THE STRUCTURE OF BANACH ALGEBRAS A, SATISFYING $xAx = x^2Ax^2$ FOR EVERY $x \in A$

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Abstract. We characterize a Banach algebra A, semiprime or not, in terms of (the generalized) Le Page condition $xAx = x^2Ax^2$ for every $x \in A$.

1. Introduction Let A be a complex Banach algebra. If A is unital and satisfies condition

$$\text{(I)} \qquad \text{“}Ax = Ax^2 \ \text{ for every } \ x \in A\text{”},$$

then A is semisimple commutative (Le Page [8]), and finite dimensional (Duncan and Tullo [3], hence isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$). Besides, B. Aupetit [1] completed the last result showing that a unital Banach algebra E is finite dimensional, commutative and semisimple if and only if, for every $x \in E$, there exists $y \in E$ with $x = x^2y$.

For the non-unital case, Esterle and Oudadess in [4] showed that A satisfies the condition (I) above if and only if $A = B \oplus R$, where B is a subalgebra of A isomorphic to $\mathbb{C}^n$, for some $n \geq 0$ and R the (Jacobson) radical of A, while $AR = (0)$.

A. Fernadez Lopez and E. Garcia Rus in [9] considered on A the condition

$$\text{(II)} \qquad \text{“}xAx = x^2Ax^2 \ \text{ for every } \ x \in A\text{”},$$

and proved that, if a complex Banach algebra A is semiprime and satisfies condition (II), then A is a (finite) diret sum $A = M_1 \oplus \cdots \oplus M_n$ of ideals each one of which is isomorphic to $\mathbb{C}$.

In this article we give a characterization of (complex) Banach algebras, semiprime or not, satisfying condition (II). We also prove that conditions (I) and (II) are equivalent when A has no non-zero nilpotents and that condition (I) always implies condition (II) for Banach algebras. Finally, we give examples of (unital and non-unital) Banach algebras satisfying condition (II) but not condition (I).

In the sequel, all vector spaces and algebras will be taken over the complex field $\mathbb{C}$. For the standard normed algebra terms employed here we refer the reader to Bonsall and Duncan [2]. We write $SocA$ for the socle of an algebra A, R for the (Jacobson) radical of the algebra and $r(x)$ for the spectral radius of $x \in A$.

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2. The Semiprime case

Lemma 2.1. *Let A be an algebra satisfying condition (II). Then the statements (i)-(iv) are equivalent: (i) A is semisimple, (ii) A is semiprime, (iii) A has no non-zero nilpotents, (iv) if $x \in A$ with $x^2 = 0$, then $x = 0$.*

Proof. $(i) \Rightarrow (ii)$ is well known. $(ii) \Rightarrow (iii)$ Suppose $x^n = 0$ for some $x \in A$, $n \geq 2$. Then $xAx = x^n A x^n = 0$, which implies $(Ax)^2 = (0)$. So, since A is semiprime, $Ax = (0)$ and thus $x = 0$. $(iii) \Leftrightarrow (iv)$ is obvious. $(iii) \Rightarrow (i)$ If $x \in R$, then by (II), there exists some $y \in A$ such that $x^3 = x^3 y x^3$. But, $x^3 y$ is an idempotent element in R, therefore $x^3 y = 0$ and hence $x = 0$. ■

Lemma 2.2. *Let A be an algebra with no non-zero nilpotents. Then the following are equivalent:*

(i) A satisfies condition (I),

(ii) A satisfies condition (II).

Proof. $(i) \Rightarrow (ii)$: *If $x \in A$ then there exists $y \in A$ with $x = yx^2 = x^2 y$.*

In fact, $x^2 = yx^3$ for some $y \in A$. Since $x^2 - xyx^2$ and $x^2 - x^2yx$ are nilpotents, we obtain $x^2 = xyx^2 = x^2yx$. Hence $x - yx^2$ and $x - x^2y$ are nilpotents and therefore $x = yx^2 = x^2y$.

¿From the claim above, we have $xAx = x^2yAyx^2 \subseteq x^2Ax^2 \subseteq xAx$, the desired.

$(ii) \Rightarrow (i)$ If $x \in A$, then there exists some $y \in A$ with $x^3 = x^2yx^2$. Thus, $(x - yx^2)^4 = 0$ and hence, $x = yx^2$. Therefore, $Ax = Ayx^2 \subseteq Ax^2 \subseteq Ax$ and this completes the proof. ■

By Lemmas 2.1, 2.2, [7: p. 267, Proposition 2.1] and [11: p.361, Proposition 3.2] we get the next.

Proposition 2.3. *Let A be a Banach algebra satisfying condition (II). Then*

(i) A/R is commutative.

(ii) $R = \{x \in A : r(x) = 0\} = \{x \in A : x^3 = 0\}$
$= \{x \in A : xax = 0 \text{ for every } a \in A\}$.

Remark 2.4. - 1) From Proposition 2.3, ii) above it follows that, if A is a Banach algebra, then the assertions (i)-(iv) in Lemma 2.1 are equivalent to: (v) A has no non-zero quasi-nilpotents.

2) If A is a Banach algebra satisfying condition (I), then semisimplicity of A is equivalent for A to be unital (see [4: p. 93, Theorem 3.2]).

3) From Lemma 2.1 and Remark 2.4, 1) it follows that, if A satisfies condition (II), then semisimplicity of A implies that A is unital. Besides, Example 2.5 below shows that there is a unital Banach algebra, which satisfies condition (II) without being semiprime.

Examples 2.5. Suppose $e = e^2$ and $0 \neq x = ex = xe$ with $x^2 = 0$. Set $A = \{\lambda e + \mu x : \lambda, \mu \in \mathrm{C}\}$. Then A is a Banach algebra under the norm $\|\lambda e + \mu x\| = |\lambda| + |\mu|$ with unit e. Moreover, A satisfies condition (II). Of course, A does not satisfy condition (I), since $Ax \neq (0)$ while $Ax^2 = (0)$. Cf. also Lemma 2.2.

Theorem 2.6. *Let A be a semiprime Banach algebra. Then the following statements are equivalent:*

(i) For every $x \in A$ there exists $n = n(x)$, $n \geq 2$, with $xAx = x^{n(x)}Ax^{n(x)}$;

(ii) A satisfies condition (II);

(iii) A satisfies condition (I);

(iv) A is unital with no non-zero nilpotent elements and $A = SocA$;

(v) A is a direct sum $A = M_1 \oplus \cdots \oplus M_n$ of ideals each one of which is isomorphic to C*;*

(vi) $A \simeq \mathrm{C}^n$.

Proof. $(i) \Rightarrow (ii)$ It is clear that $xAx = x^{n(x)}Ax^{n(x)} \subseteq x^2Ax^2 \subseteq xAx$. $(ii) \Rightarrow (iii)$ Lemma 2.2. $(iii) \Rightarrow (iv)$ By Lemmas 2.1, 2.2, A has no non-zero nilpotent elements. Moreover, for every $x \in A$, $x \neq 0$ there exists $e = e^2 \in A$, such that $x = ex = xe$. We prove that $e \in SocA$ or equivalently (see Corollary 6 in [6]) that $eAe(= Ae)$ is finite dimensional. Otherwise, Ae contains an infinite sequence of pairwise orthogonal idempotents, say $\{e_n\}$. For $x = \sum_{n=1}^{\infty} \frac{e_n}{2^n\|e_n\|}$ there exists some $y \in A$, such that $x = yx^2$ and therefore $e_n = \frac{1}{2^n\|e_n\|}ye_n$ for every n. Hence $1 = r(e_n) = \frac{1}{2^n\|e_n\|}r(y) \longrightarrow 0$, as $n \to +\infty$, which is impossible. By a similar argument we obtain in A that every set of orthogonal idempotents is finite and therefore $e_1 + \cdots + e_n$ is the unit of A, where $\{e_1, \cdots, e_n\}$ is a maximal set of pairwise orthogonal minimal idempotents of A. $(iv) \Rightarrow (ii)$ Since A is semiprime with $A = SocA$, we have that $\dim(xAx) < +\infty$ for every $x \in A$ (see [6: p. 362, Corollary 6]). Therefore, x is an algebraic element of A. Hence, there exists a polynomial $p(t) = \lambda_0 1 + \lambda_1 t + \cdots + \lambda_n t^n$ with $p(x) = 0$. Claim that $|\lambda_0| + |\lambda_1| \neq 0$, otherwise, A contains non-zero nilpotents, a contradiction. If $\lambda_0 \neq 0$, then $1 = -\lambda_0^{-1}(\lambda_1 x + \cdots + \lambda_n x^n) \equiv q(x)$. Hence $xax = x1a1x = xq(x)aq(x)x \in x^2Ax^2$. By a similar argument $xAx \subseteq x^2Ax^2$, if $\lambda_0 = 0$ and $\lambda_1 \neq 0$. The implications: (iii)$\Rightarrow$(vi), (iv)$\Leftrightarrow$(v)$\Rightarrow$(vi) and (iv)$\Leftrightarrow$(vi) are now clear. (See also Theorem 3.2 in [4] and Theorem in [9]). ■

Remark.- Notice that, if any of (i)-(vi) in Theorem 2.6 holds true, then A is an annihilator algebra (see [12: p. 38, Lemma 3.3]).

3. The general case (the main theorem)

Lemma 3.1. *Let A be an algebra, which satisfies condition (II). Then, for every $a, b \in A$ and for every $x, y \in R$, we have: (i) $xay + yax = 0$. (ii) $x^2ay = xay^2 = x^2ya = yax^2 = ayx^2 = yx^2a = 0$. (iii) $yx^2 + x^2y = 0$. (iv) $axby = xaby = bxay = xbya = xayb$.*

Proof. It is enough to prove (i). All the other equalities stem from it, after an appropriate choice of elements. So, if $x, y \in R$, then $x + y \in R$. By Proposition 2.3, $R = \{x \in xAx = 0$ for every $a \in A\}$ (the proof here is purely algebraic) hence $xax = 0 = yay$ and $(x + y)a(x + y) = 0$ for every $a \in A$. Thus (i) follows. ■

Theorem 3.2. (The main theorem). *Let A be a Banach algebra. Then the following assertions are equivalent:*

(i) A satisfies condition (II);

(ii) $A = B \oplus R$, where B is a subalgebra of A isomorphic to $\mathbb{C}^n$ for some $n \geq 0$, $xAx = (0)$ for every $x \in R$ and $ayx = yxa$ for every $a \in B$ and $x, y \in R$.

Proof. (i)$\Rightarrow$(ii) The Banach algebra $E = A/R$ is commutative semisimple and $uEu = u^2Eu^2$ for every $u \in E$. Hence $E = \mathbb{C}^n$ for some $n \geq 0$ (cf. [9: p.144, Theorem]). We ommit the trivial case, when $E = (0)$. We can find a family $\{f_1, \cdots, f_n\}$ of pairwise orthogonal idempotents in E (viz. $f_i f_j = 0$, if $i \neq j$, $i, j = 1, \cdots, n$ and $0 \neq f_i = f_i^2$ for any $i = 1, \cdots, n$), such that $E = \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \cdots \oplus \mathbb{C}f_n$. Denote by $\pi : A \longrightarrow A/R$ the natural surjection. Then, there exist $a_1, a_2, \cdots, a_n \in A$ such that $\pi(a_i) = f_i$, $1 \leq i \leq n$. So, $\pi(a_i^2 - a_i) = 0$ and therefore $a_i^2 - a_i \in R$. Notice that $a_i \notin R$ and hence, by Proposition 2.3, $r(a_i) \neq 0$.

Claim: *There exist $e_1, e_2, \cdots, e_n$ pairwise orthogonal idempotents in A such that $\pi(e_i) = f_i$, $i = 1, \cdots, n$.*

In fact, if $a_i^2 - a_i = r_1 \in R$, we put $e_1 = a_1(1 - 2r_1 + 6r_1^2) + (r_1 - 3r_1^2)$ (see also [5: p.772, the proof of Lemma 1]). Then $e_1 = e_1^2$ and $\pi(e_1) = \pi(a_1) = f_1$. Assume that $e_1, e_2, \cdots, e_m$, $1 \leq m < j \leq n$ have been defined, such that $\pi(e_i) = f_i$, $e_i e_k = 0 = e_k e_i$, $i \neq k$, $1 \leq i, k \leq m < j \leq n$ and $e_m^2 = e_m$. Put $h = e_1 + \cdots + e_m$. Then $h^2 = h$ and $e_i h = h e_i$, $i = 1, 2, \cdots, m$. Put $w_{m+1} = (1 - h)a_{m+1}(1 - h)$. Then $e_i w_{m+1} = w_{m+1} e_i = 0$ for $1 \leq i \leq m$ and $\pi(w_{m+1}) = f_{m+1}$. Hence $w_{m+1}^2 = (1 - h)a_{m+1}(1 - h)a_{m+1}(1 - h) = (1 - h)a_{m+1}^2(1 - h) - (1 - h)a_{m+1}ha_{m+1}(1 - h) = (1 - h)a_{m+1}(1 - h) + (1 - h)r'_{m+1}(1 - h) - (1 - h)a_{m+1}ha_{m+1}(1 - h)$, where $a_{m+1}^2 - a_{m+1} = r'_{m+1} \in R$ and therefore $w_{m+1}^2 - w_{m+1} = r_{m+1} \in R$. Denote $e_{m+1} = w_{m+1}(1 - 2r_{m+1}) + (r_{m+1} - 3r_{m+1}^2)$. Then $e_i e_{m+1} = 0 = e_{m+1} e_i$ for $1 \leq i < m$, $\pi(e_{m+1}) = \pi(w_{m+1}) = f_{m+1}$ and $e_{m+1}^2 = e_{m+1}$. We continue this way inductively up to n.

Now, define $B = [e_1, \cdots, e_n]$, the linear span of $\{e_1, \cdots, e_n\}$. Then $B \simeq \mathbb{C}^n$. We also have $B \cap R = (0)$. In fact, if there exists $0 \neq b \in B \cap R$, then $b = \lambda_1 e_1 + \cdots + \lambda_n e_n$, $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n$ and $\lambda_k \neq 0$ for some $1 \leq k \leq n$. Hence $be_k = \lambda_k e_k \in B \cap R$. Therefore $e_k \in B \cap R$, which implies $e_k = 0$, a contradiction. Now, if $x \in A$, there exist $\mu_1, \cdots, \mu_n \in \mathbb{C}$, such that $\pi(x) = \mu_1 f_1 + \cdots + \mu_n f_n$. Thus $x - \mu_1 e_1 - \cdots - \mu_n e_n \in R$ and hence $A = B \oplus R$. By Proposition 2.3, $xAx = (0)$ for every $x \in R$. To prove that $ayx = yxa$ for $a \in B$ and $y, x \in R$ it is enough to show that

$$eyx = yxe \quad \text{for every} \quad e = e^2 \in B. \tag{3.1}$$

Claim: $ex^2 = x^2e$ for $e = e^2 \in B$ and $x \in R$.

In fact, by condition (II), there exist $c \in B$ and $z \in R$ such that

$$(e + x)e(e + x) = (e + x)^2(c + z)(e + x)^2. \tag{3.2}$$

Since $xAx = (0)$, (3.2) implies $e+ex+xe = e+ex+xe+2exe+ex^2+x^2e+ezxe+exze+eze$ and hence $ex^2 = ex^2e = x^2e$.

Now, for $e = e^2 \in B$ and $x, y \in R$, there exist $c \in B$ and $z \in R$ with

$$(e+x)(e+y)(e+x) = (e+x)^2(c+z)(e+x)^2. \tag{3.3}$$

By (3.3), Proposition 2.3 and Lemma 3.1 we obtain

$$eyx + xye + eye = 2exe + eze + ex^2 + x^2e + ezxe + exze. \tag{3.4}$$

¿From (3.4) and the fact that $ex^2 = x^2e$ we get $eyx = eyxe$ and $xye = exye$ for every $e^2 = e \in B$ and $x, y \in R$. Hence $eyx = eyxe = yxe$ and (3.1) follows.

(ii)⇒(i) It is enough to show that $xAx \subseteq x^2Ax^2$ for every $x \in A$. If $a, b \in B$ and $x, y \in R$, then there exist idempotents $e, e' \in B$ with $a = ea = ae$ and $b = e'b = be'$ (e.g., if $a = \lambda_{k_1}e_{k_1} + \lambda_{k_2}e_{k_2} + \cdots + \lambda_{k_m}e_{k_m}$, $\lambda_{k_i} \neq 0$, $1 \leq i \leq m \leq n$ and $b = \mu_{l_1}e_{l_1} + \cdots + \mu_{l_s}e_{l_s}$, $\mu_{l_j} \neq 0$, $1 \leq j \leq s \leq n$, put $e = e_{k_1} + \cdots + e_{k_m}$ and $e' = e_{l_1} + \cdots + e_{l_s}$. Then for $d = \lambda_{k_1}^{-1}e_{k_1} + \cdots + \lambda_{k_m}^{-1}e_{k_m}$ and $h = \lambda_{k_1}^{-2}e_{k_1} + \cdots + \lambda_{k_m}^{-2}e_{k_m}$, we have $ad = e$, $a^2d = a$ and $a^2h = e$. Similarly, for e'). Now, for $c = hb$ and $z \in R$ with $z = dyd - cxd - dxc - 2d^2cx^2 + dyxd + hxyd$, we get $(a+x)(b+y)(a+x) = (a+x)^2(c+z)(a+x)^2$.

In fact, $J_1 \equiv (a+x)(b+y)(a+x) = a^2b + abx + xab + ayx + xya + aya$ and $J_2 \equiv (a+x)^2(c+z)(a+x)^2 = a^4c + a^2za^2 + a^3cx + a^2cxa + a^2cx^2 + a^2zax + a^2zxa + axa^2c + xa^3c + x^2a^2c + axza^2 + xaza^2$. By Lemma 3.1, $a^2zax + xaza^2 = 0$ and for $c = hb$, we get $a^2b = a^4c$. Therefore $J_2 \equiv a^2b + a^2za^2 + abx + xab + ebxa + axeb + ebx^2 + x^2eb + axza^2$ and for $z = dyd - cxd - dxc - 2d^2cx^2 + dyxd + hxyd$ we have $J_2 = a^2b + abx + xab + aya + ayx + xya = J_1$ and the proof is complete. ∎

¿From Theorem 3.2 above and Theorem 3.2 in [4] we have the following.

Corollary 3.3. *Let A be a Banach algebra. If A satisfies condition (I), then A satisfies condition (II), as well.*

Examples 3.4. We give an example of a non-unital Banach algebra, which satisfies condition (II), but not condition (I): Let H be a separable Hilbert space and $(e_n)_{n\geq 1}$ an orthonormal basis of H. We define $A = \{f \otimes e_1 : f \in H\}$, where $f \otimes e_1$ is the rank one operator on H given by $(f \otimes e_1)(h) = \langle h, f \rangle e_1$ for every $h \in H$. Then $xAx = x^2Ax^2$ for every $x = f \otimes e_1 \in A$ and $Ax \neq Ax^2$ for $x = f \otimes e_1$ with $f \perp e_1$. We can easily check that $A = B \oplus R$ where $B \simeq [e_1 \otimes e_1]$ and $R = \{f \otimes e_1 : f \in H, f \perp e_1\}$.

4. Condition (II) in Topological Algebras A *topological algebra* is a linear associative algebra over the complex field C, which moreover, is a topological vector space and the ring multiplication is separately continuous (see [10]). A *locally m-convex algebra* is a topological algebra whose topology is defined by a family $(p_\alpha)_{\alpha\in\Lambda}$ (where Λ is a directed index set) of submultiplicative seminorms (ibid.).

The following example yields a particular instance of a unital commutative semisimple (non-normed) topological algebra satisfying condition (II).

Examples 4.1. Consider the set C^{N} of all complex sequences. C^{N} becomes a commutative unital complex algebra under the coordinatewise operations. For each $n \in \mathrm{N}$, $p_n := |\cdot|_n \circ pr_n$, (where $|\cdot|_n$ is the usual algebra norm on $\mathrm{C}_n \equiv \mathrm{C}, n \in \mathrm{N}$ and $pr_n : \mathrm{C}^{\mathrm{N}} \longrightarrow \mathrm{C}_n$ is the canonical projection) defines on C^{N} a multiplicative seminorm. Equip C^{N} with the cartesian product topology, say τ. Then $(\mathrm{C}^{\mathrm{N}}, \tau)$ is a complete non-normed algebra. Moreover, the cartesian product topology is defined by the family $(p_n)_{n\in\mathrm{N}}$ and $(\mathrm{C}^{\mathrm{N}}, \tau)$ is a Fréchet locally m-convex algebra [10: p. 82, Lemma 1.1]. Besides, since C^{N} has an orthogonal basis, it satisfies condition (II). Finally, if $a \in \mathrm{C}^{\mathrm{N}}$ with $a^2 = 0$, then $a = 0$ and hence, by Lemma 2.1, C^{N} is semisimple.

On the other hand, one has the following topological algebra-theoretic proof (referee's remark). That is, one has $\mathrm{C}^{\mathrm{N}} = \mathcal{C}_c(\mathrm{N})$, within a topological algebra isomorphism; hence, the assertion, since the latter algebra is *"functionally semisimple"* (viz. Gel'fand map one-to-one [10]).

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