ON THE g-ORTHOGONAL PROJECTION AND THE BEST APPROXIMATION OF A VECTOR IN A QUASI-INNER PRODUCT SPACES

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Abstract. Let X be a quasi-inner product space ([7]), and $x, y \in X\backslash\{0\}$. We resolve the problem of the relations between the three vectors: so-called g-orthogonal projection of the vector y on the subspace $[x]$ $(-a(x,y)x, \text{Lemma } 2)$, the best approximation of the vector y with vector from $[x]$ $(-b(x,y)x, \text{Lemma } 1)$ and the vector $-\dfrac{g(x,y)}{\|x\|^2}x$. The equality

$$a(x,y) = b(x,y) = -\frac{g(x,y)}{\|x\|^2}\ ,$$

is valid if and only if X is an inner-product space (i.p. space)[1].

Let X be a real normed space and

$$g(x,y) := \frac{\|x\|}{2}\left(\lim_{t\to -0}\frac{\|x+ty\|-\|x\|}{t} + \lim_{t\to +0}\frac{\|x+ty\|-\|x\|}{t}\right)\ (x,y\in X)^2\ .$$

We are called that X is a quasi-inner product space (q.i.p. space) if the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8\ \left[\|x\|^2 g(x,y) + \|y\|^2 g(y,x)\right]\ (x,y\in X)^3\ , \tag{1}$$

holds ([7]).

The equality (1) holds in the space l^4, but doesn't hold in the space l^1 ([7]). On the properties of the functional g and the q.i.p. spaces see the recent papers: [3], [4], [5], [6] and [7]. We notice that a q.i.p space is uniformly smooth and uniformly convex. From this in a q.i.p. space X we have

$$g(x,y) = \|x\|\ \lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}\ ,$$

where the functional g is linear in the second argument.

In what follows we assume that X is a complete q.i.p. space. (The space l^4 is complete q.i.p. space).

For fixed $x, y \in X\backslash\{0\}$, x and y linearly independent, we consider the real functions

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[1] If X is a i.p. space then the vector $-\frac{g(x,y)}{\|x^2}x$ is the ortogonal projection of the vector y to the vector x.

$$f(t) := \|y + tx\|, \quad \varphi(t) := \|y + tx\|^2 \, g(y + tx, x) \quad (t \in \mathbb{R})\,. \tag{2}$$

Since X is smooth, the function f is differntiable and we have

$$f^{'}(t) = \frac{g(y + tx, x)}{\|y + tx\|} \quad (t \in \mathbb{R})\,. \tag{3}$$

Using (1) we get

$$\|y + (t+1)x\|^4 - \|y + (t-1)x\|^4 = 8\left[\|y + tx\|^2 \, g(y + tx, x) + \|x\|^2 \, g(x, y + tx)\right] \,. \tag{4}$$

Besides this we have

$$\varphi(t) = \frac{1}{8}\left[\|y + (t+1)x\|^4 - \|y + (t-1)x\|^4 - 8t\|x\|^4 - 8\|x\|^2 \, g(x, y)\right] \quad (t \in \mathbb{R})\,. \tag{5}$$

Lemma 1. Let X is a complete q.i.p space and $x, y \in X \backslash \{0\}$, x and y linearly independent. Then there exists a unique $b \in \mathbb{R}$ $\;(b = b(x, y))$ such that: a) $g(y + bx, x) = 0$ and $\min f(t) = f(b)$;
b) $b = 0$ and $\varphi(t) < 0 \Leftrightarrow t < 0$.

Proof. Since X is uniformly convex and complete the statement a) follows from Lemma 4 [2] and (3). The function f is convex. Hence a) implies that b) is true.

Corollary 1. Under conditions of Lemma 1, the following statements are valid: a) $b < 0 \Leftrightarrow g(y, x) > 0$ and $b = 0 \Leftrightarrow g(y, x) = 0$; b) The vector $-bx$ is the best approximation of vector y with vectors from $[x]$.

In [5] the orthogonality $\perp^g$ is defined as

$$x \perp^g y \iff \|x\|^2 \, g(x, y) + \|y\|^2 \, g(y, x) = 0\,.$$

In an i.p. space $(X, (.,.))$ we have $x \perp^g y \Leftrightarrow (x, y) = 0$.

If X is a q.i.p. space then the orthogonality $\perp^g$ is equivalent with James isocceles orthogonality i.e.

$$x \perp^g y \iff \|x + y\| = \|x - y\|\,.$$

Lemma 2 (Theorem 3.4, [5]). Let X be a q.i.p. space and $x, y \in X$, $x \neq 0$. Then there exists a unique $a \in \mathbb{R}$ $(a = a(x, y))$ such that $x \perp^g y + ax$, i.e.

$$\|y + ax\|^2 \, g(y + ax, x) + \|x\|^2 \, g(x, y + ax) = 0\,. \tag{6}$$

The vector $-ax$ we are called g-orthogonal projection of the vector y on the subspace $[x]$.

Theorem 1. Let X be a complete q.i.p. space, $x \neq 0$, x and y linearly independent. Then

$$b - 1 < a < b + 1 \quad .$$

Proof. From (4) we get $f(a+1) - f(a-1) = 0$. By Roll theorem we obtain $0 = 2f^{'}(c)$ where $c \in (a-1, a+1)$. By (3) we have $g(y + cx, x) = 0 \ \ (c \in (a-1, a+1))$. Since X is uniformly convex and $g(y + bx, x) = 0$ we find $c = b$. So, $b \in (a-1, a+1)$.

Now we resolve the problem of the relations between the three numbers: $a(x,y)$, $b(x,y)$ and the number $-\dfrac{g(x,y)}{\|x\|^2}$.

Theorem 2. Let X be a complete q.i.p. space and $x \neq 0$, x and y linearly independent. Then either the number a is between number b and $-\dfrac{g(x,y)}{\|x\|^2}$ or

$$a = b = -\frac{g(x,y)}{\|x\|^2} \ . \tag{7}$$

Proof. Using (4) and (5) we obtain

$$\varphi(a) = -\|x\|^4 \, a - \|x\|^2 \, g(x,y) \ . \tag{8}$$

In according to the property of the function φ (Lemma 1 and (6)) we have the following three possibilities

1. $a < b$. Then $\varphi(a) < 0$, i.e. $a > -\dfrac{g(x,y)}{\|x\|^2}$. So,

$$-\frac{g(x,y)}{\|x\|^2} < a < b \ .$$

2. $a > b$. Then $\varphi(a) > 0$, i.e. $a < -\dfrac{g(x,y)}{\|x\|^2}$. Hence we have

$$b < a < -\frac{g(x,y)}{\|x\|^2} \ .$$

3. $a = b$. Then $a = -\dfrac{g(x,y)}{\|x\|^2}$, i.e.

$$a = b = -\frac{g(x,y)}{\|x\|^2} \ .$$

Corollary 2. Under conditions of Theorem 2 we have

$$\|y + bx\| \leq \|y + ax\| \leq \left\| y - \frac{g(x,y)}{\|x\|^2} \right\|.$$

If x is orthogonal to y in the sens Birkhoff we denot $x \perp_B y$.

Lemma 3 (4.4, [1]). A normed space X is an i.p. space if and only if the implication

$$x \perp_B y \Longrightarrow \|x + y\| = \|x - y\| \,, \tag{9}$$

holds.

Lemma 4 (Theorem 2, [4]). A normed space X is a smooth if and only if the equivalence

$$g(x,y) = 0 \Longleftrightarrow x \perp_B y \,,$$

holds.

Finally, we give the answer of the question when (7) is valid.

Theorem 3. Let X be a complete q.i.p. space. X is a i.p. space if and only if the equality (7) valid.

Proof. If X is an i.p. space with $(.,.)$ then $g(x,y) = (x,y)$ and $\left(y - \frac{(x,y)}{\|x\|^2} x, x \right) = 0$. So, $a = -\frac{(x,y)}{\|x\|^2}$, i.e. (7) is valid.

Assum that (7) is valid. Let $g(y,x) = 0$. Then by Corollary 1 we have $b = 0$. In this case, from (7) we obtain $g(x,y) = 0$. Then by (1) we get $\|x + y\| = \|x - y\|$. Since X is smooth, by Lemma 4, we have $y \perp_B x$. Hence the implication (9) is valid.

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