ON THE BRANCH OF BH–ALGEBRAS

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ABSTRACT. In this paper, we give a normal BH-algebra, and we concider the branch in BH-algebra and investigate some related properties.

1. INTRODUCTION

Y. Imai and K. Iséki ([4]) and K. Iséki ([5]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In ([3]), Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Y. B. Jun, E. H. Roh and H. S. Kim ([6]) discussed the BH-algebras, which is a generalization of BCH-algebras. Moreover, they introduced the notions of ideal, maximal ideal and translation ideal, and investigated some properties.

In this paper, we give a normal BH-algebra, and we concider the branch in BH-algebra and investigate some related properties. This paper is the some generalization of Chaudhry's results([1]).

2. PRELIMINARIES

A *BH-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$"satisfying the following axioms:

(1) $x * x = 0$,
(2) $x * 0 = x$,
(3) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all x, y in X.

Example 2.1. (a) Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X; *, 0)$ is a BH-algebra, but not a BCH-algebra, since $(2*3)*2 = 1 \neq 2 = (2*2)*3$.

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(b) Let $\mathbb{R}$ be the set of all real numbers and define

$$x * y := \begin{cases} 0 & \text{if } x = 0, \\ \frac{(x-y)^2}{x} & \text{otherwise,} \end{cases}$$

for all $x, y \in \mathbb{R}$, where "$-$" is the usual substraction of real numbers. Then $(\mathbb{R}; *, 0)$ is a BH-algebra, but not a BCH-algebra.

The relations between BH-algebras and BCH-algebras (also, BCK/BCI- algebras) are as follows:

Theorem 2.2 ([6]). *Every BCH-algebra is a BH-algebra. Every BH-algebra satisfying the condition $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$, is a BCH-algebra.*

Theorem 2.3 ([6]). *Every BH-algebra satisfying the condition*

(c1) $((x * y) * (x * z)) * (z * y) = 0, \quad \forall x, y, z \in X,$

is a BCI-algebra.

Theorem 2.4 ([6]). *Every BH-algebra satisfying the conditions* (c1) *and*

(c2) $(x * y) * x = 0, \quad \forall x, y \in X,$

is a BCK-algebra.

A nonempty subset S of a BH-algebra X is called a *subalgebra* if $x, y \in S$ implies $x*y \in S$. A nonempty subset A of a BH-algebra X is called an *ideal* if $0 \in A$ and if $x * y, y \in A$ imply that $x \in A$.

3. Main Results

Now, we see the following examples.

Example 3.1. Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:

$*$	0	1	2
0	0	2	0
1	1	0	2
2	2	2	0

Then $(X; *, 0)$ is a BH-algebra, but X is not satisfied the identity $0 * (x * y) = (0 * x) * (0 * y)$ since $0 * (1 * 2) = 0 \neq 2 = (0 * 1) * (0 * 2)$.

Example 3.2. Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:

$*$	0	1	2
0	0	0	0
1	1	0	2
2	2	2	0

Then $(X; *, 0)$ is a BH-algebra and X satisfies the identity $0 * (x * y) = (0 * x) * (0 * y)$.

By Examples 3.1 and 3.2, we will define the following definition.

Definition 3.1. A BH-algebra X is called a BH_1-*algebra* if it satisfying the following conditions:

(4) $0 * (x * y) = (0 * x) * (0 * y)$.

Definition 3.2. Let X be a BH-algebra. Then the set

$$M(X) = \{x \in X | 0 * (0 * x) = x\}$$

is called a *medial part* of X and an element of $M(X)$ is called a *medial element* of X.

Obviously $0 \in M(X)$ and so $M(X)$ is nonempty. In general, $M(X)$ is not a subalgebra of a BH-algebra. But we have the following Theorem.

Theorem 3.1. *If X is a BH_1-algebra, then $M(X)$ is a subalgebra of X.*

Proof. Clearly $0 \in M(X)$. Let $x, y \in M(X)$. Then we have $0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = x * y$. Thus $x * y \in M(X)$ and so $M(X)$ is a subalgebra of X. □

Theorem 3.2. *Let X be a BH_1-algebra and let*

$$A = \{x \in X | 0 * x = 0\}.$$

Then A is an ideal and subalgebra of X.

Proof. Clearly $0 \in A$. Let $x, y \in X$ be such that $x * y \in A$ and $y \in A$. Then $0 * (x * y) = 0$ and $0 * y = 0$. Thus we have $0 * x = (0 * x) * (0 * y)) = 0 * (x * y) = 0$, and hence $x \in A$. Therefore A is an ideal of X. Obviously, A is a subalgebra of X. □

Next, we see the following examples.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	3	0	2
3	3	3	0	0

Then $(X; *, 0)$ is a BH-algebra in which satisfies the identity $(x * y) * x = 0 * y$, but not satisfied the identity $(x * (x * y)) * y = 0$ because $(2 * (2 * 1)) * 1 = 3 \neq 0$.

Example 3.4. Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:

*	0	1	2
0	0	0	1
1	1	0	1
2	2	2	0

Then $(X; *, 0)$ is a BH-algebra in which satisfies the identity $(x * (x * y)) * y = 0$, but not satisfied the identity $(x * y) * x = 0 * y$ because $(1 * 2) * 1 \neq 0 * 2$.

Example 3.5. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	2	0
2	2	0	0	0
3	3	3	2	0

Then $(X; *, 0)$ is a BH-algebra in which satisfies the identities $(x * y) * x = 0 * y$ and $(x * (x * y)) * y = 0$.

By Examples 3.3, 3.4 and 3.5, next conditions (5) and (6) are independent. We give the following definition.

Definition 3.3. A BH-algebra X is said to be *normal* if it satisfying the following condition: (4) and

(5) $(x * y) * x = 0 * y$,

(6) $(x * (x * y)) * y = 0$.

Example 3.6. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	1	0	0
1	1	0	1	1
2	2	1	0	0
3	3	1	3	0

Then $(X; *, 0)$ is a BH-algebra in which satisfies the identities (4), (5) and (6).

Theorem 3.3. *Let X be a normal BH-algebra. Then for each $x \in X$, there is a unique $x_m \in M(X)$ such that $x_m * x = 0$.*

Proof. Let $x \in X$, then $(0*(0*x))*x = 0$ by (6). We take $x_m = 0*(0*x)$, then $x_m*x = 0$. To prove that x_m is in $M(X)$. By (5), we have $0*(0*(0*x)) = ((0*(0*x))*x)*(0*(0*x)) = 0*x$. Thus $0 * (0 * x_m) = 0 * (0 * (0 * (0 * x))) = 0 * (0 * x) = x_m$, and so $x_m \in M(X)$. To prove uniqueness we assume that $y_m \in M(X)$ be such that $y_m * x = 0$. Then by (5), we get $0 * y_m = (y_m * x) * y_m = 0 * x$. Thus $0 * (0 * x) = 0 * (0 * y_m) = y_m$, and hence $x_m = y_m$. □

Corollary 3.4. *Let X be a normal BH-algebra and let $x, y \in X$ be such that $x * y = 0$. Then $x_m = y_m$ where $x_m, y_m \in M(X)$.*

Remark. Let X be a normal BH-algebra. If $x_m \in M(X)$ and $y * x_m = 0$, then $y = x_m$. Thus each medial point of a normal BH-algebra is also minimal point.

Theorem 3.5. *Let X be a normal BH-algebra. Then for any $x, y \in X$, we have*

$$(x * y)_m = x_m * y_m.$$

where $(x * y)_m, x_m, y_m \in M(X)$.

Proof. By Theorem 3.1, $M(X)$ is a subalgebra of X, we get $x_m * y_m \in M(X)$. Then by (4) and (6) we have $(x_m*y_m)*(x*y) = ((0*(0*x))*(0*(0*y)))*(x*y) = (0*(0*(x*y)))*(x*y) = 0$. By Theorem 3.3, we know that $(x * y)_m = x_m * y_m$. □

Definition 3.4. Let X be a normal BH-algebra and let $x_m \in M(X)$. The set

$$\{x \in X | x_m * x = 0\}$$

is called the *branch* of X determined by x_m and is denoted by $V(x_m)$.

Theorem 3.6. *Let X be a normal BH-algebra. Then*

(i) $X = \bigcup_{x_m \in M(X)} V(x_m)$

(ii) $V(x_m) \cap V(y_m) = \emptyset$ *if* $x_m \neq y_m$ and $x_m, y_m \in M(X)$.

Proof. (i). Clearly $V(x_m) \subseteq X$ for all $x_m \in M(X)$. Thus $\bigcup_{x_m \in M(X)} V(x_m) \subseteq X$. Let $y \in X$, then there is $y_m \in M(X)$ such that $y_m * y = 0$. Thus $y \in V(y_m) \subseteq \bigcup_{x_m \in M(X)} V(x_m)$. Hence $X \subseteq \bigcup_{x_m \in M(X)} V(x_m)$. Therefore $X = \bigcup_{x_m \in M(X)} V(x_m)$.

(ii). Let $z \in V(x_m) \cap V(y_m)$ where $x_m \neq y_m$ in M(X). Then $x_m * z = 0$ and $y_m * z = 0$. Thus z has two medial points, a contradiction to Theorem 3.3. Hence $V(x_m) \cap V(y_m) = \emptyset$ if $x_m \neq y_m$. □

Theorem 3.7. *Let X be a normal BH-algebra. Then*

(i) *If $x*y \in A$ and $y*x \in A$, then $x, y \in V(x_m)$ for some $x_m \in M(X)$,*

(ii) *If $x \in V(x_m)$, $y \in V(y_m)$ and $x_m \neq y_m$, then $x*y, y*x \in X - A$.*

Proof. (i). Let $x*y \in A$ and $y*x \in A$. If $x \in V(x_m)$ and $y \in V(y_m)$. Then by Theorem 3.5 gives $(x*y)_m = x_m * y_m$ and $(y*x)_m = y_m * x_m$. Since $x*y, y*x \in A = V(0)$, we have $(x*y)_m = 0 = (y*x)_m$. Now uniqueness of medial point gives $x_m * y_m = 0 = y_m * x_m$. Thus $x_m = y_m$. Hence $x, y \in V(x_m)$ for some $x_m \in M(X)$.

(ii). Let $x \in V(x_m)$, $y \in V(y_m)$ and $x_m \neq y_m$. If $x*y \in A = V(0)$, then by Theorem 3.5, we get $(x*y)_m = x_m * y_m$. Thus $x*y \in V(x_m * y_m)$. Hence $x_m * y_m = 0$. Thus $(x_m * y_m) * x_m = 0 * x_m$, which gives $0 * y_m = 0 * x_m$ and hence $0*(0*y_m) = 0*(0*x_m)$. Thus $x_m = y_m$, a contradiction. Hence $x*y \in X - A$. Similarly we can be shown that $y*x \in X - A$. □

Remark. We know that every BCH-algebra satisfies conditions (1) - (6). Thus this note is the generalization of Chaudhry's results.

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References

[1] M. A. Chaudhry, *On BCH-algebras*, Math. Japonica **36, No. 4** (1991), 665-676.

[2] W. A. Dudek and J. Thomys, *On decomposition of BCH-algebras*, Math. Japonica **35, No. 6** (1990), 1131-1138.

[3] Q. P. Hu and X. Li, *On BCH-algebras*, Mathematics Seminar Notes **11** (1983), 313-320.

[4] Y. Imai and K. Iséki, *On axiom systems of propositional calculi XIV*, Proc. Japan Academy **42** (1966), 19-22.

[5] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Academy **42** (1966), 26-29.

[6] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Scientiae Mathematicae **1, No. 3** (1998), 347-354.

[7] Y. B. Jun, E. H. Roh and Q. Zhang, *On fuzzy translation BH-ideals in BH-algebras*, J. Fuzzy Math. **8(2)** (2000), 361-370.

[8] E. H. Roh, *On (fuzzy) ideals in BCK/BCI-algebras*, Ph. D. Thesis, Gyeongsang National University, Korea, 1996.

[9] E. H. Roh and S. Y. Kim, *On BH^*-subalgebras of transtive BH^*-algebras*, Far East J. Math. Sci. **1(2)** (1999), 255-263.

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