R_p$ and $pR_p + J_{[\alpha]}R_p = R_p$. Applying Theorem 1.3 to $A_p = R_p[\alpha, \ \alpha^{-1}]$, we obtain $pA_p = A_p$. This shows that $p \notin \mathrm{Im}(\phi)$. This is a contradiction.

(ii) $\Longrightarrow$ (i). We have only to prove that $pA_p \neq A_p$ for arbitrary prime ideal p of R. Assume that $pA_p = A_p$. Then Theorem 1.3 asserts that $pR_p + J_{[\alpha]}R_p = R_p$. Hence $J_{[\alpha]}R_p = R_p$. Besides, there exists an integer i $(1 \leq i \leq d)$ such that $pR_p \supset J_{[\alpha], \ i}R_p$ by Theorem 1.3. Hence we see that $\mathrm{rad}(J_{[\alpha]}R_p) \underset{\neq}{\supset} \mathrm{rad}(J_{[\alpha], \ i}R_p)$, hence $\mathrm{rad}(J_{[\alpha]}) \underset{\neq}{\supset} \mathrm{rad}(J_{[\alpha], \ i})$. This is absurd. □

Corollary 1.6. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree d. Set $A = R[\alpha, \ \alpha^{-1}]$. Then A/R is a faithfully flat extension if and only of $J_{[\alpha], \ i} = R$ for every i $(1 \leq i \leq d)$.*

Proof. Let ϕ the contraction mapping of Spec(A) into Spec(R).

($\Longrightarrow$). Since A/R is faithfully flat, the homomorphism ϕ is surjective by H. Matsumura [2, (4D) Theorem 3]. Then Proposition 1.5 implies that $\mathrm{rad}(J_{[\alpha]}) = \mathrm{rad}(J_{[\alpha],\ i})$ for every integer i $(1 \leq i \leq d)$. Furthermore, by Lemma 1.2, $J_{[\alpha]} = R$ because A/R is a flat extension. Hence $J_{[\alpha],\ i} = R$ for every i $(1 \leq i \leq d)$.

($\Longleftarrow$). It is easily verified that $J_{[\alpha]} = R$ because $J_{[\alpha]} \supset J_{[\alpha],\ i}$. Then A/R is a flat extension by Lemma 1.2. Moreover, we know that $\mathrm{rad}(J_{[\alpha]}) = R = \mathrm{rad}(J_{[\alpha],\ i})$. By Proposition 1.5, the contraction mapping ϕ is surjective. Hence A/R is a faithfully flat extension by H. Matsumura [2, (4D) Theorem 3]. $\square$

The following holds about $R[\alpha]$.

Theorem 1.7. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree d. Set $B = R[\alpha]$. Then the following are equivalent.*

(i) $J_{[\alpha]} = R$.

(ii) $J_{[\alpha]}B = B$.

(iii) $\widetilde{J_{[\alpha]}}B = B$.

Proof. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i). Since $J_{[\alpha]}B = B$, we know that α is in $\mathrm{rad}(J_{[\alpha]}B)$ by Lemma 1.2, we have $J_{[\alpha]} = R$.

(iii) $\Longrightarrow$ (ii) is clear from the fact $J_{[\alpha]} \supset \widetilde{J_{[\alpha]}}$.

(ii) $\Longrightarrow$ (iii). Let p be a prime divisor of $\widetilde{J_{[\alpha]}}$. By (ii) $\Longrightarrow$ (i), we get $J_{[\alpha]} = R$. Therefore Lemma 1.1 implies that $pB = B$. Hence $\mathrm{rad}(\widetilde{J_{[\alpha]}}B) = B$, and so $\widetilde{J_{[\alpha]}}B = B$. $\square$

An analogous result to Theorem 1.7 holds in the case $R[\alpha,\ \alpha^{-1}]$.

Theorem 1.8. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree d. Set $A = R[\alpha,\ \alpha^{-1}]$. Then the following are equivalent.*

(i) *$J_{[\alpha]} = R$, i.e., the extension A/R is a flat extension.*

(ii) $J_{[\alpha]}A = A$.

(iii) $\widetilde{J_{[\alpha]}}A = A$.

Proof. (i) $\Longrightarrow$ (ii) is immediate.

(ii) $\Longrightarrow$ (i). We will prove that A/R is a flat extension. Let P be a prime ideal of A and set $p = P \cap R$. Then $p \not\supset J_{[\alpha]}$ because $J_{[\alpha]}A = A$. Hence $J_{[\alpha]}R_p = R_p$. By Lemma 1.2 we see that $A_p = R_p[\alpha, \alpha^{-1}]/R_p$ is a flat extension. Moreover, A_P/A_p is also a flat extension. Therefore A_P/R_p is a flat extension. So is A/R.

(iii) $\Longrightarrow$ (ii) is clear from $\widetilde{J_{[\alpha]}} \subset J_{[\alpha]}$.

(ii) $\Longrightarrow$ (iii). Let p be a prime divisor of $\widetilde{J_{[\alpha]}}$. Then $J_{[\alpha]} = R$ by (ii) $\Longrightarrow$ (i). Hence Lemma 1.1 shows that $\alpha R[\alpha] = R[\alpha]$. This means that $pA = A$. Therefore $\mathrm{rad}(\widetilde{J_{[\alpha]}})A = A$, and so $\widetilde{J_{[\alpha]}}A = A$. $\square$

Remark 1.9. $\widetilde{J_{[\alpha]}} = R$ does not hold necessarily even if $J_{[\alpha]} = R$.

Corollary 1.10. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree d. Set $A = R[\alpha, \alpha^{-1}]$. Then the following are equivalent to the conditions in Theorem 1.8.*

(iv) *There exists an integer i $(1 \leq i \leq d)$ such that $J_{[\alpha],\ i}A = A$.*

(v) $J_{[\alpha],\ i}A = A$ *for every* i $(1 \leq i \leq d)$.

Proof. (iv) $\Longrightarrow$ (ii) is clear from $J_{[\alpha],\ i} \subset J_{[\alpha]}$.
(iii) $\Longrightarrow$ (iv) is immediate from $\widetilde{J_{[\alpha]}} = J_{[\alpha],\ d}$.
(v) $\Longrightarrow$ (iv) is obvious.
(i) $\Longrightarrow$ (v). Let p be a prime divisor of $J_{[\alpha],\ i}$. By the condition (i) we know that $J_{[\alpha]} = R$. Hence by Lemma 1.1, we get $pR[\alpha] = R[\alpha]$. Hence $pA = A$. This implies that $J_{[\alpha],\ i}A = A$. □

§ 2. Shifting a generator by an element of A.

We denote by U(A) the unit group of A. We will find a condition for A to coincide with $R[a\alpha, (a\alpha)^{-1}]$.

Lemma 2.1. *Let R be a Noetherian integral domain with the quotient field K. Let α be an element of an algebraic field extension over K and set $A = R[\alpha,\ \alpha^{-1}]$. If a is an element of A and $A = R[a\alpha,\ a\alpha^{-1}]$, then a is in* U(A).

Proof. Since $a^{-1} = \alpha(a\alpha)^{-1}$ is in A, we know that a is in U(A). □

Proposition 2.2. *Let R be a Noetherian domain and α an anti-integral element over R of degree d. Set $A = R[\alpha,\ \alpha^{-1}]$. If* grade$(J_{[\alpha],\ i}) > 1$ *for every i $(1 \leq i \leq d)$, then* U$(A) \cap R =$ U(R).

Proof. It is clear that U$(A) \cap R \supset$ U(R). Assume that

$$\mathrm{U}(A) \cap R \underset{\neq}{\supset} \mathrm{U}(R).$$

Then there exists an element a of U$(A) \cap R$ such that $a \notin$ U(R). Since a is in U(A), we have $aA = A$. Hence there exists a prime divisor p of rad(aR) such that $pA = A$. Then by Theorem 1.3, we see that $p \supset J_{[\alpha],\ i}$ for some i with $1 \leq i \leq d$. By K.Yoshida [6, Proposition 1.10], we obtain depth$(R_p) = 1$ because p is a prime divisor of rad(aR). On the other hand grade$(J_{[\alpha],\ i}) > 1$ and $p \supset J_{[\alpha],\ i}$. This shows that depth$(R_p) > 1$, and we reach a contradiction. □

Theorem 2.3. *Let R be a Noetherian domain and α an anti-integral element over R of degree d. Let a be an element of R. Set $A = R[\alpha,\ \alpha^{-1}]$ and assume that* grade $(J_{[\alpha],\ i}) > 1$ *for every i with $1 \leq i \leq d$. Then $A = R[a\alpha, (a\alpha^{-1})]$ if and only if a is in* U(R).

Proof. ($\Longrightarrow$) Lemma 2.1 implies that a is in U(A). Hence a is in U$(A) \cap R$. By Proposition 2.2, we know that U$(A) \cap R =$ U(R). Therefore a is in U(R).
($\Longleftarrow$) Since a is in U(R), we see that $(a\alpha)^{-1}$ is in A. Hence

$$R[a\alpha, (a\alpha^{-1})] \subset A.$$

Note that $\alpha = (a\alpha)a^{-1}$ and $\alpha^{-1} = (a\alpha)^{-1}a$. Then we get $A \subset R[a\alpha, (a\alpha^{-1})]$. Therefore $A = R[a\alpha, (a\alpha^{-1})]$. □

§3. Generalized denominator ideals.

We will consider the ring $R[\eta_1, \cdots, \eta_d]$. We can refer to S.Oda and K.Yoshida [3, Corollary 15.2] and S.Oda and K.Yoshida [4, Corollary 1.2] for the condition $I_{[\alpha]}R[\alpha] = R[\alpha]$ and the ring $R[\eta_1, \cdots, \eta_d]$. In this section we will study the ring $R[\eta_1, \cdots, \eta_d]$ in the case the condition $I_{[\alpha]}R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$ holds.

Lemma 3.1. *Set $C = R[\eta_1, \cdots, \eta_d]$. If $I_{[\alpha]}R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$, then $C \subset R[\alpha, \alpha^{-1}]$.*

Proof. By definition of $I_{[\alpha]}$, it is easily seen that $\eta_1, \cdots, \eta_d$ are in $I_{[\alpha]}^{-1}$. We know that

$$I_{[\alpha]}^{-1} \subset I_{[\alpha]}^{-1}R[\alpha, \alpha^{-1}] = I_{[\alpha]}^{-1}I_{[\alpha]}R[\alpha, \alpha^{-1}] \subset R[\alpha, \alpha^{-1}].$$

Hence $C \subset R[\alpha, \alpha^{-1}]$. □

Proposition 3.2. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree d. If $I_{[\alpha]}R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$, then $J_{[\alpha]} = R$ and $R[\alpha, \alpha^{-1}]/R$ is a flat extension.*

Proof. We will show that $R[\alpha, \alpha^{-1}]/R$ is a flat extension. Let P be a prime ideal of $R[\alpha, \alpha^{-1}]$ and set $p = P \cap R$. By the condition $I_{[\alpha]}R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$, we know that $I_{[\alpha]} \not\subset p$. So $J_{[\alpha]} \not\subset p$ because $I_{[\alpha]} \subset J_{[\alpha]}$. Hence $J_{[\alpha]}R_p = R_p$. Then Lemma 1.2 shows that $R_p[\alpha, \alpha^{-1}]/R_p$ is a flat extension. Hence $R[\alpha, \alpha^{-1}]/R$ is also a flat extension. By Lemma 1.2, we get $J_{[\alpha]} = R$. □

The following is an analogous result to S. Oda and K. Yoshida [3, Theorem 11].

Theorem 3.3. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree d. Set $C = R[\eta_1, \cdots, \eta_d]$. If $I_{[\alpha]}R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$, then the following conditions hold.*
(1) $C \subset R[\alpha, \alpha^{-1}]$.
(2) $I_{[\alpha]}C = C$.
(3) C/R is a birational and flat extension.

Proof. We have already proved (1) in Lemma 3.1.
(2) Proposition 3.2 says that $J_{[\alpha]} = R$.
Moreover, $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \cdots, \eta_d)$. Then we get $I_{[\alpha]}C = C$.
(3) It is clear that C/R is a birational extension. Let p be a prime ideal p of R. Then we will show that $pC = C$ or $C \subset R_p$. From this fact it is easily seen that C/R is a flat extension. If $p \supset I_{[\alpha]}$, then $I_{[\alpha]}C = C$ means that $pC = C$. If $p \not\supset I_{[\alpha]}$, then $\eta_1, \cdots, \eta_d$ are in R_p because $I_{[\alpha]} = \cap_{i=1}^d I_{\eta_i}$. Hence $C \subset R_p$. □

Theorem 3.4. *Let R be a Noetherian integral domain and α an anti-integral element over R of degree d. Assume that $J_{[\alpha]} = R$. Then the following two statements hold.*
(1) $I_{[\alpha]}R[\alpha] = R[\alpha]$ if and only if $\mathrm{rad}(I_{[\alpha]}) = \mathrm{rad}(J_{[\alpha],\ d})$.
(2) $I_{[\alpha]}R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$ if and only if $\mathrm{rad}(I_{[\alpha]}) = \mathrm{rad}(\cap_{i=1}^d J_{[\alpha],\ i})$.

Proof. (1) ($\Longrightarrow$) It is immediate from $\mathrm{rad}(I_{[\alpha]}) \subset \mathrm{rad}(J_{[\alpha],\ d})$ that $I_{[\alpha]} \subset J_{[\alpha],d}$. Let p be a prime divisor of $I_{[\alpha]}$. Then we have $pR[\alpha] = R[\alpha]$ because $I_{[\alpha]}R[\alpha] = R[\alpha]$. By Lemma 1.1, we get $p \supset J_{[\alpha],\ d}$, hence $p \supset \mathrm{rad}(J_{[\alpha],\ d})$. Therefore $\mathrm{rad}(J_{[\alpha],\ d}) \subset \mathrm{rad}(I_{[\alpha]})$, and so $\mathrm{rad}(I_{[\alpha]}) = (J_{[\alpha],\ d})$.
($\Longleftarrow$) Let p be a prime divisor of $I_{[\alpha]}$. Then

$$p \supset \mathrm{rad}(I_{[\alpha]}) = \mathrm{rad}(J_{[\alpha],\ d}) \supset J_{[\alpha],\ d}.$$

Hence by Lemma 1.1, we obtain $pR[\alpha] = R[\alpha]$. Therefore $\mathrm{rad}(I_{[\alpha]})R[\alpha] = R[\alpha]$. This means that $I_{[\alpha]}R[\alpha] = R[\alpha]$.

(2) Since $J_{[\alpha]} = R$, by Lemma 1.3, the following holds.

$pR[\alpha,\ \alpha^{-1}] = R[\alpha,\ \alpha^{-1}]$ if and only if there exists an integer i $(1 \leq i \leq d)$ such that $p \supset J_{[\alpha],\ i}$.

Note that $p \supset \cap_{i=1}^{d} J_{[\alpha],\ i}$ if and only if there exists an integer i $(1 \leq i \leq d)$ satisfying $p \supset J_{[\alpha],\ i}$. By making use of these facts we can prove the assertion (2) in a similar way to the proof of (1). □

Remark 3.5. In the case $d = 1$, Proposition 1.5, Corollary 1.6, 1.10, Proposition 2.2 and Theorem 2.3 hold by rewriting $i = 0, 1$ instead of $i = 1, \cdots, d$. Theorem 3.4 (2) does not hold even if we rewrite $i = 0, 1$ because $I_{[\alpha]} \not\subset I_{[\alpha]}\eta_1$.

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