ON TOPOLOGICAL BCI–ALGEBRAS II

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Abstract. As a continuation of [7], we introduce the notions of topological subalgebras, topological ideals and topological homomorphisms in topological BCI-algebras and study some related properties. In the section 3, we investigate the compactness in a TBCI-algebra X and quotient TBCI-algebra X/I where I is a topological ideal of X. In the section 4, we introduce the notion of topological homomorphisms, study some properties for this notion and show that an open topological homomorphism f from TBCI-algebra X to TBCI-algebra Y gives rise to a one-to-one correspondence between the closed c-ideals of Y and the closed c-ideals of X which contains $ker f$.

1. Introduction

The notion of BCK-algebras was proposed by Y. Iami and K. Iséki in 1966. In the same year, K. Iséki [4] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universial structures including lattices and Boolean algebras. R. A. Alo and E. Y. Deeba [1] attempted to study the topological aspects of the BCK-structures, and they initiated the study of various topologies on BCK-algebras analogous to that which has already been studied on lattices, but no attempts have been made to study the topological structures making the BCK-operation continuous. In [7], Y. B. Jun et al. initiated the study of topological BCI-algebras (briefly, TBCI-algebras) and some properties of this structure, and gave a characterization of a TBCI-algebra in terms of neighborhoods, and showed that a TBCI-algebra X is Hausdorff if and only if $\{0\}$ is closed in X. They also gave a filter base Ω generating a BCI-topology, and maked a BCI-algebra X into a TBCI-algebra for which Ω is a fundamental system of neighborhoods of 0. As a continuation of [7], we introduce the notions of topological subalgebras, topological ideals and topological homomorphisms in topological BCI-algebras and study some related properties. In the section 3, we get the following results: (i) If I is a topological ideal of a TBCI-algebra X, then the natural projection Φ_I from X to X/I is open and continuous. (ii) Every topological subalgebra of a compact (locally compact) TBCI-algebra is compact (locally compact). (iii) Let I be a topological ideal of a compact TBCI-algebra X, then I and X/I are compact. (iv) Let I be a topological c-ideal of a locally compact TBCI-algebra X, then X/I is locally compact. (v) Let I be a compact topological ideal of a transitive open TBCI-algebra X, $\beta = \{I_x \mid x \in X\}$ be a base for topological $\mathcal{T}$ and I_x compact for each $x \in X$. Then we have that if $Q \subseteq X/I$ is compact, so is $P = \Phi_I(Q)$; and if X/I is locally compact, then X is locally compact. In

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the section 4, we introduce the notion of topological homomorphisms, study some properties for this notion and show that an open topological homomorphism f from TBCI-algebra X to TBCI-algebra Y gives rise to a one-to-one correspondence between the closed c-ideals of Y and those closed c-ideals of X which contains $ker f$.

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

Recall that a *BCI-algebra* is an algebra $(X, *, 0)$ of type (2, 0) satisfying the following axioms for every $x, y, z \in X$,

(I) $((x*y)*(x*z))*(z*y) = 0$,

(II) $(x*(x*y))*y = 0$,

(III) $x*x = 0$,

(IV) $x*y = 0$ and $y*x = 0$ imply $x = y$.

A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x*y = 0$.

In a BCI-algebra X, the following hold:

(1) $x*0 = x$.

(2) $(x*y)*z = (x*z)*y$.

(3) $0*(x*y) = (0*x)*(0*y)$.

(4) $x*y = 0$ implies $(x*z)*(y*z) = 0$ and $(z*y)*(z*x) = 0$.

(5) $x*0 = 0$ implies $x = 0$.

A nonempty subset S of a BCI-algebra X is called a *BCI-subalgebra* of X if $x*y \in S$ whenever $x, y \in S$.

A non-empty subset I of a BCI-algebra X is called a *BCI-ideal* of X if

(i) $0 \in I$,

(ii) $x*y \in I$ and $y \in I$ imply $x \in I$.

A BCI-ideal I of a BCI-algebra X is said to be *closed* if $0*x \in I$ whenever $x \in I$. Here we call this a *c-ideal* of X.

3. Compactness in topological BCI-algebras

In this paper we shorten the statement "U is an open set containing x" to the phrase "U is a neighborhood of x".

Definition 3.1. (Jun et al. [7]) A topology $\mathcal{T}$ on a BCI-algebra X is said to be a *BCI-topology*, and X, furnished with $\mathcal{T}$, is called a *topological BCI-algebra* (briefly, *TBCI-algebra*) if $(x, y) \mapsto x*y$ is continuous from $X \times X$, furnished with the cartesian product topology defined by $\mathcal{T}$, to X. Moreover if X is a p-semisimple BCI-algebra, we say that X, furnished with $\mathcal{T}$, is a *p-semisimple TBCI-algebra.*

Example 3.2. (Jun et al. [7]) (1) A BCI-algebra with discrete or indiscrete topology is a TBCI-algebra.

(2) Consider a BCI-algebra $X = \{0, a, b, c, d\}$ with Cayley table (Table 1) and Hasse diagram (Figure 1):

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	b	d
c	c	c	c	0	d
d	d	d	d	d	0

Table 1

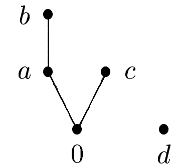

Figure 1

Note that $\mathcal{T} := \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{0,a\}, \{b,c\}, \{b,d\}, \{c,d\}, \{0,a,b\}, \{0,a,c\}, \{0,a,d\}, \{b,c,d\}, \{0,a,b,c\}, \{0,a,b,d\}, \{0,a,c,d\}\}$ is a BCI-topology. By routine calculations we know that X furnished with $\mathcal{T}$ is a TBCI-algebra.

We first give a characterization of a TBCI-algebra in terms of neighborhoods.

Lemma 3.3. (Jun et al. [7]) *Let $\mathcal{T}$ be a BCI-topology for a BCI-algebra X. Then X, furnished with $\mathcal{T}$, is a TBCI-algebra if and only if for each x and y in X and each neighborhood W of $x * y$ there are neighborhoods U of x and V of y such that $U * V \subseteq W$.*

Lemma 3.4. (Jun et al. [7]) *Let X be a TBCI-algebra. If A is both an open subset in topological space X and an ideal of BCI-algebra X, then it is a closed subset in topological space X.*

Lemma 3.5. (Jun et al. [7]) *Let X be a TBCI-algebra. Then $\{0\}$ is closed subset if and only if X is Hausdorff in topological space X.*

Definition 3.6. A nonempty subset S of a TBCI-algebra X is called a *topological subalgebra* of X if

(i) S is a BCI-subalgebra of X considered as an algebraic BCI-algebra,
(ii) S is a closed set in the topological space X.

Definition 3.7. A nonempty subset I of a TBCI-algebra X is called a *topological ideal* of X if

(i) I is a BCI-ideal of X considered as an algebraic BCI-algebra,
(ii) I is open in the topological space X.

Definition 3.8. A nonempty subset I of a TBCI-algebra X is called a *topological c-ideal* of X if

(i) I is a c-ideal of X considered as an algebraic BCI-algebra,
(ii) I is open in the topological space X.

Example 3.9. Let $X = \{0,a,b,c,d\}$ be a TBCI-algebra as in Example 3.2(2). By routine calculations we know that $A = \{0,c\}$ is a topological subalgebra which is not a topological ideal, and $B = \{0,a\}$ is a topological ideal.

Proposition 3.10. *Every topological ideal of a TBCI-algebra X is a closed subset of X considered as a topological space.*

Proof. Follows from Lemma 3.4. $\square$

Proposition 3.11. *Let X be a TBCI-algebra and I a BCI-ideal of X considered as an algebraic BCI-algebra. Define a map $\Phi_I : X \to X/I$ by $\Phi_I(x) = I_x$ for each $x \in X$ where I_x is the equivalence class containing x. Then the set*

$$\mathcal{T}_{X/I} := \{O \subseteq X/I \mid \Phi_I^{-1}(O) \text{ is open in } X\}$$

is a topology on X/I, which is called the quotient topology on X/I.

Proof. Straightforward. □

Proposition 3.12. *If I is a topological ideal of a TBCI-algebra X, then the mapping Φ_I as in Proposition 3.11 is open and continuous.*

Proof. Clearly Φ_I is continuous. In order to prove Φ_I is open, let W be an open set in X and $W' = \Phi_I(W)$. We shall show that W' is open in X/I or equivalently $\Phi_I^{-1}(W')$ is open in X. Let $y \in \Phi_I^{-1}(W')$. Then $\Phi_I(y) \in W' = \Phi_I(W)$; whence $\Phi_I(y) = \Phi_I(w)$ for some $w \in W$. It follows that $y * w \in I$ and $w * y \in I$. Note that $f : X \times X \to X$, $(x, y) \mapsto x * y$, is continuous. Then $B := \{(a, b) \in X \times X | a * b \in I\} = \{(a, b) \in X \times X | f(a, b) \in I\} = f^{-1}(I)$ is open in $X \times X$ since I is open. Hence $B = \bigcup_{i\in\Lambda} (U_i \times V_i)$ for some open sets U_i and V_i in X where Λ is an index set and $i \in \Lambda$. It follows from $y * w \in I$ that there exists $i_0 \in \Lambda$ such that $y \in U_{i_0}$, $w \in V_{i_0}$ and $U_{i_0} * V_{i_0} \subseteq I$. Similarly $w * y \in I$ implies that there exists $j_0 \in \Lambda$ such that $w \in U_{j_0}$, $y \in V_{j_0}$ and $U_{j_0} * V_{j_0} \subseteq I$. Thus $y \in U_{i_0} \cap V_{j_0}$ and $w \in U_{j_0} \cap V_{i_0}$. Note that $(U_{i_0} \cap V_{j_0}) * (U_{j_0} \cap V_{i_0}) \subseteq U_{i_0} * V_{i_0} \subseteq I$ and $(U_{j_0} \cap V_{i_0}) * (U_{i_0} \cap V_{j_0}) \subseteq U_{j_0} * V_{j_0} \subseteq I$. Let $x \in G$ where $G = U_{i_0} \cap V_{j_0}$. It follows that $x * w \in I$ and $w * x \in I$. Hence $I_x = I_w = \Phi_I(w) \in \Phi_I(W) = W'$, and so $G \subseteq \Phi_I^{-1}(W')$. Note that $y \in G \subseteq \Phi_I^{-1}(W')$ we have $\Phi_I^{-1}(W')$ is open in X. □

Theorem 3.13. *If I is a topological ideal of a TBCI-algebra X with topology $\mathcal{T}_X$, then X/I is a TBCI-algebra with topology $\mathcal{T}_{X/I}$.*

Proof. It is sufficient to prove that $(I_x, I_y) \mapsto I_x * I_y$ is continuous. Let $x, y \in X$ and let W be a neighborhood of $I_x * I_y$. Then $\Phi_I^{-1}(W)$ is open in X and $x * y \in \Phi_I^{-1}(W)$. Since X is a TBCI-algebra, there exists neighborhoods U' of x and V' of y such that $U' * V' \subseteq \Phi_I^{-1}(W)$ by Lemma 3.3. Put $U := \Phi_I(U')$ and $V := \Phi_I(V')$. Then U and V are open in X/I since Φ_I is open. Note that $I_x \in U$, $I_y \in V$ and

$$\begin{aligned} U * V &= \{\Phi_I(u) * \Phi_I(v) | u \in U' \text{ and } v \in V'\} \\ &= \{\Phi_I(u * v) | u \in U' \text{ and } v \in V'\} \\ &= \Phi_I(U' * V') \subseteq W \end{aligned}$$

so from Lemma 3.3 that $(I_x, I_y) \mapsto I_x * I_y$ is continuous. □

Theorem 3.14. *Let X be a TBCI-algebra. Then*

(i) *If X is compact* (resp. *locally compact*)*, then every topological subalgebra S of X is compact* (resp. *locally compact*)*.*

(ii) *If X is compact and I is a topological ideal, then I and X/I are compact.*

(iii) *If X is locally compact, then every topological ideal I is locally compact.*

(iv) *If X is locally compact and I is topological c-ideal, then X/I is locally compact.*

Proof. (i) Note that S is closed so that S is compact if X is compact. If X is locally compact, then $\forall x \in S$ $\exists$ a neighborhood G of x whose closure is compact in X. Then $U := G \cap S$ is a neighborhood of x in S and $\overline{U} \subseteq \overline{G} \cap \overline{S} = \overline{G} \cap S$. Since $\overline{G} \cap S$ is compact in S, it follows that $\overline{U}$ is compact in S. Hence S is locally compact.

(ii) Let X be compact and I a topological ideal. Then I is closed by Proposition 3.10. Hence I is compact. Note that Φ_I is a continuous mapping from X onto X/I. Therefore X/I is compact.

(iii) It is similar to the proof of (i).

(iv) Assume that X is locally compact and I is a topological c-ideal. Let $a \in X$, $A = \Phi_I(a)$ and let U be a neighborhood of a in X such that $\overline{U}$ is compact. Since I is a c-ideal of X, we have $\Phi_I(I) = \{I_0\}$ which is an open set in X/I because I is open in X and Φ_I is open. Note that $\{I_0\}$ is a BCI-ideal of X/I so that $\{I_0\}$ is a closed set in X/I by Lemma 3.4. It follows from Lemma 3.5 that X/I is Hausdorff. Since $\overline{U}$ is compact, the continuous image $\Phi_I(\overline{U})$ is compact, and hence $\Phi_I(\overline{U})$ is closed in X/I. From $\Phi_I(U) \subseteq \Phi_I(\overline{U})$ it follows that $\overline{\Phi_I(U)} \subseteq \Phi_I(\overline{U})$ so that $\overline{\Phi_I(U)}$ is compact. On the other hand, put $U^* = \{I_x \mid I_x \cap U \neq \emptyset\}$. Since $a \in I_a \cap U = A \cap U$, we get $A \in U^*$. Now we claim that $U^* = \Phi_I(U)$ and then U^* is a neighborhood of A. Let $I_x \in U^*$. Then $I_x \cap U \neq \emptyset$ and so $\exists y \in I_x \cap U$. Hence $I_x = I_y \in \Phi_I(U)$ and therefore $U^* \subseteq \Phi_I(U)$. Conversely let $I_x = \Phi_I(x) \in \Phi_I(U)$. Then $x \in U$ and thus $x \in I_x \cap U$, i.e., $I_x \cap U \neq \emptyset$ which means that $I_x \in U^*$. Consequently $U^* = \Phi_I(U)$ and $\overline{U^*} = \overline{\Phi_I(U)}$ is a compact set in X/I. Therefore X/I is locally compact. This completes the proof. □

For a fixed element a of a TBCI-algebra X, define a self-map $f_a : X \to X$ by $f_a(x) = a*x$ for all $x \in X$.

Definition 3.15. A TBCI-algebra X is said to be *transitive open* if for each $a \in X$ the self-map f_a is open and continuous.

Lemma 3.16. *Let U be an open set in a transitive open TBCI-algebra X and let $a \in X$. Then*

(i) *$a * U$ is open.*
(ii) *$f_a^{-1}(U) = \{x \in X \mid a * x \in U\}$ is open.*
(iii) *$A * U$ is open for every subset A of X.*

Proof. Note that $f_a(U) = a * U$, $f_a^{-1}(U) = V$ and $A * U = \cup_{a \in A}(a * U)$. Hence we get the desired results. □

Theorem 3.17. *Every p-semisimple TBCI-algebra X is transitive open.*

Proof. Let $a \in X$. We shall prove that f_a is open and continuous. Let $x \in X$ and let W be a neighborhood of $f_a(x) = a * x$. Then there exist neighborhoods U and V of a and x, respectively, such that $U*V \subseteq W$ by Lemma 3.3. It follows that $f_a(V) = a*V \subseteq U*V \subseteq W$, which means that f_a is continuous. Now let G be an open set in X. We claim that $f_a(G)$ is open. Let $x \in f_a(G)$. Then $x = f_a(u) = a * u$ for some $u \in G$, and so $u = a * (a * u) = a * x$ since X is p-semisimple. It follows that $a * x \in G$ so that there exist neighborhoods P and Q of a and x, respectively, such that $P * Q \subseteq G$, which implies $f_a(Q) = a * Q \subseteq P * Q \subseteq G$. Thus $f_a(q) \in G$ for all $q \in Q$, and so $q = a * (a * q) = a * f_a(q) \in f_a(G)$, i.e., $Q \subseteq f_a(G)$. Therefore f_a ia an open map, ending the proof. □

Proposition 3.18. *Let $\triangle_0$ be a collection of closed subsets of a TBCI-algebra X having the finite intersection property. Then there is a maximal collection $\triangle'$ of closed subsets of X having the finite intersection property such that $\triangle_0 \subseteq \triangle'$.*

Proof. Let $\mathcal{F} = \{\triangle \mid \triangle$ is a collection of closed subsets of X having the finite intersection property and $\triangle_0 \subseteq \triangle\}$. We shall prove that $\mathcal{F}$ has a maximal element with respect to the partial order "$\subseteq$ " on $\mathcal{F}$. Let $\triangle_1 \subseteq \triangle_2 \subseteq \ldots \subseteq \triangle_n \subseteq \ldots$ be a chain in $\mathcal{F}$ and let $\triangle = \cup_{i=1}^{\infty} \triangle_i$. Then $\triangle_0 \subseteq \triangle$. We also can show that $\triangle$ has the finite intersection property. In fact, let $U_1, U_2, \cdots, U_n \in \triangle$, then there is an integer m such that $U_1, U_2, \cdots, U_n \in \triangle_m$ and so $\cap_{i=1}^{n} U_i \neq \emptyset$ since $\triangle_m \in \mathcal{F}$. It follows that $\triangle \in \mathcal{F}$ and hence it is a upper bound of the chain $\{\triangle_i : i = 1, 2, \ldots, n, \ldots\}$. By Zorn's Lemma, $\mathcal{F}$ has a maximal element $\triangle'$, it is the desired one.

Theorem 3.19. *Let X be a transitive open TBCI-algebra and I a compact topological ideal of X. Let $\beta := \{I_x \mid x \in X\}$ be a base for topology $\mathcal{T}$, where I_x is compact for each $x \in X$. Then*

(i) *If $Q \subseteq X/I$ is compact, then so is $P = \Phi_I^{-1}(Q)$.*

(ii) *If X/I is compact* (resp. *locally compact*), *then X is compact* (resp. *locally compact*).

Proof. (i) By Proposition 3.18, we only need to prove the theorem for a maximal collection $\triangle$ of the closed subsets in the subspace P having the finite intersection property. We shall show that it also has non-empty intersection.

In the space Q, consider the collection $\triangle^*$ of all sets of the form $\Phi_I(F)$ such that $F \in \triangle$. Since $\triangle$ has the finite intersection property, so is the system $\triangle^*$. Let $\overline{\triangle^*}$ denote the collection $\{\overline{\Phi_I(F)} \cap Q \mid F \in \triangle\}$. Then $\overline{\triangle^*}$ is a collection of closed subsets in the space Q having the finite intersection property. By hypothesis, Q is compact, and so there exist a common point $A \in \overline{\Phi_I(F)}$ for each $F \in \triangle$. Now let U be an arbitrary neighborhood of zero in X and U^* a set of all equivalence classes contained in $A * U$, i.e., $U^* = \{I_a \in X/I \mid I_a \subseteq A * U\}$. By Lemma 3.16, $A * U$ is open in X and hence $\Phi_I(A * U)$ is open in X/I since Φ_I is an open mapping. Clearly $U^* \subseteq \Phi_I(A * U)$. On the other hand, by hypothesis, $\{I_x \mid x \in X\}$ is a base for topology $\mathcal{T}$ and so $A * U = \bigcup_{i\in\Lambda} I_{x_i}$ for some index set Λ. Thus for any $I_a = \Phi_I(a) \in \Phi_I(A*U)$, we have $a \in A*U = \bigcup_{i\in\Lambda} I_{x_i}$ and hence $a \in I_{x_{i_0}}$ for some $i_0 \in \Lambda$. Therefore $I_a = I_{x_{i_0}} \subseteq A * U$ and so $I_a \in U^*$, which shows that $\Phi_I(A * U) = U^*$. Hence U^* is open in X/I. Since $A \subseteq A * U$, we get $A \in U^*$ and consequently U^* meets every set of the system $\triangle^*$. We claim that $A * U$ meets every set of $\triangle$. Let $F \in \triangle$. Then $\Phi_I(F) \in \triangle^*$ and so $U^* \cap \Phi_I(F) \neq \emptyset$. Thus we can find $a \in F$ such that $I_a \in \Phi_I(F)$ and $I_a \in U^*$. Note that $I_a \in U^*$ implies $I_a \subseteq A * U$ and hence $a \in A * U$. Therefore $(A * U) \cap F \neq \emptyset$. It follows that $A * U$ meets every set of $\triangle$. Put $FU^{-1} := \{x \in X \mid x * U \in F$ for some $u \in U\}$. Since $(A * U) \cap F \neq \emptyset$, we have $FU^{-1} \cap A \neq \emptyset$ for each $F \in \triangle$. Indeed let $a \in A * U \cap F$, then there is $x \in A$ and $u \in U$ such that $a = x * u \in A * U$ and $a = x * u \in F$. Hence $x \in FU^{-1} \cap A$. Now let $\triangle' = \{FU^{-1} \cap A \mid F \in \triangle, U$ is neighborhood of the zero in $X\}$. From the above fact we have that the system $\triangle'$ also has the finite intersection property. In fact, if $F_1U_1^{-1} \cap A$ and $F_2U_2^{-1} \cap A$ are in $\triangle'$, then taking $F = F_1 \cap F_2$ and $U = U_1 \cap U_2$ we have $FU^{-1} \cap A \in \triangle'$ since $F \in \triangle$ by the maximal property of $\triangle$. Note that

$$\emptyset \neq FU^{-1} \cap A \subseteq (F_1U_1^{-1} \cap A) \cap (F_2U_2^{-1} \cap A).$$

Therefore $\triangle'$ has the finite intersection property. Since I_x is compact for each $x \in X$ by hypothesis, A ($= I_{x_0}$ for some $x_0 \in X$) is compact in X and so there exists $a \in \overline{FU^{-1} \cap A}$ for each $FU^{-1} \cap A \in \triangle'$. For any neighborhood V of zero in X, we have that $a * V$ and $(a*V)*U$ are open by Lemma 3.16. Hence $\overline{FU^{-1} \cap A}$ and $a*V$ have nonempty intersection and consequently $FU^{-1} \cap A$ and $a*V$ have nonempty intersection. It follows that F meets the open set $(a * V) * U$. By Lemma 3.3, we have that for any neighborhood W_a of a, there exist neiborhoods V_a and U_0 of a and 0, respectively, such that $V_a * U_0 \subseteq W_a$. Put $V_0 = \{x \in X \mid a * x \in V_a\}$, then V_0 is an open set by Lemma 3.16 and thus V_0 is a neighborhoot of zero. Since $a * V_0 \subseteq V_a$, we have $(a * V_0) * U_0 \subseteq V_a * U_0 \subseteq W_a$. By the above argument we have $((a * V_0) * U_0) \cap F \neq \emptyset$, and hence $W_a \cap F \neq \emptyset$. It follows that $a \in \overline{F}$ and so $a \in F$ since F is closed. So a is common to all the sets of the system $\triangle$, which shows that P is compact.

(ii) It follows from (i) that if X/I is compact, so is X. Now let X/I be locally compact and $a \in X$. Then $I_x \in X/I$ and thus there is a neighborhood U of I_x such that it's closure $\overline{U}$ is compact. Put $P = \Phi_I^{-1}(U)$. Then P is a neighborhood of x. Noticing that $P \subseteq \Phi_I^{-1}(\overline{U})$, we have $\overline{P} \subseteq \Phi_I^{-1}(\overline{U})$ since $\Phi_I^{-1}(\overline{U})$ is closed. Since $\Phi_I^{-1}(\overline{U})$ is compact by (i) and since $\overline{P}$ is a closed subset of $\Phi_I^{-1}(\overline{U})$, therefore $\overline{P}$ is compact. Hence X is locally compact. □

Corollary 3.20. *Let X be a p-semisimple TBCI-algebra, I a compact topological c-ideal and the system $\beta := \{I_x \mid x \in X\}$ a base for the topology $\mathcal{T}$. If $Q \subseteq X/I$ is compact, then $P = \Phi_I^{-1}(Q)$ is compact in X. In particular, if X/I is compact (locally compact), then X is compact (locally compact).*

Proof. By Theorem 3.17, X is transitive open and then for each $a \in X$, f_a is continuous and open. Since X is a p-semisimple, we can see that f_a is bijective. Indeed, if $f_a(x) = f_a(y)$, then $a * x = a * y$ and so $x = a * (a * x) = a * (a * y) = y$, which implies f_a is injective. Moerover for each $x \in X$, $f_a(a * x) = a * (a * x) = x$ and so f_a is surjective. Therefore f_a is a homeomorphism. Now we claim that $I_a = f_a(I)$, for each $a \in X$. Let $x \in f_a(I)$. Then $x = f_a(y)$ for some $y \in I$. Since I is c-ideal, $x * a = (a * y) * a = 0 * y \in I$. Note that $a * x = a * (a * y) = y \in I$. Then we have $x \in I_a$. Conversely if $x \in I_a$, then $x * a \in I$ and $a * x \in I$. Thus there exists $y \in I$ such that $y = a * x$. Hence $x = a * (a * x) = a * y = f_a(y) \in f_a(I)$. Therefore $I_a = f_a(I)$ and hence I_a is compact. This shows that X satisfies the hypothesis of Theorem 3.19 and so Corollary 3.20 holds. □

4. Topological homomorphisms in topological BCI-algebras

Definition 4.1. Let X and Y be two TBCI-algebras. A mapping $g : X \to Y$ is called a *topological homomorphism* if

(i) g is a homomorphism from X to Y as BCI-algebras,

(ii) g is a continuous mapping in the topological spaces.

A topological homomorphism g from X to Y is said to be *open* if g is an open mapping of the topological spaces.

Definition 4.2. Let X and Y be two TBCI-algebras. A mapping $f : X \to Y$ is called a *topological isomorphism* if

(i) f is an isomorphism of the BCI-algebras,

(ii) f is a homeomorphism of the topological spaces.

If $X = Y$, the topological isomorphism of X into Y is called a *topological automorphism*. Two TBCI-algebras are said to be *topological isomophic* if there exists a topological isomorphism of X into Y.

Proposition 4.3. *Let X and Y be transitive open TBCI-algebras and g be a homomorphism of a BCI-algebra X into BCI-algebra Y.*

(i) *If for each neighborhood U^* of 0^* in Y, there exists a neighborhood U of 0 in X such that $g(U) \subseteq U^*$, then g is continuous.*

(ii) *If for each neighborhood V of 0 in X, there exists a neighborhood V^* of 0^* in Y such that $g(V) \supseteq V^*$, then g is open.*

Proof. (i) Let $a \in X$ and $g(a) = a^*$ in Y. Assume that W^* is an arbitrary neighborhood of a^*. Then $U^* = \{y^* \in Y \mid a^* * y^* \in W^*\}$ is an open set by Lemma 3.16 and $0^* \in U^*$, i.e., U^* is a neighborhood of 0^*. Hence by hypothesis, there exists a neighborhood U of 0 in X such that $g(U) \subseteq U^*$. Thus the open set $a * U$ contains a and $g(a * U) = g(a) * g(U) = a^* * g(U) \subseteq a^* * U^* \subseteq W^*$. Therefore g is continuous.

(ii) Let $a \in X$ and $g(a) = a^*$. Assume that W is an arbitrary neighborhood of a. Then $V = \{x \in X \mid a * x \in W\}$ is an open set containing 0 by Lemma 3.16. By hypothesis there exists a neighborhood V^* of 0^* in Y such that $g(V) \supseteq V^*$. Note that $a^* * V^*$ is an open set containing a^* and $a^* * V^* \subseteq g(a) * g(V) = g(a * V) \subseteq g(W)$. It follows that $g(W)$ is open and so g is an open mapping. $\square$

Theorem 4.4. *Let X be a TBCI- algebra and I a topological c-ideal of X. Then the natural projection Φ_I is an open topological homomorphism of X onto X/I and $\{I_0\}$ is an open set in X/I.*

Proof. Obviouly Φ_I is a homomorphism of BCI-algebras, and it is open and continuous by Proposition 3.12. Hence Φ_I is an open topological homomorphism of X onto X/I. Since I is a topological c-ideal, we have $\Phi_I(I) = \{I_0\}$ and so $\{I_0\}$ is open in X/I. $\square$

The following theorem provides the converse of Theorem 4.4.

Theorem 4.5. *Let X and Y be two TBCI-algebras. Assume g is an open topological homomorphism of X onto Y having $kerg = I$, and the zero element $\{0'\}$ of Y is an open set in Y. Then we have*

(i) *I is a topological ideal of a TBCI- algebra X.*

(ii) *if we define $f : X/I \to Y$ by $f(I_x) = g(x)$, then f is a topological isomorphism, i.e., X/I is topological isomorphic to Y.*

Proof. (i) It is easy to see that I is a BCI-ideal of X. Moreover since g is continuous and $\{0'\}$ is open in Y, we have that $I = kerg = g^{-1}(0')$ is open in X. Therefore I is a topological ideal of X.

(ii) Note that $f(I_x * I_y) = f(I_{x*y}) = g(x * y) = g(x) * g(y) = f(I_x) * f(I_y)$ for all $I_x, I_y \in X/I$. Thus f is a homomorphism of the BCI-algebra X/I onto Y. Next for each $y \in Y$, since g is onto, there is $x \in X$ such that $g(x) = y$. Hence $f(I_x) = g(x) = y$, which implies that f is onto. Finally if $f(I_x) = f(I_y)$, then $g(x) = g(y)$. It follows that $g(x * y) = g(x) * g(y) = 0'$ and $g(y * x) = g(y) * g(x) = 0'$ and hence $x * y \in kerg = I$ and $y * x \in kerg = I$. Hence $I_x = I_y$. This shows that f is injective. Combining the above arguments we have that f is an isomorphism of the BCI-algebra X/I onto the BCI-algebra Y. Finally we show that f is a homeomorphism of the topological space X/I into the topological space Y. We shall verify that both f and f^{-1} are continuous. Let $A = I_a \in X/I$ and $f(A) = a'$. Assume that U^* is an arbitrary neyghborhood of a'. Then $g(a) = f(I_a) = f(A) = a'$. Since g is continuous, there exists a neighborhood V of a such that $g(V) \subseteq U^*$. Denote by V^* the neighborhood of A in X/I consisting of all I_x with $x \in V$. Then $f(V^*) = \{f(I_x) \mid I_x \in V^*\} = \{g(x) \mid x \in V\} = g(V) \subseteq U^*$, and hence f is continuous. Next let $a' \in Y$ and $f^{-1}(a') = A$. Denote by U^* an arbitrary neighborhood of A in the space X/I. Then $U := \Phi_I^{-1}(U^*)$ is an open set in X. Let a be an element in U such that $A = I_a$. Then $g(a) = f(I_a) = f(A) = a'$. Since g is open, there exists a neighborhood V^* of a' such that $V^* \subseteq g(U)$. Now we claim that $f^{-1}(V^*) \subseteq U^*$. Indeed, for each $f^{-1}(x) \in f^{-1}(V^*)$, we have $x \in V^*$ and so $x = g(u)$ for some $u \in U$. Thus $f^{-1}(x) = f^{-1}(g(u)) = f^{-1}(f(I_u)) = I_u \in U^*$ which shows that $f^{-1}(V^*) \subseteq U^*$. Therefore f^{-1} is continuous, ending the proof. $\square$

¿From Theorem 4.5 we have the following corollary.

Corollary 4.6. Let X and Y be two TBCI-algebras with $\{0'\}$ being open set in Y. If g is an open topological homomorphism of X onto Y having $kerg = \{0\}$, then g is a topological isomorphism and thus X and Y are topological isomorphic.

Theorem 4.7. *Let X and Y be two TBCI-algebras and f an open topological homomorphism of X onto Y. Denote by N' the kernel of f. Then f gives rise to a one-to-one correspondence between the closed c-ideals of Y and the closed c-ideals of X which contain N' as follows:*

(i) *if N^* is a closed c-ideal of Y, then the closed c-ideal N of X corresponding to it is just the inverse image $N = f^{-1}(N^*)$;*

(ii) *if N is a closed c-ideal of X containing N', then the closed c-ideal N^* of Y corresponding to it is just the image $N^* = f(N)$.*

(iii) *The two correspondences thus defined are mutually inverse to one another. Moreover topological ideals correspond to one another. Finally if N and N^* are two topological ideals corresponding to one another in this fashion, then the quotient TBCI-algebras X/N and Y/N^* are topological isomorphic.*

Proof. We consider first the correspondence of N to N^*. Let N^* be a closed c-ideal of Y. Then $N = f^{-1}(N^*)$ is also closed and contains N' since f is continuous. Moreover for each $y, x * y \in N$, we have $f(y), f(x) * f(y) = f(x * y) \in N^*$. Since N^* is an ideal of BCI-algebra Y, it follows that $f(x) \in N^*$ and hence $x \in f^{-1}(N^*) = N$. This shows that N is an ideal of BCI-algebra X. Finally let $x \in N$, then $f(x) \in N^*$ and thus $f(0 * x) = f(0) * f(x) = 0' * f(x) \in N^*$ since N^* is a c-ideal, where $0'$ is the zero element in Y. Therefore $0 * x \in f^{-1}(N^*) = N$ and N is a closed c-ideal of X. Now we show that the topological ideals correspond to one another and the quotient TBCI-algebras are topological isomorphic. If N^* is a topological ideal of Y and if Φ_{N^*} denotes the natural projection of Y onto Y/N^*, then $h := \Phi_{N^*} \circ f$ is an open topological homomorphism of X onto Y/N^*, with kernel N, and $\{N^*\} = \Phi_{N^*}(N^*)$ is an open set in Y/N^*. By Theorem 4.5, $N = ker h$ is a topological ideal of X and X/N is topological isomorphic to Y/N^*.

Conversely let N be a closed c-ideal of X containing N' and consider the corresponence of N^* to N where $N^* = f(N)$ and $N' \subseteq N$. We first show that the inverse image of N^* under the mapping f coincides with N. Indeed if $f(a) \in N^*$, then there exists $b \in N$ such that $f(a) = f(b)$. Thus $f(a * b) = f(a) * f(b) = 0'$, i.e., $a * b \in N' \subseteq N$ so that $a \in N$ since N is a BCI-ideal and $b \in N$. Hence $f^{-1}(N^*) \subseteq N$ and so $f^{-1}(N^*) = N$. From this fact it follows that $f(X \setminus N) = Y \setminus N^*$. Since f is open and $X \setminus N$ is an open set, $Y \setminus N^*$ is also open and hence the set N^* is closed in Y. Finally we show that N^* is a c-ideal of Y. Let $y', x' * y' \in N^*$, then there exist $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$. It follows that $f(y) = y' \in N^*$ and $f(x * y) = f(x) * f(y) = x' * y' \in N^*$ so that $y, x * y \in N$. Since N is a BCI-ideal of X, we have $x \in N$, which shows that $f(x) = x' \in N^*$ and thus N^* is an ideal of BCI-algebra Y. Moreover for each $x' \in N^*$, there exists $x \in N$ such that $f(x) = x'$ and hence $0' * x' = f(0) * f(x) = f(0 * x) \in f(N) = N^*$. It follows that N^* is a c-ideal of BCI-algebra Y. Therefore N^* is a closed c-ideal of Y, ending the proof. □

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