A NECESSARY AND SUFFICIENT CONDITION FOR A COMPLETE SPREAD

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Abstract. In this paper, we give a necessary and sufficient condition for a spread to be complete, that is, an External Characterization Theorem for a complete spread: a spread $f : X \longrightarrow Z$ is complete if and only if, whenever $j : X \longrightarrow Y$ is a dense embedding with $j(X)$ locally connected in Y and $g : Y \longrightarrow Z$ is a spread such that $f = g \circ j$, then j is a homeomorphism.

1. Introduction and preliminaries

Spreads (the precise definition of which is given below) were introduced into topological spaces by Fox [1] as an encompassing notion for branched and unfolded coverings. The purpose of this note is to give an external characterization of Fox's complete spread, namely, that a spread $f : X \longrightarrow Z$ is complete if and only if, whenever $j : X \longrightarrow Y$ is a dense embedding with $j(X)$ locally connected in Y and $g : Y \longrightarrow Z$ is a spread such that $f = g \circ j$, then j is a homeomorphism.

Definition 1.1 (Fox [1]). *A continuous function $f : X \longrightarrow Y$ between locally connected T_1-spaces is called a spread if the components V_W of inverse images $f^{-1}(W)$, where W is open in Y, form a basis of X.*

Let $f : X \longrightarrow Z$ be a spread. Given a point $z \in Z$, we denote by η_z the collection

$$\eta_z = \{V \subseteq Z \mid V \text{ is a neighbourhood of } z\}$$

and, for each $V \in \eta_z$,

$$\mathcal{C}_z^V = \{K \subseteq X \mid K \text{ is a component of } f^{-1}(V)\}.$$

Definition 1.2 (Hunt [2]). *A function $\chi : \eta_z \longrightarrow \mathcal{C}_z^V$ is called a spread point of $z \in Z$ if it satisfies the following condition:*

$$\text{for any } V_1,\ V_2 \in \eta_z, \text{ if } V_1 \subseteq V_2 \text{ then } \chi(V_1) \subseteq \chi(V_2).$$

Remark 1.3. Given $z \in Z$, suppose $f^{-1}(z) \neq \emptyset$. Pick $x \in X$ such that $f(x) = z$. Now, let $\chi(V)$ be that component of $f^{-1}(V)$ which contains x. Given any neighbourhoods W_1 and W_2 of z, suppose $W_1 \subseteq W_2$. Since $f^{-1}(W_1) \subseteq f^{-1}(W_2)$, each of $x \in \chi(W_1) \cap \chi(W_2)$ and $\chi(W_1) \subseteq f^{-1}(W_1) \subseteq f^{-1}(W_2)$. Since $\chi(W_2)$ is that component of $f^{-1}(W_2)$ which contains x, we must have $\chi(W_1) \subseteq \chi(W_2)$. So, χ is a spread point of z.

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Definition 1.4 (Fox [1]).
(i) *A topological space* X *is said to be locally connected in another topological space* Y *if there is a basis* $\mathfrak{B}$ *of* Y *such that* $X \cap B$ *is connected for each* $B \in \mathfrak{B}$.
(ii) *A spread* $g : Y \longrightarrow Z$ *is called an extension of a spread* $f : X \longrightarrow Z$ *if there is an embedding* $i : X \longrightarrow Y$ *such that* $g \circ i = f$.
(iii) *A spread* $f : X \longrightarrow Z$ *is said to be complete if every* $z \in Z$*satisfies the following property: if for every open set* W *containing* z *there is associated a component* V_W *of* $f^{-1}(W)$ *such that* $W_1 \subseteq W_2$ *in* Z *implies* $V_{W_1} \subseteq V_{W_2}$ *in* X*, then*

$$\bigcap_W V_W \neq \emptyset.$$

(iv) *A completion of a spread* $f : X \longrightarrow Z$ *is an extension* $g : Y \longrightarrow Z$ *which is complete such that* X *is dense and locally connected in* Y.

It is clear that for a complete spread $f : X \longrightarrow Y$, the intersection $\bigcap_W V_W$ is a singleton.

Theorem 1.5 (Fox's Theorem). *Every spread between locally connected* T_1*-spaces has a completion; unique up to equivalence.*

Whereas Fox constructed a unique spread completion between locally connected T_1-spaces, Michael [4] has shown that such a completion is achievable without local connectedness. (Michael [5] refers to Fox's spread completion as *Fox's Canonical Completion.*) In his paper, Fox gave a difficult but elegant construction. And in an effort to extend Fox's spreads to uniform spaces Hunt [2] introduced the concept of *a spread point* which not only simplified Fox's original construction but also helped in the construction of the uniform spread completion via a collection of spread points. See also [3]. Of significance in Hunt's construction is the proof of the equivalence of $Hunt'$s spread points to minimal Cauchy filters.

Indeed, all other characteristics of Fox's spreads (including those by Michael [5] and Hunt [2]) deal with internal properties of spreads. In this paper, we seek to give an external character of complete spreads.

2. A necessary and sufficient condition for a complete spread

In this section we prove the main result of this paper:

Theorem 2.1. *A spread* $f : X \longrightarrow Z$ *is complete if and only if, whenever* $j : X \longrightarrow Y$ *is a dense embedding,* $j(X)$ *is locally connected in* Y *and* $g : Y \longrightarrow Z$ *is a spread such that* $f = g \circ j$ *then* j *is a homeomorphism.*

Proof.
($\Longleftarrow$): Suppose the condition holds. Given a spread $f : X \longrightarrow Z$, consider its Fox's completion $g : Y \longrightarrow Z$. Then X is dense and locally connected in Y; furthermore, X is embedded in Y, say by $j : X \longrightarrow Y$. That f is complete is obvious, since j is a homeomorphism and g is complete.

($\Longrightarrow$): Now suppose that X is densely embedded in Y, $j(X)$ is locally connected in Y and $g : Y \longrightarrow Z$ is a spread such that $f = g \circ j$. Since j is an embedding, it needs only be shown that j is also onto. This is done in Lemma 2.4 whose proof is dependent on the following propositions. □

Proposition 2.2. Let $y \in Y$. *For each neighbourhood N of $g(y) \in Z$, there is a component V_N of $f^{-1}(N)$ and a basic open set B_N in Y such that $j^{-1}(B_N) \subseteq V_N \subseteq f^{-1}(N)$.*

Proof. Pick a basis $\mathfrak{B}$ of Y such that $j(X) \cap B$ is connected, for every $B \in \mathfrak{B}$. Given an open neighbourhood N of $g(y)$, find a component C_N of $g^{-1}(N)$ such that $y \in C_N \subseteq g^{-1}(N)$. Again there exists $B_N \in \mathfrak{B}$ such that $y \in B_N \subseteq C_N \subseteq g^{-1}(N)$.

Since $j(X)$ is homeomorphic to a subset of Y and $j(X) \cap B_N$ is connected, the set $j^{-1}[B_N \cap j(X)]$ is connected. But $j^{-1}[B_N \cap j(X)] = j^{-1}(B_N)$, so $j^{-1}(B_N)$ is connected. We also have

$$j^{-1}(B_N) \subseteq j^{-1}(C_N) \subseteq f^{-1}(N).$$

Now choose V_N to be the (connected!) component of $f^{-1}(N)$ for which it holds that

$$j^{-1}(B_N) \subseteq V_N \subseteq f^{-1}(N). \qquad \square$$

Proposition 2.3. *Under the hypothesis of the previous proposition, for any neighbourhoods N_1 and N_2 of $g(y) \in Z$, if $N_1 \subseteq N_2$ then $V_{N_1} \subseteq V_{N_2}$, where V_{N_1} and V_{N_2} satisfy the conclusion of Proposition 2.2.*

Proof. Find basic open sets $B_{N_1},\ B_{N_2} \in \mathfrak{B}$ such that

$$y \in B_{N_1} \subseteq g^{-1}(N_1) \text{ and } y \in B_{N_2} \subseteq g^{-1}(N_2),$$

as well as the respective components V_{N_1} and V_{N_2} of $f^{-1}(N_1)$ and $f^{-1}(N_2)$ such that

$$j^{-1}(B_{N_1}) \subseteq V_{N_1} \subseteq f^{-1}(N_1) \text{ and } j^{-1}(B_{N_2}) \subseteq V_{N_2} \subseteq f^{-1}(N_2).$$

Since $j(X)$ is dense, it follows that $B_{N_1} \cap B_{N_2} \cap j(X) \neq \emptyset$ which implies $j^{-1}[B_{N_1} \cap B_{N_2} \cap j(X)] \neq \emptyset$. Thus $j^{-1}(B_{N_1}) \cap j^{-1}(B_{N_2}) \neq \emptyset$. Since $j^{-1}(B_{N_1}) \subseteq V_{N_1} \subseteq f^{-1}(N_1) \subseteq f^{-1}(N_2)$, and $j^{-1}(B_{N_2}) \subseteq f^{-1}(N_2)$, we have $j^{-1}(B_{N_1}) \cap j^{-1}(B_{N_2}) \subseteq f^{-1}(N_2)$. Now $j^{-1}(B_{N_1}) \cup j^{-1}(B_{N_2})$ is connected, so that

$$j^{-1}(B_{N_1}) \cup j^{-1}(B_{N_2}) \subseteq V_{N_1} \subseteq f^{-1}(N_1) \subseteq f^{-1}(N_2).$$

But by choice, V_{N_2} is the component of $f^{-1}(N_2)$ which contains $j^{-1}(B_{N_2})$, therefore $V_{N_1} \subseteq V_{N_2}$. □

Let us return to the proof of the surjectiveness of j. Let $y \in Y$. Then by Propositions 2.2, 2.3 and that f is complete, we may suppose without loss of generality that $\bigcap_N V_N = \{x\}$, where V_N is a component in Proposition 2.2 for each neighbourhood N of $g(y)$.

Lemma 2.4. *Claim that $f(x) = g(y)$ and $j(x) = y$.*

Proof. Suppose, for a contradiction, that $f(x) \neq g(y)$. Then $g(y) \in Z \backslash \{f(x)\}$, so $Z \backslash \{f(x)\}$ is an open neighbourhood of $g(y)$. Associated with $Z \backslash \{f(x)\}$ is a component $V_{Z \backslash \{f(x)\}}$ of $f^{-1}(Z \backslash \{f(x)\})$ which contains x. Then $f(x) \in Z \backslash \{f(x)\}$, a contradiction; thus $f(x) = g(y)$.

On the other hand, suppose that $j(x) \neq y$. Then $Y - \{j(x)\}$ is an open set containing y. But $g : Y \longrightarrow Z$ is a spread, so for some open set W in Z, there is a component C of $g^{-1}(W)$ such that $y \in C \subseteq g^{-1}(W)$ and $C \subseteq Y - \{j(x)\}$. Since $j(X)$ is locally connected in Y, there is a basis $\mathfrak{B}$ of Y such that $j(X) \cap B$ is connected, for each $B \in \mathfrak{B}$. Choose a basic open set $B_W \in \mathfrak{B}$ such that $y \in B_W \subseteq C \subseteq g^{-1}(W)$, where $\mathfrak{B}$ is some basis of Y each of whose basic elements has a connected intersection with $j(X)$. Therefore

$$j^{-1}(B_W) = j^{-1}[j(X) \cap B_W] \subseteq f^{-1}(W)$$

and $j^{-1}(B_W)$ is connected. We select the component V_W of $f^{-1}(W)$ for which $j^{-1}(B_W) \subseteq V_W \subseteq f^{-1}(W)$ from which we have $j(V_W) \subseteq g^{-1}(W)$. Also $B_W \subseteq C$ implies $j^{-1}(B_W) \subseteq j^{-1}(C)$. Thus $j^{-1}(C) \cap V_W \neq \emptyset$, i.e. $C \cap j(V_W) \neq \emptyset$. Therefore $j(V_W) \subseteq C \subseteq Y - \{j(x)\}$. But then $j(x) \in j(V_W)$ ensures that $j(x) \in Y - \{j(x)\}$, which is a contradiction. Hence $j(x) = y$. □

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